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On α_*-ψ-Contractive Multi-Valued Operators in b-Metric Spaces and Applications

Monica-Felicia Bota[1], *Veronica Ilea*[1], *Erdal Karapınar*[2,3,*] *and Oana Mleşniţe*[1]

[1] Department of Mathematics, Babeş-Bolyai University Kogălniceanu Street No.1, 400084, Cluj-Napoca, Romani
[2] Department of Mathematics, Atilim University, 06836, İncek, Ankara, Turkey
[3] Nonlinear Analysis and Applied Mathematics Research Group (NAAM), King Abdulaziz University, 21589, Jeddah, Saudi Arabia.

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Abstract: In this paper, we will prove some existence and uniqueness theorems for α_*-ψ contractive type operators defined over b-metric spaces. In particular, we will provide results related to Ulam-Hyers stability, well-posedness and limit shadowing. The theorems presented will extend, generalize or unify several statements currently exist in the literature on those topics. We will also give examples to illustrate the applications our results.
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1 Introduction and Preliminaries

Very recently, Samet et al. introduced in [28] the notion of α-ψ-contractive type mapping and proved a number of fixed point results for such mappings. As an extension of α-ψ-contractive type mapping, Asl et al. later proposed α_*-ψ-contractive type of multivalued mappings. Motivated by their results several authors have started working on this subject area and successfully managed to improve this notion in various ways by proving theorems for single and multi-valued operators in the setting of different abstract spaces, see e.g. [17,30,31,32,33,34,35,36].

The aim of this paper is to provide results that establish the existence and/or uniqueness of fixed points of α_*-ψ-contractive type mappings in the context of b-metric spaces. Czerwik introduced b-metric spaces in [16,15] as a generalization of metric spaces. Czerwik's generalization can be considered as a continuation of other approaches studied earlier by Bourbaki [14], Bakhtin [4], Heinonen [19]. Following Czerwik's papers, b-metric spaces and related fixed point theorems have been heavily investigated by many authors, see e.g. Boriceanu et al.[10], Boriceanu [11,12], Bota [13], Aydi et al. [2,3].

As a secondary purpose, we consider Ulam-Hyers stability under the light of the fixed point results we prove on b-metric spaces. The Ulam-Hyers stability problem of functional equations, which was originated from a question of Ulam [29] in 1940, deals with the stability of group homomorphisms. The first affirmative partial answer to the question of Ulam for Banach spaces was given by Hyers [18] in 1941. Thereafter, this type of stability is called the Ulam-Hyers stability and have attracted attention from several authors. In particular, Ulam-Hyers stability results in fixed point theory have been studied densely by Bota-Boriceanu and Petruşel [8,9,21,24,26]. Finally, we discuss well-posedness of the fixed problems and limit shadowing property of the multivalued operators.

Throughout this paper, the standard notations and terminologies in nonlinear analysis will be used. We recollect some essential definitions and fundamental results in this section. First, we begin with the definition of a b-metric space:

Definition 1.(*Bakhtin [4], Czerwik [16]) Let X be a set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is said to be a b-metric if the following conditions are satisfied:*

1.$d(x,y) = 0$ if and only if $x = y$,

* Corresponding author e-mail: erdalkarapinar@yahoo.com

2. $d(x,y) = d(y,x)$,
3. $d(x,z) \leq s[d(x,y) + d(y,z)]$,

for all $x,y,z \in X$. *A pair* (X,d) *is called a b-metric space.*

Note that a b-metric is a metric when $s = 1$. Hence, the class of b-metric spaces contains the class of metric spaces. For more details and examples on b-metric spaces, see e.g. [4,5,14,15,16,19]. In particular, we would like to give a specific example:

Example 1.[5] Let X be a set with the cardinal $card(X) \geq 3$. Suppose that $X = X_1 \cup X_2$ is a partition of X such that $card(X_1) \geq 2$. Let $s > 1$ be arbitrary. Then, the functional $d : X \times X \to [0,\infty)$ defined by:

$$d(x,y) := \begin{cases} 0, & x = y \\ 2s, & x,y \in X_1 \\ 1, & \text{otherwise.} \end{cases}$$

is a b-metric on X with coefficient $s > 1$.

