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Onset of Hyperbolic Macroscopic Behavior in Complex Systems Subjected to External Agents

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Abstract: This paper deals with the derivation of hyperbolic equations from a space inhomogeneous thermostatted kinetic equation which can be proposed for the modeling of complex systems subjected to the external actions at the microscopic and macroscopic scales. The particles of the system are able to perform an activity which is modeled by introducing a specific variable. The derivation of the hyperbolic equations is obtained by performing different scalings into the time and space variables and letting the scaling parameter goes to zero. Applications and future research direction are discussed into the last section of the paper.

Keywords: Thermostats, Nonlinearity, Integro-differential equation, Hyperbolic scaling, Asymptotic limit

1 Introduction

The interest in modeling complex systems dates back to the last century during which scientists coming from different applied sciences, e.g. mathematician, physicist, computer scientist, have spent much research activity with the main aim of identifying the fundamental properties of these systems. Specifically the research activity has been focused on the understanding of the interactions occurring among the components of the system which give rise to the collective behaviors and on how the system interacts and forms relationships with its environment [1]. Collective behaviors are usually consequence of the ability of individuals to develop specific and autonomous strategies. Moreover the collective behavior that emerges in complex systems is usually in response to external actions that can affect the whole dynamics, e.g., tumor growth can be stopped by vaccine cells, the dynamics of swarms of insect can be modified by the attack of a predator. Different mathematical and computational approaches have been developed, adapted and employed in an attempt to describe collective behaviors and macroscopic features as the result of microscopic interactions, see the references section of the review paper [2].

In this context, in kinetic theory, the asymptotic analysis has been proposed for deriving macroscopic information (evolution of local density and momenta of the distribution function) by microscopic (kinetic) interactions, see among others, papers [3,4,5,6,7,8,9,10]. The asymptotic analysis consists in defining an appropriate time and space scaling and letting the inter-particle distances tend to those of the macroscopic level. Depending on the choice of the scaling the asymptotic limit of kinetic equations leads to parabolic or hyperbolic equations [11,12,13].

This paper deals with the derivation of hyperbolic equations for the local density and the first velocity momentum of the system by employing an asymptotic analysis of a kinetic equation that is a generalization of the mathematical framework proposed and analyzed in [14]. Specifically, the underlying equation consists in a thermostatted kinetic equation, which includes the role of the external agents (open systems) at the macroscopic and microscopic scales, and a velocity jump-process. The underlying equation belongs to the class of nonlinear partial integro-differential equation with quadratic nonlinearity. The derivation of the hyperbolic equations is obtained by performing a scaling into the time and space variables and, under suitable assumptions on the terms of

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the equation, letting the scaling parameter goes to zero. The interested reader in the derivation of macroscopic equations from thermostatted kinetic models for closed systems is referred to the recent contributions [15,16,17].

The present paper is organized into four more sections that follow this introduction. Specifically, Section 2 is concerned with the derivation of the underlying thermostatted kinetic equation for open systems and the definition of the local macroscopic quantities. Section 3 is devoted to the derivation of the asymptotic equation by performing a time-space scaling of the thermostatted kinetic equation introduced in Section 1. In particular Section 3 contains the main result of the paper which consists in showing the onset of hyperbolic behavior. The derivation of asymptotic equations by generalized time and space scalings is performed in Section 4. Finally Section 5 concludes the paper by focusing on the applications which include, but are not limited, to biological systems, vehicular traffic, crowds and swarms dynamics. This section highlights also research perspectives from the mathematical and modeling viewpoint.

2 The Thermostatted Kinetic Equation for Open Systems

This section deals with the derivation of the space-inhomogeneous thermostatted kinetic equation for open systems that acts as a general paradigm for the derivation of specific models.

The system is composed by a large number of particles whose evolution is described by the distribution function $f = f(t,\mathbf{x},\mathbf{v},u) : [0,\infty) \times D_{\mathbf{x}} \times D_{\mathbf{v}} \times D_u$, where $\mathbf{x}$ is the space variable, $\mathbf{v}$ is the velocity variable and u is the variable which models the activity of the particles. These variables constitute the microscopic state of the particles and $\Omega = D_{\mathbf{x}} \times D_{\mathbf{v}} \times D_u$ denotes the domain of all possible microscopic states; $d\Omega = d\mathbf{x}\,d\mathbf{v}\,du$ denotes the Lebesgue measure on Ω. An external force fields $\mathscr{F} : D_u \to \mathbb{R}$ acts on the system thereby moving the system out of equilibrium. Moreover an external agent at the microscopic scale has the ability to modify the variable u by a particular action related to the variable $\omega \in D_u$; the action is modeled by the distribution function $g = g(t,\mathbf{x},\mathbf{v},\omega) : [0,\infty) \times D_{\mathbf{x}} \times D_{\mathbf{v}} \times D_u \to \mathbb{R}^+$, which is a known function of its arguments.