In the rest of this work, we shall need the families of subsets of a b-metric space (X,d) listed below:

$$\begin{aligned} \mathscr{P}(X) &:= \{Y \mid Y \subset X\}; \\ P(X) &:= \{Y \in \mathscr{P}(X) \mid Y \neq \emptyset\}; \\ P_b(X) &:= \{Y \in P(X) \mid Y \text{ is bounded }\}; \\ P_{cp}(X) &:= \{Y \in P(X) \mid Y \text{ is compact}\}; \\ P_{cl}(X) &:= \{Y \in P(X) \mid Y \text{ is closed}\}; \\ P_{b,cl}(X) &:= P_b(X) \cap P_{cl}(X). \end{aligned}$$

Additionally, we will also need the gap, excess generalized, Pompeiu-Hausdorff and δ-functionals which will be used to define certain b- metric spaces to study various multi-valued operators. The definition of those functionals are given as follows:

The gap functional:

$$(1)\ \ D : \mathscr{P}(X) \times \mathscr{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$D(A,B) = \begin{cases} \inf\{d(a,b) \mid a \in A,\ b \in B\}, & A \neq \emptyset \neq B, \\ 0, & A = \emptyset = B, \\ +\infty, & \text{otherwise.} \end{cases}$$

The excess generalized functional:

$$(2)\ \ \rho : \mathscr{P}(X) \times \mathscr{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$\rho(A,B) = \begin{cases} \sup\{D(a,B) \mid a \in A\}, & A \neq \emptyset \neq B, \\ 0, & A = \emptyset, \\ +\infty, & B = \emptyset \neq A. \end{cases}$$

Pompeiu-Hausdorff generalized functional:

$$(3)\ \ H : \mathscr{P}(X) \times \mathscr{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$H(A,B) = \begin{cases} \max\{\rho(A,B), \rho(B,A)\}, & A \neq \emptyset \neq B, \\ 0, & A = \emptyset = B, \\ +\infty, & \text{othewise.} \end{cases}$$

δ functional:

$$(4)\ \ \delta : \mathscr{P}(X) \times \mathscr{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$\delta(A,B) = \begin{cases} \sup\{d(a,b) \mid a \in A,\ b \in B\}, & A \neq \emptyset \neq B, \\ 0, & A = \emptyset = B, \\ +\infty, & \text{otherwise.} \end{cases}$$

Remark If $x_0 \in X$ in (1), then $D(x_0,B) := D(\{x_0\},B)$. Moreover, if $A = B$ in (4), then we have $\delta(A,A) := \delta(A)$. Notice that $(P_{b,cl}(X),H)$ is a complete b-metric space provided that (X,d) is a complete b-metric space (see Czerwik [16]).

We will also require the following four lemmas in the proof of the main result:

Lemma 1. *Let* (X,d) *be a b-metric space and let* $A,B \in P(X)$. *We suppose that there exists* $\eta > 0$ *such that:*
(i) for each $a \in A$ *there is* $b \in B$ *such that* $d(a,b) \leq \eta$;
(ii) for each $b \in B$ *there is* $a \in A$ *such that* $d(a,b) \leq \eta$.
Then, $H(A,B) \leq \eta$.

Lemma 2. *Let* (X,d) *be a b-metric space,* $A \in P(X)$ *and* $x \in X$. *Then* $D(x,A) = 0$ *if and only if* $x \in \overline{A}$.

Lemma 3. *(Czerwik [16]) Let* (X,d) *be a b-metric space. Also let* $\{x_k\}_{k=0}^n \subset X$. *Then*

1. $D(x,A) \leq s[d(x,y) + D(y,A)]$, for all $x,y \in X$ *and* $A \subset X$.
2. $d(x_n,x_0) \leq sd(x_0,x_1) + \ldots + s^{n-1}d(x_{n-2},x_{n-1}) + s^n d(x_{n-1},x_n)$.
3. $H(A,C) \leq s[H(A,B) + H(B,C)]$, *for all* $A,B,C \in P(X)$.

Lemma 4. *Let* (X,d) *be a b-metric space and let* $A,B \in P(X)$. *For each* $q > 1$ *and for all* $a \in A$ *there exists* $b \in B$ *such that*

$$d(a,b) \leq qH(A,B).$$

Proof. Supposing the contrary: there exist $q > 1$ and $a \in A$ such that for all $b \in B$ we have $d(a,b) > qH(A,B)$. Taking $\inf_{b \in B}$ we get $D(a,B) \geq qH(A,B)$. But $H(A,B) \geq D(a,B) \geq qH(A,B)$. Hence we obtain $q \leq 1$, which is a contradiction. □

Since a b-metric d on X induces a structure of an L-space on X, we have the following concepts (see also [27]):

Definition 2. *Let* (X,d) *be a b-metric space and* $T : X \to P_{cl}(X)$ *be a multivalued operator. An operator* T *is called a multivalued weakly Picard (briefly MWP) if for each* $x \in X$ *and each* $y \in T(x)$ *there exists a sequence* $\{x_n\}_{n \in \mathbb{N}}$ *such that*

(i) $x_0 = x$, $x_1 = y$;
(ii) $x_{n+1} \in T(x_n)$, *for each* $n \in \mathbb{N}$;
(iii) the sequence $(x_n)_{n \in \mathbb{N}}$ *is convergent and its limit is a fixed point of* T.

Remark. A sequence $(x_n)_{n \in \mathbb{N}}$ satisfying the condition *(i)* and *(ii)*, in the Definition 2 is called a sequence of successive approximations of T starting from (x_0,x_1).

We next give some examples of MWP operators in b-metric spaces.

Example 2. Let (X,d) be a b-metric space and $t_i : X \to X$, $i \in \{1,2,\ldots,n\}$, be single valued k_i-contractions, i.e., $k_i \in [0,1[$ and $d(t_i(x),t_i(y)) \leq k_i d(x,y)$, for each $x,y \in X$. Then the multivalued operator $T : X \to P_{cl}(X)$, $T(x) = \{t_1(x),\ldots,t_n(x)\}$, is a MWP operator.

Example 3. Let (X,d) be a b-metric space and $T : X \to P_{cl}(X)$ be a multivalued a-contraction, i.e., $a \in [0,1[$ and $H(T(x),T(y)) \leq ad(x,y)$, for each $x,y \in X$. Then T is a MWP operator. Indeed, if we analyze the proof of Theorem 5 in Czerwik [16] we may remark that for $(x,y) \in Graph(T)$ there exists a sequence $x_{n+1} \in T(x_n)$, $n \in \mathbb{N}$ such that $x_0 = x, x_1 = y$ and $x_n \xrightarrow{d} x^* \in Fix(T)$ as $n \to +\infty$.

A mapping $\varphi : [0,\infty) \to [0,\infty)$ is called a *comparison function* if it is increasing and $\varphi^n(t) \to 0$, $n \to \infty$, for any $t \in [0,\infty)$. We denote by Φ, the class of the comparison functions $\varphi : [0,\infty) \to [0,\infty)$. For more details and examples , see e.g. [25,7]. Among them, we recall the following essential result.

Lemma 5. *(Berinde [7], Rus [25]) If $\varphi : [0,\infty) \to [0,\infty)$ is a comparison function, then:*

(1) each iterate φ^k of φ, $k \geq 1$, is also a comparison function;
(2) φ is continuous at 0;
(3) $\varphi(t) < t$, for any $t > 0$.

Let Ψ denotes the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ. It is clear that if $\Phi \subset \Psi$ (see e.g. [20]) and hence, by Lemma 5 (3), for $\psi \in \Psi$ we have $\psi(t) < t$, for any $t > 0$. Later, Berinde [7] introduced the concept of (c)-*comparison function* as follows:

Definition 3. *(Berinde [7]) A function $\varphi : [0,\infty) \to [0,\infty)$ is said to be a (c)-comparison function if*

(c_1) φ is increasing,
(c_2) there exists $k_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $\varphi^{k+1}(t) \leq a\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0,\infty)$.

The notion of a (c)-comparison function was improved as a (b)-comparison function by Berinde [6] in order to extend some fixed point results to the class of b-metric spaces.

Definition 4. *(Berinde [6]) Let $s \geq 1$ be a real number. A mapping $\varphi : [0,\infty) \to [0,\infty)$ is called a (b)-comparison function if the following conditions are fulfilled*

(b_1) φ is monotone increasing;
(b_2) there exist $k_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $b^{k+1}\varphi^{k+1}(t) \leq ab^k\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0,\infty)$.

We denote by Ψ_b the class of (b)-comparison functions. Notice that (b)-comparison function is a (c)-comparison function when $b = 1$. Also notice that any (b)-comparison function is a comparison function due to the lemma below:

Lemma 6. *(Berinde [5]) If $\varphi : [0,\infty) \to [0,\infty)$ is a (b)-comparison function, then we have the following*

(1) the series $\sum_{k=0}^{\infty} b^k\varphi^k(t)$ converges for any $t \in \mathbb{R}_+$;
(2) the function $s_b : [0,\infty) \to [0,\infty)$ defined by $s_b(t) = \sum_{k=0}^{\infty} b^k\varphi^k(t)$, $t \in [0,\infty)$, is increasing and continuous at 0.