The evolution of the system is obtained by equating the time derivative of f to the balance of the particle interactions that occur in nonlinear matter. Specifically we have:

$$(\partial_t + \mathbf{v}\cdot\nabla_{\mathbf{x}}) f + \partial_u\left(\mathscr{F}\left(1 - u\int_{\Omega} u\, f\, d\Omega\right) f\right) =$$

$$\eta J[f] + \eta^e Q[f,g] + \nu V[f], \qquad (1)$$

where the operator $J[f] = J[f](t,\mathbf{x},\mathbf{v},u)$, which models the gain-loss of particles due to transitions in the activity variable, reads:

$$J[f] = \int_{(D_u)^2} \mathscr{A}(u_*,u^*,u)\, f(t,\mathbf{x},\mathbf{v},u_*)\, f(t,\mathbf{x},\mathbf{v},u^*)\, du_*\, du^*$$

$$- f(t,\mathbf{x},\mathbf{v},u)\int_{D_u} f(t,\mathbf{x},\mathbf{v},u^*)\, du^*. \qquad (2)$$

where:

- η models the probability that a particle with microscopic state $(\mathbf{x},\mathbf{v},u_*)$ interacts instantaneously with a particle with microscopic state $(\mathbf{x},\mathbf{v},u^*)$;
- $\mathscr{A} = \mathscr{A}(u_*,u^*,u) : D_u \times D_u \times D_u \to \mathbb{R}^+$ is the density function modeling the probability that particle with microscopic state $(\mathbf{x},\mathbf{v},u_*)$ interacting with particles with microscopic state $(\mathbf{x},\mathbf{v},u^*)$ reaches the microscopic state $(\mathbf{x},\mathbf{v},u)$. In particular $\mathscr{A}(u_*,u^*,u)$ satisfies the following identity:

$$\int_{D_u} \mathscr{A}(u_*,u^*,u)\, du = 1, \quad \forall u_*,u^* \in D_u.$$

The operator $V[f] \equiv V[f](t,\mathbf{x},\mathbf{v},u)$ models the velocity-jump process, and it reads:

$$V[f] = \int_{D_{\mathbf{v}}} T(\mathbf{v}^*,\mathbf{v})\, f(t,\mathbf{x},\mathbf{v}^*,u)\, d\mathbf{v}^*$$

$$- \int_{D_{\mathbf{v}}} T(\mathbf{v},\mathbf{v}^*)\, f(t,\mathbf{x},\mathbf{v},u)\, d\mathbf{v}^*, \qquad (3)$$

where $T(\mathbf{v}^*,\mathbf{v})$ is the turning kernel which gives the probability that, if a jump occurs, the velocity $\mathbf{v}^* \in D_{\mathbf{v}}$ jumps into the velocity $\mathbf{v} \in D_{\mathbf{v}}$. The domain $D_{\mathbf{v}}$ is assumed to be bounded and spherically symmetric with respect to origin (i.e. $\mathbf{v}$ and $-\mathbf{v} \in D_{\mathbf{v}}$). In particular ν is the turning rate or turning frequency of the velocity-jump, hence $1/\nu$ is the mean run time.

The operator $\mathscr{T}_{\mathscr{F}}[f] = \mathscr{T}_{\mathscr{F}}[f](t,\mathbf{x},\mathbf{v},u)$ is the transport term that models the Gaussian thermostat [18,19], and it reads:

$$\mathscr{T}_{\mathscr{F}}[f] := \partial_u\left(\mathscr{F}\left(1 - u\int_{\Omega} u\, f\, d\Omega\right) f\right). \qquad (4)$$

In particular (4) is a damping operator adjusted to control the following moment of f related to the energy

$$\int_{D_u} u^2 f(t,\mathbf{x},\mathbf{v},u)\, du.$$

The operator $Q_f^g = Q[f,g](t,\mathbf{x},\mathbf{v},u)$, which models the interaction of the system with the external agent, reads:

$$Q_f^g = \int_{(D_u)^2} \mathscr{B}(u_*,\omega^*,u)\, f(t,\mathbf{x},\mathbf{v},u_*)\, g(t,\mathbf{x},\mathbf{v},\omega^*)\, du_*\, d\omega^* \\ - f(t,\mathbf{x},\mathbf{v},u) \int_{D_u} g(t,\mathbf{x},\mathbf{v},\omega^*)\, d\omega^*, \quad (5)$$

where:

- η^e is the inner-outer encounter rate between the agent with state $(\mathbf{x},\mathbf{v},\omega^*)$, and the particle with state $(\mathbf{x},\mathbf{v},u_*)$.
- $\mathscr{B}(u_*,\omega^*,u)$ is the inner-outer transition probability density which describes the probability density that a particle with state $(\mathbf{x},\mathbf{v},u_*)$, falls into the state $(\mathbf{x},\mathbf{v},u)$ after an interaction with the external agent whose state is $(\mathbf{x},\mathbf{v},\omega^*)$.

The density $\mathscr{B}$ satisfies the following condition:

$$\int_{D_u} \mathscr{B}(u_*,\omega^*,u)\, du = 1, \quad \forall u_*,\omega^* \in D_u. \quad (6)$$

The mathematical analysis developed in the present paper is addressed to obtain the evolution equation of the local density $\rho[f](t,\mathbf{x},u)$ of the system defined at time t in the position $\mathbf{x}$ and activity u, as follows:

$$\rho := \rho[f](t,\mathbf{x},u) = \int_{D_\mathbf{v}} f(t,\mathbf{x},\mathbf{v},u)\, d\mathbf{v}, \quad (7)$$

and the evolution equation of the relative mass velocity of particles $\mathbb{U}(t,\mathbf{x},u)$ defined on $[0,\infty[\times D_\mathbf{x} \times D_u$ by

$$\mathbb{U} := \mathbb{U}[f](t,\mathbf{x},u) = \frac{1}{\rho[f](t,\mathbf{x},u)} \int_{D_\mathbf{v}} \mathbf{v}\, f(t,\mathbf{x},\mathbf{v},u)\, d\mathbf{v}. \quad (8)$$

The thermostatted kinetic equation for open systems (1) constitutes the underlying equation for the derivation of the hyperbolic equations. In particular, Eq. (1) is a nonlinear partial integro-differential equation with quadratic nonlinearity.