We will need the following generalized Cauchy lemma proved by Păcurar in [22]:

Lemma 7. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a $b-$comparison function with constant $s \geq 1$ and $a_n \in \mathbb{R}_+$, $n \in \mathbb{N}$ such that $a_n \to 0$, as $n \to \infty$ then*

$$\sum_{k=0}^{n} s_{n-k}\varphi^{n-k}(a_k) \to 0, \text{ as } n \to \infty$$

Next, we shall present the definition of α_*-ψ-contractive and α_*-admissible mappings introduced by Hasanzade et al.[17]:

Definition 5. *[17] Let (X,d) be a metric space and $T : X \to P(X)$ be a multivalued operator. We say that T is an α_*-ψ-contractive multivalued operator if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that*

$$\alpha_*(T(x),T(y))H(T(x),T(y)) \leq \psi(d(x,y)), \text{ for all } x,y \in X. \tag{1}$$

where $\alpha_*(A,B) = inf\{\alpha(a,b),\ a \in A,\ b \in B\}$.

Definition 6. *[17] Let $T : X \to P(X)$ and $\alpha : X \times X \to [0,\infty)$. We say that T is α_*-admissible if*

$$x,y \in X,\ \ \alpha(x,y) \geq 1 \Longrightarrow \alpha_*(T(x),T(y)) \geq 1.$$

2 Main results

In this section, we shall state and prove our main results. First we start with b-metric version of Definition 5 for the Pompeiu-Hausdorff generalized functional:

Definition 7. *Let (X,d) be a b-metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. We say that T is an α_*-ψ-contractive multivalued operator of type (b) if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi_b$ such that*

$$\alpha_*(T(x),T(y))H(T(x),T(y)) \leq \psi(d(x,y)), \text{ for all } x,y \in X. \tag{2}$$

Our first main result is the following.

Theorem 1.*Let (X,d) be a complete b-metric space with constant $s>1$. Let $T: X\to P_{cl}(X)$ be an α_*-ψ-contractive multivalued operator of type-(b) satisfying the following conditions:*

(i)T is α_*-admissible;
(ii)there exists $x_0\in X$ and $x_1\in T(x_0)$ such that $\alpha(x_0,x_1)\geq 1$;
(iii)if x_n is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ and $x_n\to x$ then $\alpha(x_n,x)\geq 1, \forall n$.

Then T is MWP operator.

*Proof.*Let $x_0\in X$ and $x_1\in T(x_0)$. If $x_0=x_1$ we obtain the desired conclusion. Let $x_0\neq x_1$ and $x_1\notin T(x_1)$. Then, by the properties of the functional H there exist $q_1>1$ and $x_2\in T(x_1)$ such that $d(x_1,x_2)\leq q_1 H(T(x_0),T(x_1))$ and $q_1\psi(d(x_0,x_1))<d(x_0,x_1)$.

By using the fact that T is an α_*-ψ-contraction we obtain

$$\begin{aligned} d(x_1,x_2) &\leq q_1 H(T(x_0),T(x_1)) \leq q_1\alpha_*(T(x_0),T(x_1))H(T(x_0),T(x_1)) \\ &\leq q_1\psi(d(x_0,x_1)) < d(x_0,x_1). \end{aligned}$$

We know that $\alpha(x_0,x_1)\geq 1$. By (i), we have $\alpha_*(T(x_0),T(x_1))\geq 1$. This implies that $\alpha(x_1,x_2)\geq 1$. Thus we get $\alpha_*(T(x_1),T(x_2))\geq 1$.

From the fact that ψ is an increasing function, we find that

$$\psi(d(x_1,x_2)) < \psi(d(x_0,x_1)).$$

In a similar way there exist $q_2>1$ and $x_3\in T(x_2)$ such that $d(x_2,x_3)\leq q_2 H(T(x_1),T(x_2))$ and $q_2\psi(d(x_1,x_2))<d(x_0,x_1)$. So we have

$$\begin{aligned} d(x_2,x_3) &\leq q_2 H(T(x_1),T(x_2)) \leq q_2\alpha_*(T(x_1),T(x_2))H(T(x_1),T(x_2)) \\ &\leq q_2\psi(d(x_1,x_2)) < \psi(d(x_0,x_1)). \end{aligned}$$

Since ψ is an increasing function, we obtain that

$$\psi(d(x_2,x_3)) < \psi^2(d(x_0,x_1)).$$

Inductively, we see that there exists $x_{n+1}\in T(x_n)$ such that $\alpha(x_{n+1},x_{n+2})\geq 1$ and

$$d(x_{n+1},x_{n+2}) < \psi^n(d(x_0,x_1)), \text{ for each } n\in\mathbb{N}.$$

We shall prove that $\{x_n\}$ is a Cauchy sequence.

$$\begin{aligned} d(x_n,x_{n+p}) &\leq s\cdot d(x_n,x_{n+1}) + s^2\cdot d(x_{n+1},x_{n+2}) + \ldots \\ &+ + s^{p-2}\cdot d(x_{n+p-3},x_{n+p-2}) + \\ &+ s^{p-1}\cdot d(x_{n+p-2},x_{n+p-1}) + s^p\cdot d(x_{n+p-1},x_{n+p}) \\ &< s\cdot\psi^{n-1}(d(x_0,x_1)) + s^2\cdot\psi^n(d(x_0,x_1)) + \ldots + \\ &+ s^{p-2}\cdot\psi^{n+p-4}(d(x_0,x_1)) + s^{p-1}\cdot\psi^{n+p-3}(d(x_0,x_1)) + \\ &+ s^p\cdot\psi^{n+p-2}(d(x_0,x_1)) \\ &= \frac{1}{s^{n-2}} \\ &\cdot\left[s^{n-1}\cdot\psi^{n-1}(d(x_0,x_1)) + \ldots + s^{n+p-3}\cdot\psi^{n+p-3}(d(x_0,x_1)) + \right. \\ &\left. + s^{n+p-2}\cdot\psi^{n+p-2}(d(x_0,x_1))\right] \\ &= \frac{1}{s^{n-2}}\cdot\sum_{k=n-1}^{n+p-2} s^k\cdot\psi^k(d(x_0,x_1)). \end{aligned}$$

Let $S_n=\sum_{k=0}^{n} s^k\psi^k(d(x_0,x_1))$, $n\geq 1$. Then we find that

$$d(x_n,x_{n+p}) \leq \frac{1}{s^{n-1}}[S_{n+p-1}-S_{n-1}],\ n\geq 1,\ p\geq 1. \quad (3)$$

Due to Lemma 6, we conclude that the series $\sum_{k=1}^{n-1} s^k\psi^k(d(x_0,x_1))$ is convergent.

Say $S=\lim_{n\to\infty} S_n$. Regarding $s\geq 1$ and (6), we obtain that $\{x_n\}$ is a Cauchy sequence in the b-metric space (X,d). Since (X,d) is complete, there exists $x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$.

Since $\alpha(x_n,x^*)\geq 1$ we have from hypothesis (i) and (iii) that $\alpha_*(T(x_n),T(x^*))\geq 1$. As a consequence, we derive that

$$\begin{aligned} D(x^*,T(x^*)) &\leq s[d(x^*,x_{n+1}) + D(x_{n+1},T(x^*))] \\ &\leq s[d(x^*,x_{n+1}) + H(T(x_n),T(x^*))] \\ &\leq s[d(x^*,x_{n+1}) + \alpha_*(T(x^*), \\ &\quad T(x_n))H(T(x_n),T(x^*))] \\ &\leq s[d(x^*,x_{n+1}) + \psi(d(x_n,x^*))]. \end{aligned}$$

From the properties of ψ we have $D(x^*,T(x^*))=0$. Since $T(x)$ is closed we obtain $x^*\in T(x^*)$.

We give another characterization of Definition 5 for a b-metric via the gap functional:

Definition 8.*Let (X,d) be a b-metric space and $T: X\to P_{cl}(X)$ be a multivalued operator. We say that T is a generalized α_*-ψ-contractive multivalued operator of type (b) if there exist two functions $\alpha: X\times X\to[0,\infty)$ and $\psi\in\Psi_b$ such that*

$$\alpha_*(T(x),T(y))D(y,T(y)) \leq \psi(d(x,y)), \quad \text{for all } x\in X,\ y\in T(x). \quad (4)$$

We have the following result:

Theorem 2. *Let (X,d) be a complete b-metric space and let $T : X \to P_{cl}(X)$ be an α^*-admissible strictly generalized $\alpha^* - \psi$-contractive mapping. Assume that there exist $x_0 \in X$ and $x_1 \in T(x_0)$ such that $\alpha(x_0,x_1) \geq 1$. Then there exists an orbit $\{x_n\}$ of T at x_0 and $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Moreover, x^* is a fixed point of T if and only if $f(\xi) = D(\xi, T\xi)$ is lower semi-continuous at x.*

Proof. Following the same lines of argument given in the proof of Theorem 1 we can construct a Cauchy sequence $\{x_n\}$ which converges to x^*, as $n \to \infty$. By the lower semicontinuity of the function D we obtain the desired conclusion.