In what follows we assume that the solutions of (1) are bounded and belong to a functional spaces where all needed convergence results will be true. Moreover the average of the function φ with respect to the variable $\mathbf{v}$ is denoted by

$$\langle \varphi \rangle := \int_{D_\mathbf{v}} \varphi(\mathbf{v})\, d\mathbf{v}.$$

Finally the following Kronecker delta will be used:

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

3 The Hyperbolic Equations

This section is concerned with the derivation of the hyperbolic equations from the thermostatted kinetic equation for open systems (1). Specifically we introduce a scaling parameter ε and we consider the following scaling for the time variable, the space variable and the macroscopic external force field:

$$(t,\mathbf{x},\mathbf{v},u,F) \to \left(\frac{t}{\varepsilon},\frac{\mathbf{x}}{\varepsilon},\mathbf{v},u,\varepsilon^\ell F\right), \quad l \geq 1, \quad (9)$$

that implies that the mean run time $1/\nu$ is small compared to the typical mechanical time τ. Bearing the paper [20] in mind, we set:

$$\eta = \varepsilon^r, \quad \eta^e = \varepsilon^q, \quad \nu = \frac{1}{\varepsilon^p}, \quad (10)$$

where $p,q,r \geq 1$. Accordingly the thermostatted kinetic equation for open system (1) is scaled as follows:

$$\varepsilon(\partial_t f_\varepsilon + \mathbf{v}\cdot\nabla_\mathbf{x} f_\varepsilon) + \varepsilon^\ell \mathscr{T}_F[f_\varepsilon] = \\ \varepsilon^r J[f_\varepsilon] + \varepsilon^q Q[f_\varepsilon,g_\varepsilon] + \frac{1}{\varepsilon^p} V[f_\varepsilon], \quad (11)$$

where the meaning of each operator can be recovered by section 2 and where, with a slight abuse of notation, we have set

$$f_\varepsilon(t,\mathbf{x},\mathbf{v},u) = f\left(\frac{t}{\varepsilon},\frac{\mathbf{x}}{\varepsilon},\mathbf{v},u\right), \quad (12)$$

$$g_\varepsilon(t,\mathbf{x},\mathbf{v},u) = g\left(\frac{t}{\varepsilon},\frac{\mathbf{x}}{\varepsilon},\mathbf{v},u\right). \quad (13)$$

The following lemma holds true.

Lemma 1. *Let $f_\varepsilon(t,\mathbf{x},\mathbf{v},u)$ be a sequence of solutions of the thermostatted kinetic equation (11). Assume that:*

$(\boldsymbol{A}_1)$ $\langle V[f]\rangle = \langle \mathbf{v} V[f]\rangle = 0$, $\forall \mathbf{x} \in D_\mathbf{x}, u \in D_u$.

$(\boldsymbol{A}_2)$ *For all $\rho \in [0,+\infty)$ and $\mathbb{U} \in \mathbb{R}^3$ there exists a unique function*

$$F_{\rho,\mathbb{U}} = F_{\rho,\mathbb{U}}(\mathbf{v}) \in L^1(D_\mathbf{v},(1+|\mathbf{v}|)\, d\mathbf{v})$$

such that:

$$V[F_{\rho,\mathbb{U}}] = 0, \int_{D_\mathbf{v}} F_{\rho,\mathbb{U}}(\mathbf{v}) d\mathbf{v} = \rho,$$

$$\int_{D_\mathbf{v}} \mathbf{v} F_{\rho,\mathbb{U}}(\mathbf{v})\, d\mathbf{v} = \rho\, \mathbb{U}. \quad (14)$$

$(\boldsymbol{A}_3)$ *When $\varepsilon \to 0$:*

$$f_\varepsilon \longrightarrow f \quad \text{a.e. in } [0,\infty) \times D_\mathbf{x} \times D_\mathbf{v} \times D_u, \quad (15)$$

$$g_\varepsilon \longrightarrow g \quad \text{a.e. in } [0,\infty) \times D_\mathbf{x} \times D_\mathbf{v} \times D_u, \quad (16)$$

$$V[f_\varepsilon] \longrightarrow V[f] \quad (17)$$

and the following quantities

$$\langle f_\varepsilon\rangle, \langle \mathbf{v} f_\varepsilon\rangle, \langle \mathbf{v}\otimes\mathbf{v} f_\varepsilon\rangle,$$

$$\langle J[f_\varepsilon]\rangle, \langle \mathscr{T}_{\mathscr{F}}[f_\varepsilon]\rangle, \langle Q[f_\varepsilon, g_\varepsilon]\rangle,$$

$$\langle \mathbf{v}J[f_\varepsilon]\rangle,\ \langle \mathbf{v}\mathscr{T}_{\mathscr{F}}[f_\varepsilon]\rangle,\ \langle \mathbf{v}Q[f_\varepsilon, g_\varepsilon]\rangle,$$

converge, in the sense of distributions on $\mathbb{R}_+^* \times D_\mathbf{x} \times D_u$*, to the corresponding quantities*

$$\langle f\rangle, \langle \mathbf{v}f\rangle, \langle \mathbf{v}\otimes\mathbf{v}f\rangle,$$

$$\langle J[f]\rangle, \langle \mathscr{T}_{\mathscr{F}}[f]\rangle, \langle Q[f,g]\rangle,$$

$$\langle \mathbf{v}J[f]\rangle, \langle \mathbf{v}\mathscr{T}_{\mathscr{F}}[f]\rangle,\ \langle \mathbf{v}Q[f,g]\rangle.$$