Next, we propose a version of Definition 5 for *b*-metric via δ-functional:

Definition 9. *Let (X,d) be a b-metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. We say that T is an α_*-ψ-δ-contractive multivalued operator of type (b) if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi_b$ such that*

$$\alpha_*(T(x),T(y))\delta(y,T(y)) \leq \psi(d(x,y)),\quad \text{for all } x \in X,\ y \in T(x). \tag{5}$$

Finally, we have the following result:

Theorem 3. *Let (X,d) be a complete b-metric space and let $T : X \to P_b(X)$ be an α^*-admissible generalized $\alpha^* - \psi - \delta$-contractive mapping. Assume that there exist $x_0 \in X$ and $x_1 \in T(x_0)$ such that $\alpha(x_0,x_1) \geq 1$. Then there exists an orbit $\{x_n\}$ of T at x_0 and $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Moreover, $\{x^*\} = Tx^*$ if and only if $f(\xi) = \delta(\xi, T\xi)$ is lower semi-continuous at x.*

Proof. Similar to the proof of Theorem 1, we let $x_0 \in X$ and $x_1 \in T(x_0)$. We obtain the desired conclusion if $x_0 = x_1$. Let $x_0 \neq x_1$ and $x_1 \notin T(x_1)$. By the hypothesis we have that $\alpha(x_0,x_1) \geq 1$. As T is α_*-admissible we have that $\alpha_*(T(x_0),T(x_1)) \geq 1$. Then

$$\delta(x_1,T(x_1)) \leq \alpha_*(T(x_0),T(x_1))\delta(x_1,T(x_1)) \leq \psi(d(x_0,x_1)).$$

There exists $x_2 \in T(x_1)$ such that

$$0 < d(x_1,x_2) \leq \delta(x_1,T(x_1)) < \psi(d(x_0,x_1)).$$

From the fact that ψ is increasing we obtain that

$$\psi(d(x_1,x_2)) < \psi^2(d(x_0,x_1)).$$

By an inductive procedure we have that there exists $x_{n+1} \in T(x_n)$ such that $\alpha(x_n,x_{n+1}) \geq 1$ and

$$d(x_n,x_{n+1}) < \delta(x_n,T(x_n)) \leq \psi^n(d(x_0,x_1)),\ \text{for each } n \in \mathbb{N}.$$

We shall prove that $(x_n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence.

$$\begin{aligned}
d(x_n,x_{n+p}) &\leq s\cdot d(x_n,x_{n+1}) + s^2\cdot d(x_{n+1},x_{n+2}) \\
&+ \ldots + s^{p-2}\cdot d(x_{n+p-3},x_{n+p-2}) + \\
&+ s^{p-1}\cdot d(x_{n+p-2},x_{n+p-1}) + s^p\cdot d(x_{n+p-1},x_{n+p}) \\
&< s\cdot\psi^n(d(x_0,x_1)) + s^2\cdot\psi^{n+1}(d(x_0,x_1)) + \ldots + \\
&+ s^{p-2}\cdot\psi^{n+p-3}(d(x_0,x_1)) + s^{p-1}\cdot\psi^{n+p-2}(d(x_0,x_1)) + \\
&+ s^p\cdot\psi^{n+p-1}(d(x_0,x_1)) \\
&= \frac{1}{s^{n-1}}\cdot[s^n\cdot\psi^n(d(x_0,x_1)) + \ldots + s^{n+p-2}\cdot\psi^{n+p-2}(d(x_0,x_1)) + \\
&+ s^{n+p-1}\cdot\psi^{n+p-1}(d(x_0,x_1))] \\
&= \frac{1}{s^{n-1}}\cdot\sum_{k=n}^{n+p-1} s^k\cdot\psi^k(d(x_0,x_1)).
\end{aligned}$$

Denoting $S_n = \sum_{k=0}^{n} s^k\psi^k(d(x_0,x_1))$, $n \geq 1$ we obtain:

$$d(x_n,x_{n+p}) \leq \frac{1}{s^{n-1}}[S_{n+p-1} - S_{n-1}],\ n \geq 1,\ p \geq 1. \tag{6}$$

Due to Lemma 6, we conclude that the series $\sum_{k=1}^{n-1} s^k\psi^k(d(x_0,x_1))$ is convergent.

Thus there exists $S = \lim_{n\to\infty} S_n$. Regarding $s \geq 1$ and by (6), we obtain that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in the *b*-metric space (X,d). Since (X,d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Letting $n \to \infty$ in $\delta(x_n,T(x_n)) \leq \psi^n(d(x_0,x_1))$ we obtain that

$$\delta(x_n,T(x_n)) \to 0,\ \text{as},\ n \to \infty.$$

By the semicontinuity of the functional δ we obtain the desired conclusion.

Remark. It is clear that Definition 7 reduces to Definition 5 and (b)-comparison function becomes (c)-comparison function. Consequently, Theorem 2.1 in [17] is a consequence of Theorem 3. Further, if we replace multivalued operator with a single valued self-mapping we get Theorem as a corollary of Theorem 3. Hence, several results in these papers can be concluded also from Theorem 3.

3 Applications

Definition 10. *Let (X,d) be a b-metric space and $T : X \to P(X)$ be a multivalued operator. The fixed point inclusion*

$$x \in T(x),\ x \in X \tag{7}$$

is called generalized Ulam-Hyers stable if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous in 0 and $\psi(0) = 0$

such that for each $\varepsilon > 0$ and for each solution $y^* \in X$ of the inequality

$$D(y,T(y)) \leq \varepsilon \quad (8)$$

there exists a solution x^* of the fixed point inclusion (7) such that

$$d(y^*,x^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$, then the fixed point inclusion (7) is said to be Ulam-Hyers stable.

For Ulam-Hyers stability results in the case of fixed point problems see [8], [21], [24], [26].

In order to prove the next theorem we need the following hypothesis:

$$(H): \quad \text{for all } x,y \in X, \text{ there exists } z \in X \text{ such that}$$

$$\alpha(x,z) \geq 1 \text{ and } \alpha(y,z) \geq 1.$$

Regarding the Ulam-Hyers stability problem the ideas given in Petru et al. [23] allow us to obtain the following result.

Theorem 4.*Let (X,d) be a complete b-metric space with constant $s > 1$. Suppose that all the hypotheses of Theorem 1 hold and additionally that the function $\beta : [0,\infty) \to [0,\infty)$, $\beta(r) := r - s\psi(r)$ is strictly increasing and onto. Suppose also that $SFixT \neq \emptyset$ then we have:*

(i)The fixed point inclusion (7) is generalized Ulam-Hyers stable.
(ii)$Fix(T) = SFix(T) = \{x^*\}$.
(iii)Let $S : X \to P_{cl}(X)$ be a multivalued operator, $\eta > 0$ such that $Fix(S) \neq \emptyset$ and $H(S(x),T(x)) \leq \eta$, for all $x \in X$. Then

$$H(Fix(S),Fix(T)) \leq \beta^{-1}(s\eta).$$

(iv)(Well-posedness of the fixed point problem with respect to D) If $\{x_n\}$ is a sequence in X such that $D(x_n,T(x_n)) \to 0$ as $n \to \infty$, then $x_n \xrightarrow{d} x^*$ as $n \to \infty$.
(v)(Well-posedness of the fixed point problem with respect to H) If $\{x_n\}$ is a sequence in X such that $H(x_n,T(x_n)) \to 0$ as $n \to \infty$, then $x_n \xrightarrow{d} x^*$ as $n \to \infty$.
(vi)(Limit shadowing property of the multivalued operator) Suppose that φ is a sub-additive b-comparison function. If $\{y_n\}$ is a sequence in X such that $D(y_{n+1},T(y_n)) \to 0$, as $n \to \infty$, then there exists a sequence $\{x_n\} \subset X$ of successive approximations of T, such that $d(x_n,y_n) \to 0$, as $n \to \infty$.

Proof.

(i)Since $T : X \to P_{cl}(X)$ is a multivalued weakly Picard operator, so $\{x^*\} \in Fix(T)$. Let $\varepsilon > 0$ and $y^* \in X$ be a solution of (8), i.e,

$$D(y^*,T(y^*)) \leq \varepsilon.$$

Since T is α_*-ψ-contractive multivalued mapping of type-(b) and since $x^* \in Fix(T)$, from (H) there exists $y^* \in X$ such that $\alpha(x^*,y^*) \geq 1$ and taking into account that T is α_*-admissible we have $\alpha_*(T(x^*),T(y^*)) \geq 1$. So, we obtain:

$$d(x^*,y^*) = D(T(x^*),y^*) \leq s[H(T(x^*),T(y^*)) + D(T(y^*),y^*)] \leq$$

$$\leq s[\alpha_*(T(x^*),T(y^*))H(T(x^*),T(w^*)) + \varepsilon] \leq s[\psi(d(x^*,y^*)) + \varepsilon].$$

Therefore,

$$\beta(d(x^*,y^*)) := d(x^*,y^*) - s\psi(d(x^*,y^*)) \leq s\cdot\varepsilon \Longrightarrow d(x^*,y^*) \leq \beta^{-1}(s\cdot\varepsilon).$$

Consequently, the fixed point inclusion (7) is β^{-1}-generalized Ulam-Hyers stable.