Then the asymptotic limit f *admits the following form*

$$f(t,\mathbf{x},\mathbf{v},u) = F_{\rho(t,\mathbf{x},u),\mathbb{U}(t,\mathbf{x},u)}(\mathbf{v}). \tag{18}$$

Proof. Multiplying by ε^p the left and the right hand-sides of Eq (11) and passing to the limit when $\varepsilon \to 0$, yields $V[f_0] = 0$. Setting

$$\rho_\varepsilon(t,\mathbf{x},u) = \int_{D_\mathbf{v}} f_\varepsilon(t,\mathbf{x},\mathbf{v},u)\,d\mathbf{v}, \tag{19}$$

$$\rho_\varepsilon(t,\mathbf{x},u)\mathbf{U}_\varepsilon(t,\mathbf{x},u) = \int_{D_\mathbf{v}} \mathbf{v} f_\varepsilon(t,\mathbf{x},\mathbf{v},u)\,d\mathbf{v}. \tag{20}$$

According to $(\mathbf{A}_4)$, there exists the unique function $F_{\rho,\mathbb{U}}$, where $\mathbf{U}$ depends on $(t,\mathbf{x},u)$, verifying the conditions (14). Therefore

$$f_0(t,\mathbf{x},\mathbf{v},u) = F_{\rho(t,\mathbf{x},u),\mathbb{U}(t,\mathbf{x},u)}(\mathbf{v}). \tag{21}$$

The main result of this section is the following theorem.

Theorem 1. *Let* $f_\varepsilon(t,\mathbf{x},\mathbf{v},u)$ *be a sequence of solutions to the scaled thermostatted kinetic equation (11). Assume that* $(\boldsymbol{A}_1$-$\boldsymbol{A}_2$-$\boldsymbol{A}_3)$ *hold true and that every formally small term in* ε *vanishes. Then* ρ *and* $\rho\mathbb{U}$ *are the weak solutions of the following equations:*

$$\partial_t\rho + \mathrm{div}_\mathbf{x}(\rho\mathbb{U}) + \delta_{\ell,1}\partial_u\left(\mathscr{F}\left(1-u\mathbb{A}[\rho](t)\right)\rho\right) =$$

$$\delta_{r,1}\langle J[f]\rangle + \delta_{q,1}\langle Q[f,g]\rangle,$$

$$\partial_t(\rho\mathbb{U}) + \delta_{\ell,1}\partial_u\left(\mathscr{F}\left(1-u\mathbb{A}[\rho](t)\right)\rho\mathbb{U}\right) +$$

$$\mathrm{div}_\mathbf{x}(\rho\mathbb{U}\otimes\mathbb{U}+\mathbb{P}) = \delta_{r,1}\langle \mathbf{v}J[f]\rangle + \delta_{q,1}\langle \mathbf{v}Q[f,g]\rangle,$$

where $\mathbb{P} := \mathbb{P}[f](t,\mathbf{x},u)$ *is the following pressure tensor:*

$$\mathbb{P}[f](t,\mathbf{x},u) = \int_{D_\mathbf{v}} (\mathbf{v}-\mathbb{U})\otimes(\mathbf{v}-\mathbb{U})\, f(t,\mathbf{x},\mathbf{v},u)\,d\mathbf{v}, \tag{22}$$

and

$$\mathbb{A}[\rho](t) = \int_{D_\mathbf{x}\times D_u} u\rho(t,\mathbf{x},u)\,d\mathbf{x}\,du.$$

Proof. Let $\psi \in \{1,\mathbf{v}\}$. Then multiplying the scaled Eq. (11) by ψ and integrating with respect to the velocity variable $\mathbf{v}$ we have

$$\begin{aligned}\varepsilon\int_{D_\mathbf{v}}(\partial_t + \mathbf{v}\cdot\nabla_\mathbf{x})\, f_\varepsilon\,\psi\,d\mathbf{v} + \varepsilon^\ell\partial_u\int_{D_\mathbf{v}}\psi\,\mathscr{T}_{\mathscr{F}}[f_\varepsilon]\,d\mathbf{v} \\ = \varepsilon^r\int_{D_\mathbf{v}}\psi J[f_\varepsilon]\,d\mathbf{v} + \varepsilon^q\int_{D_\mathbf{v}}\psi\,Q[f_\varepsilon,g_\varepsilon]\,d\mathbf{v} \\ +\frac{1}{\varepsilon^p}\int_{D_\mathbf{v}}\psi\tilde{V}[f_\varepsilon]\,d\mathbf{v}.\end{aligned} \tag{23}$$

Therefore we have

$$\partial_t\langle f_\varepsilon\rangle + \langle \mathbf{v}\cdot\nabla_\mathbf{x} f_\varepsilon\rangle + \varepsilon^{\ell-1}\langle\partial_u\mathscr{T}_{\mathscr{F}}[f_\varepsilon]\rangle =$$

$$\varepsilon^{r-1}\langle J[f_\varepsilon]\rangle + \varepsilon^{q-1}\langle Q[f_\varepsilon,g_\varepsilon]\rangle, \tag{24}$$

$$\partial_t\langle \mathbf{v}f_\varepsilon\rangle + \mathrm{div}_\mathbf{x}\langle\mathbf{v}\otimes\mathbf{v}f_\varepsilon\rangle + \varepsilon^{\ell-1}\langle\mathbf{v}\partial_u\mathscr{T}_{\mathscr{F}}[f_\varepsilon]\rangle =$$

$$\varepsilon^{r-1}\langle\mathbf{v}J[f_\varepsilon]\rangle + \varepsilon^{q-1}\langle\mathbf{v}Q[f_\varepsilon,g_\varepsilon]\rangle. \tag{25}$$

Since

$$\begin{aligned}\partial_t\langle f_\varepsilon\rangle + \langle\mathbf{v}\cdot\nabla_\mathbf{x} f_\varepsilon\rangle &\longrightarrow \partial_t\rho + \mathrm{div}_\mathbf{x}(\rho\mathbb{U}),\\ \partial_t\langle\mathbf{v}f_\varepsilon\rangle + \mathrm{div}_\mathbf{x}\langle\mathbf{v}\otimes\mathbf{v}f_\varepsilon\rangle &\longrightarrow \\ \partial_t(\rho\mathbf{U}) + \nabla_\mathbf{x}\cdot&\left(\int_{D_\mathbf{v}}\mathbf{v}\otimes\mathbf{v}F_{\rho,\mathbf{U}}\right),\end{aligned}$$

and

$$\int(\mathbf{v}\otimes\mathbf{v})F_{\rho,\mathbf{U}}\,d\mathbf{v} = \rho\mathbf{U}\otimes\mathbf{U} + \mathbb{P},$$

we have

$$\partial_t\langle\mathbf{v}f_\varepsilon\rangle + \mathrm{div}_\mathbf{x}\langle\mathbf{v}\otimes\mathbf{v}f_\varepsilon\rangle \longrightarrow \partial_t(\rho\mathbf{U}) + \nabla_\mathbf{x}\cdot(\rho\mathbf{U}\otimes\mathbf{U}+\mathbb{P}).$$

The last computations refer to the limit of the thermostatted operator. Specifically, letting ε goes to zero and bearing Lemma 1 in mind, we have:

$$\langle\partial_u\mathscr{T}_{\mathscr{F}}[f_\varepsilon]\rangle \underset{\varepsilon\to 0}{\longrightarrow} \mathscr{T}_1$$

where

$$\begin{aligned}\mathscr{T}_1 &= \left\langle\partial_u\left(\mathscr{F}\left(1-u\int_\Omega uF_{\rho,\mathbb{U}}\,d\mathbf{x}\,d\mathbf{v}\,du\right)F_{\rho,\mathbb{U}}\right)\right\rangle\\ &= \left\langle\partial_u\left(\mathscr{F}\left(1-u\int_{D_\mathbf{x}\times D_u}u\rho\,d\mathbf{x}\,du\right)F_{\rho,\mathbb{U}}\right)\right\rangle. \qquad (26)\\ &= \partial_u\left(\mathscr{F}\left(1-u\mathbb{A}[\rho](t)\right)\rho\right). \qquad (27)\end{aligned}$$

Moreover

$$\langle\mathbf{v}\,\partial_u\mathscr{T}_{\mathscr{F}}[f_\varepsilon]\rangle \underset{\varepsilon\to 0}{\longrightarrow} \mathscr{T}_2$$

where

$$\begin{aligned}\mathscr{T}_2 &= \left\langle\mathbf{v}\partial_u\left(\mathscr{F}\left(1-u\int_\Omega uF_{\rho,\mathbb{U}}\,d\mathbf{x}\,d\mathbf{v}\,du\right)F_{\rho,\mathbb{U}}\right)\right\rangle\\ &= \left\langle\partial_u\left(\mathscr{F}\left(1-u\int_{D_\mathbf{x}\times D_u}u\rho\,d\mathbf{x}\,du\right)\mathbf{v}F_{\rho,\mathbb{U}}\right)\right\rangle\\ &= \partial_u\left(\mathscr{F}\left(1-u\mathbb{A}[\rho](t)\right)\rho\mathbb{U}\right).\end{aligned} \tag{28}$$

Therefore the proof is concluded.

4 On the Generalized Time and Space Scalings

This section is devoted to the problem of deriving asymptotic equation by performing different scalings in the time and space variables. Specifically we consider the following scaling

$$(t,\mathbf{x},\mathbf{v},u,F) \to \left(\frac{t}{\varepsilon^{\alpha}}, \frac{\mathbf{x}}{\varepsilon^{\beta}}, \mathbf{v}, u, \varepsilon^{\ell+\alpha-1}F\right), \quad (29)$$

with $\alpha, \beta, l \geq 1$ and $\beta < \alpha$. Moreover we set:

$$\eta = \varepsilon^{r+\alpha-1}, \quad \eta^e = \varepsilon^{q+\alpha-1}, \quad \nu = \frac{1}{\varepsilon^p}, \quad (30)$$

where $p,q,r \geq 1$. Accordingly the thermostatted kinetic equation for open system (1) is scaled as follows:

$$\varepsilon^{\alpha}\partial_t f_{\varepsilon} + \varepsilon^{\beta}\mathbf{v}\cdot\nabla_{\mathbf{x}} f_{\varepsilon} + \varepsilon^{\ell+\alpha-1}\mathscr{T}_F[f_{\varepsilon}] =$$

$$\varepsilon^{r+\alpha-1}J[f_{\varepsilon}] + \varepsilon^{q+\alpha-1}Q[f_{\varepsilon}, g_{\varepsilon}] + \frac{1}{\varepsilon^p}V[f_{\varepsilon}], \quad (31)$$

where, with a slight abuse of notation, we have set

$$f_{\varepsilon}(t,\mathbf{x},\mathbf{v},u) = f\left(\frac{t}{\varepsilon^{\alpha}}, \frac{\mathbf{x}}{\varepsilon^{\beta}}, \mathbf{v}, u\right),$$

$$g_{\varepsilon}(t,\mathbf{x},\mathbf{v},u) = g\left(\frac{t}{\varepsilon^{\alpha}}, \frac{\mathbf{x}}{\varepsilon^{\beta}}, \mathbf{v}, u\right).$$

Bearing all above in mind, Eqs (24) and (25) now read

$$\partial_t \langle f_{\varepsilon}\rangle + \frac{\langle \mathbf{v}\cdot\nabla_{\mathbf{x}} f_{\varepsilon}\rangle}{\varepsilon^{\alpha-\beta}} + \varepsilon^{\ell-1}\langle \partial_u \mathscr{T}_{\mathscr{F}}[f_{\varepsilon}]\rangle =$$

$$\varepsilon^{r-1}\langle J[f_{\varepsilon}]\rangle + \varepsilon^{q-1}\langle Q[f_{\varepsilon}, g_{\varepsilon}]\rangle, \quad (32)$$

$$\partial_t \langle \mathbf{v} f_{\varepsilon}\rangle + \frac{\mathrm{div}_{\mathbf{x}}\langle \mathbf{v}\otimes\mathbf{v} f_{\varepsilon}\rangle}{\varepsilon^{\alpha-\beta}} + \varepsilon^{\ell-1}\langle \mathbf{v}\partial_u \mathscr{T}_{\mathscr{F}}[f_{\varepsilon}]\rangle =$$

$$\varepsilon^{r-1}\langle \mathbf{v}J[f_{\varepsilon}]\rangle + \varepsilon^{q-1}\langle \mathbf{v}Q[f_{\varepsilon}, g_{\varepsilon}]\rangle. \quad (33)$$

In order to perform the limit of Eqs (32) (33) when ε goes to zero, it is fundamental now to obtain the convergence of the terms

$$\frac{\langle \mathbf{v}\cdot\nabla_{\mathbf{x}} f_{\varepsilon}\rangle}{\varepsilon^{\alpha-\beta}}, \quad \frac{\mathrm{div}_{\mathbf{x}}\langle \mathbf{v}\otimes\mathbf{v} f_{\varepsilon}\rangle}{\varepsilon^{\alpha-\beta}},$$

which is a hard problem that requires further assumptions on the turning operator and whose limit depends on the choice of values of α e β.

Preliminary to the derivation of the asymptotic equation is the following result, see [15] for the proof.

Lemma 2. *Assume that assumption* $(\boldsymbol{A}_1)$ *holds and:*

$(\boldsymbol{A}_4)$ *There exists a bounded equilibrium velocity distribution* $F(\mathbf{v}) : D_{\mathbf{v}} \to \mathbb{R}^+$*, independent of* t *and* $\mathbf{x}$*, such that:*

$$T(\mathbf{v}^*,\mathbf{v})F(\mathbf{v}) = T(\mathbf{v},\mathbf{v}^*)F(\mathbf{v}^*). \quad (34)$$

and $\langle \mathbf{v}F(\mathbf{v})\rangle = \langle F(\mathbf{v})\rangle = 0$, $\forall \mathbf{x}\in D_{\mathbf{x}}, u \in D_u$.

$(\boldsymbol{A}_5)$ *The kernel* $T(\mathbf{v},\mathbf{v}^*)$ *is bounded, and there exists a constant* $\sigma > 0$ *such that*

$$T(\mathbf{v},\mathbf{v}^*) \geq \sigma F(\mathbf{v}), \quad \forall (\mathbf{v},\mathbf{v}^*) \in D_{\mathbf{v}}\times D_{\mathbf{v}}. \quad (35)$$

Then

- *The following equality holds*

$$-\int_{D_{\mathbf{v}}} V[f]\, f F^{-1} d\mathbf{v} =$$

$$\frac{1}{2}\int_{D_{\mathbf{v}}\times D_{\mathbf{v}}} T(\mathbf{v},\mathbf{v}^*)F^*\left(\frac{f^*}{F^*} - \frac{f}{F}\right)^2 d\mathbf{v}\, d\mathbf{v}^* \geqslant 0. \quad (36)$$

- *The null-space of* V *is spanned by a unique normalized and nonnegative function* $F(\mathbf{v})$*:*

$$Ker(V) = \mathrm{Span}\{F\}.$$

- *For any* $h \in H$ *satisfying* $\int_{D_{\mathbf{v}}} h\, d\mathbf{v} = 0$*, there exists a unique* $f \in H$ *such that* $V[f] = h$ *and* $\int_{D_{\mathbf{v}}} f\, d\mathbf{v} = 0$.

Multiplying Eq. (31) by ε^p and letting ε goes to zero by Lemma 2 we obtain the existence of a function $\rho = \rho(t,\mathbf{x},u) : [0,\infty[\times D_{\mathbf{x}}\times D_u \to \mathbb{R}^+$ independent of $\mathbf{v}$ and such that

$$f(t,\mathbf{x},\mathbf{v},u) = \rho(t,\mathbf{x},u)F(\mathbf{v}). \quad (37)$$

Now we compute the asymptotic limit of the transport term. Let $\chi(\mathbf{v})$ be the only solution of the equation $V[f] = \mathbf{v}F(\mathbf{v})$, according to Lemma 2, we have:

$$\begin{aligned}\frac{\langle \mathbf{v}\cdot\nabla_{\mathbf{x}} f_{\varepsilon}\rangle}{\varepsilon^{\alpha-\beta}} &= \nabla_{\mathbf{x}}\cdot\left\langle \frac{\mathbf{v} f_{\varepsilon}}{\varepsilon^{\alpha-\beta}}\right\rangle \\ &= \frac{1}{\varepsilon^{\alpha-\beta}}\mathrm{div}_{\mathbf{x}}\left\langle \mathbf{v} f_{\varepsilon}\frac{F(\mathbf{v})}{F(\mathbf{v})}\right\rangle \\ &= \mathrm{div}_{\mathbf{x}}\left\langle \frac{V[f_{\varepsilon}]}{\varepsilon^{\alpha-\beta}}\frac{\chi(\mathbf{v})}{F(\mathbf{v})}\right\rangle. \quad (38)\end{aligned}$$

Moreover, multiplying the right-hand side and the left-hand side of Eq. (31) by $\frac{\varepsilon^p}{\varepsilon^{\alpha-\beta}}$, we obtain

$$\frac{V[f_{\varepsilon}]}{\varepsilon^{\alpha-\beta}} = \varepsilon^{p+\beta}\partial_t f_{\varepsilon} + \varepsilon^{\ell+\beta+p-1}\mathscr{T}_{\mathscr{F}}[f_{\varepsilon}]$$

$$+\varepsilon^{p+2\beta-\alpha-1}\mathbf{v}\cdot\nabla_{\mathbf{x}} f_{\varepsilon}$$

$$-\varepsilon^{r+\beta+p-1}J[f_{\varepsilon}] - \varepsilon^{q+\beta+p-1}Q[f_{\varepsilon}, g_{\varepsilon}].$$

Therefore

$$\frac{\langle \mathbf{v}\cdot\nabla_{\mathbf{x}} f_\varepsilon\rangle}{\varepsilon^{\alpha-\beta}} \underset{\varepsilon\to 0}{\longrightarrow} \delta_{p+2\beta,\alpha+1}\nabla_{\mathbf{x}}\cdot\left\langle\left(\frac{\chi(\mathbf{v})}{F(\mathbf{v})}\otimes\mathbf{v}\right)\nabla_{\mathbf{x}} f\right\rangle$$
$$=\delta_{p+2\beta,\alpha+1}\nabla_{\mathbf{x}}\cdot\langle(\chi(\mathbf{v})\otimes\mathbf{v})\nabla_{\mathbf{x}}\rho\rangle.$$

The above limit shows that the first asymptotic equation is of parabolic type. Then in the case $\beta<\alpha$ we are not able to obtain hyperbolic equations.

It is easy to show that when ε goes to zero, for $\beta>\alpha$, the two following terms

$$\varepsilon^{\beta-\alpha}\langle\mathbf{v}\cdot\nabla_{\mathbf{x}} f_\varepsilon\rangle,\quad \varepsilon^{\beta-\alpha}\mathrm{div}_{\mathbf{x}}\langle\mathbf{v}\otimes\mathbf{v} f_\varepsilon\rangle,$$

vanish and the resulting asymptotic equations now read:

$$\partial_t\rho+\delta_{\ell,1}\partial_u\left(\mathscr{F}\left(1-u\,\mathbb{A}[\rho](t)\right)\rho\right)=$$
$$\delta_{r,1}\langle J[f]\rangle+\delta_{q,1}\langle Q[f,g]\rangle,$$

$$\partial_t(\rho\mathbb{U})+\delta_{\ell,1}\partial_u\left(\mathscr{F}\left(1-u\,\mathbb{A}[\rho](t)\right)\rho\mathbb{U}\right)=$$
$$\delta_{r,1}\langle\mathbf{v}J[f]\rangle+\delta_{q,1}\langle\mathbf{v}Q[f,g]\rangle.$$

5 Applications and Future Research Directions

The mathematical analysis developed into the present paper has been focused on the derivation of macroscopic hyperbolic behaviour from the microscopic interactions occurring among the particles of a complex open system that has been modeled by the thermostatted kinetic theory approach. The asymptotic analysis has shown that the genuine hyperbolic dynamics arises when a specific time and space variables scaling is performed. The hyperbolic equations for the local density and first velocity momentum also show a term related to the control action of the thermostatted operator and the role of the interactions at the macroscopic scale.

Applications of the asymptotic analysis proposed in this paper refer to the mathematical modeling of biological and chemical systems, with particular attention to the cancer-immune system competition [21,22,23,24] where the macroscopic equation is referred to the evolution of the tumor at tissue scale [25,26,27]. In particular as shown in Section 4, also diffusive behavior can occur depending on the power of the scaling parameter. Moreover applications refer to the modeling of vehicular traffic flow and crowd and swarm dynamics where the onset of hyperbolic behaviour usually appears [28,29,30].

From the perspectives point of view, the underlying thermostatted kinetic equation for open systems can be generalized in order to take into account the modeling of complex open systems characterized by heterogeneous particles carrying out different functions. According to the system biology approach proposed in [31,32], the whole open system is decomposed into different functional subsystems whose particles are able to express the same activity (function, strategy).

Let $f_i=f_i(t,\mathbf{x},\mathbf{v},u)$ be the distribution function of the ith functional subsystem, for $i\in\{1,2,\ldots,n\}$, then the thermostatted kinetic equation for the ith open functional systems now reads:

$$(\partial_t+\mathbf{v}\cdot\nabla_{\mathbf{x}})f_i+\partial_u\left(\mathscr{F}_i\left(1-u\sum_{j=1}^{n}\int_{\Omega}u\,f_j\,d\Omega\right)f_i\right)=$$
$$\eta J_i[\mathbf{f}]+\eta^e Q_i[\mathbf{f},\mathbf{g}]+\nu V_i[f_i], \tag{39}$$

where $\mathbf{f}=(f_1,f_2,\ldots,f_n)$ is the vector whose components are the distribution functions of the functional subsystems, $\mathbf{g}=(g_1,g_2,\ldots,g_m)$ is the vector whose components are the distribution functions of the external actions, and

$$J_i[\mathbf{f}]=\sum_{j=1}^{n}\int_{D_u\times D_u}\mathscr{A}_{ij}(u_*,u^*,u)f_i(t,\mathbf{x},\mathbf{v},u_*)\times$$
$$\times f_j(t,\mathbf{x},\mathbf{v},u^*)\,du_*\,du^*$$
$$-f_i(t,\mathbf{x},\mathbf{v},u)\sum_{j=1}^{n}\int_{D_u}f_j(t,\mathbf{x},\mathbf{v},u^*)\,du^*, \tag{40}$$

$$Q_i[\mathbf{f},\mathbf{g}]=\sum_{j=1}^{m}\int_{D_u\times D_u}\mathscr{B}_{ij}(u_*,\omega^*,u)f_i(t,\mathbf{x},\mathbf{v},u_*)\times$$
$$\times g_j(t,\mathbf{x},\mathbf{v},\omega^*)\,du_*\,du^*$$
$$-f_i(t,\mathbf{x},\mathbf{v},u)\sum_{j=1}^{m}\int_{D_u}g_j(t,\mathbf{x},\mathbf{v},\omega^*)\,du^*, \tag{41}$$

$$V_i[f_i]=\int_{D_{\mathbf{v}}}T_i(\mathbf{v}^*,\mathbf{v})f_i(t,\mathbf{x},\mathbf{v}^*,u)\,d\mathbf{v}^*$$
$$-\int_{D_{\mathbf{v}}}T_i(\mathbf{v},\mathbf{v}^*)f_i(t,\mathbf{x},\mathbf{v},u)\,d\mathbf{v}^*, \tag{42}$$

with obvious meaning of each term into the operators.

Moreover with special attention to biological and chemical systems [33], the Eq (39) can be further generalized by including the possibility of proliferation/destruction of particles with microscoscopic state $(\mathbf{x},\mathbf{v},u)$, that is modeled by the following operator:

$$N_i[\mathbf{f}]=\eta\,\mu\,f_i(t,\mathbf{x},\mathbf{v},u)\sum_{j=1}^{n}\int_{D_u}f_j(t,\mathbf{x},\mathbf{v},u^*)\,du^*, \tag{43}$$

where μ is the net proliferation/destruction rate. Finally we can also consider the role of mutations that are modeled by the following operator:

$$M_i[\mathbf{f}]=\eta\sum_{h,k=1}^{n}\int_{D_u\times D_u}\varphi^i_{hk}\,f_h(t,\mathbf{x},\mathbf{v},u_*)\,f_k(t,\mathbf{x},\mathbf{v},u^*)\,du_*\,du^*. \tag{44}$$

where φ_{hk}^{i} is the net mutative rate into the ith functional subsystem, due to interactions that occur with rate η between the particle $(\mathbf{x}, \mathbf{v}, u_*)$ of the hth functional subsystem and the particle $(\mathbf{x}, \mathbf{v}, u^*)$ of the kth functional subsystem.

Therefore further research directions include the derivation of the asymptotic equation for the local density and momentum from the thermostatted kinetic equation for open system and with proliferative/destructive and mutative interactions. It is expected that these interactions occurring at the microscopic scale will be new source terms into the asymptotic equations.

As already mentioned in Section 2, the underlying thermostatted kinetic equation considered in this paper comprises a term related to the Gaussian thermostat which prevents an uncontrolled increase of the 2nd-order moment of local activity. It is worth stressing that, recently, the term related to the thermostat has been generalized in [34] for controling the following pth-order moment of local activity:

$$\int_{D_u} u^p f(t, \mathbf{x}, \mathbf{v}, u)\, du, \quad p \in \mathbb{N}. \tag{45}$$

In this case the thermostat term reads:

$$\mathscr{T}_{\mathscr{F},p}[f] := \partial_u \left(\mathscr{F}(u) \left(1 - u \int_{\Omega} u^{p-1} f \, d\Omega \right) f \right). \tag{46}$$

The analysis performed in the present paper needs further developments in the case of thermostatted kinetic equations characterized by nonlinear interactions. The relaxation of the binary assumption opens a new research direction in the modeling problem; a first attempt to this step has been proposed in [35].

Finally research directions refer also to the possibility to derive asymptotic equations for a mixture of functional subsystems f_1 and f_2 where the distribution f_1 is rescaled with a time scaling and the distribution function f_2 is rescaled with a time-space scaling. In this case it is expected that the asymptotic equation for the functional subsystem with distribution f_1 will be a parabolic equation and the the asymptotic equation for the functional subsystem with distribution f_2 will be a parabolic or hyperbolic equation. This work is in progress and results will be presented in due course.

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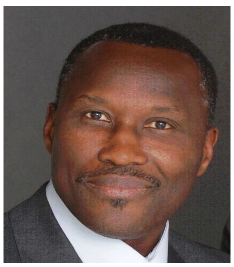

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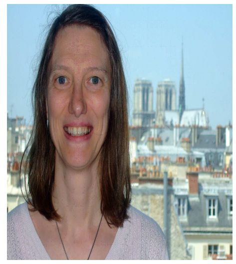

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