(ii)From Theorem 1, we get that $Fix(T) \neq \emptyset$. Let $x^* \in SFix(T)$. Notice first that $SFix(T) = \{x^*\}$. We want to prove that $Fix(T) = \{x^*\}$. Let $y \in Fix(T)$, i.e. $y \in T(y)$ with $y \neq x^*$. Using the hypothesis (H) we can estimate the following distance

$$\begin{aligned} d(x^*,y) = D(T(x^*),y) &\leq H(T(x^*),T(y)) \\ &\leq \alpha_*(T(x^*),T(y))H(T(x^*),T(y)) \\ &\leq \psi(d(x^*,y)) < d(x^*,y). \end{aligned} \quad (9)$$

We obtain that $d(x^*,y) = 0$, and so $x^* = y$. Hence $Fix(T) \subset SFix(T)$. Since $SFix(T) \subset Fix(T)$ we obtain that $SFIx(T) = Fix(T)$.

The uniqueness condition of the strict fixed point can be proved with the same method.

(iii)Let $y \in Fix(S)$ and $x^* \in SFix(T)$. Then using the hypothesis (H) we obtain:

$$\begin{aligned} d(y,x^*) &\leq H(S(y),T(x^*)) \leq \\ &\leq s\cdot[H(S(y),T(y)) + H(T(y),T(x^*))] \leq \\ &\leq s\cdot\alpha_*(T(x^*),T(y))\,[H(S(y),T(y)) + H(T(y),T(x^*))] \leq \\ &\leq s\cdot[\eta + \psi(d(y,x^*))], \end{aligned}$$

from where we obtain that $d(y,x^*) - s\cdot\psi(d(y,x^*)) \leq s\cdot\eta$. Hence

$$H(Fix(S),Fix(T)) \leq \beta^{-1}(s\eta).$$

(iv)Let $x^* \in SFix(T)$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $D(x_n,T(x_n)) \to 0$, as $n \to \infty$. Then for $v_n \in T(x_n)$ such that $d(x_n,v_n) = D(x_n,T(x_n))$, $n \in \mathbb{N}$, using the hypothesis (H) we have:

$$\begin{aligned} d(x_n,x^*) &\leq s\,[d(x_n,v_n) + d(v_n,x^*)] = \\ &= s\,[d(x_n,v_n) + D(v_n,T(x^*))] \leq \\ &\leq s\,[d(x_n,v_n) + H(T(x_n),T(x^*))] \\ &\leq s\,[D(x_n,T(x_n)) + \alpha_*(T(x^*),T(y))H(T(x_n),T(x^*))] \\ &\leq s\,[D(x_n,T(x_n)) + \psi(d(x_n,x^*))]. \end{aligned}$$

Thus we get

$$d(x_n,x^*) - s\cdot\psi(d(x_n,x^*)) \leq s\cdot D(x_n,T(x_n)).$$

From the above inequalities we obtain that

$$d(x_n,x^*) \le \beta^{-1}(s\cdot D(x_n,T(x_n))) \to 0, \text{as } n\to\infty.$$

Hence we see that $x_n \xrightarrow{d} x^*$, as $n\to\infty$.
(v)It follows from (iii).
(vi)Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in X such that $D(y_{n+1},T(y_n)) \to 0$, as $n\to\infty$. Then there exists $u_n \in T(y_n), n\in\mathbb{N}$ such that $d(y_{n+1},u_n)\to 0$, as $n\to\infty$. Using the hypothesis (H) we obtain:

$$\begin{aligned}
& d(x^*,y_{n+1})\\
&\le s[D(y_{n+1},T(y_n))+\alpha_*(T(x^*),T(y))H(T(y_n),T(x^*))]\\
&\le s\cdot D(y_{n+1},T(y_n))+s\cdot\psi(d(y_n,x^*))\\
&\le s\cdot D(y_{n+1},T(y_n))+s\cdot\psi(s\cdot D(y_n,T(y_{n-1}))\\
&\qquad +s\cdot\psi(d(y_{n-1},x^*)))\\
&\le s\cdot D(y_{n+1},T(y_n))+s^2\cdot\psi(D(y_n,T(y_{n-1})))\\
&\quad +s^2\cdot\psi^2(s\cdot D(y_{n-1},T(y_{n-2}))+s\cdot\psi(d(y_{n-2},x^*)))\le\ldots\le\\
&\le s\cdot D(y_{n+1},T(y_n))+s^2\cdot\psi(D(y_n,T(y_{n-1})))+\\
&\quad +s^3\cdot\psi^2(D(y_{n-1},T(y_{n-2})))+\ldots+s^n\cdot\psi^{n-1}(D(y_2,T(y_1)))+\\
&\quad +s^{n+1}\cdot\psi^n(D(y_1,T(y_0)))+s^{n+1}\cdot\psi^{n+1}(d(y_0,x^*))=\\
&= s\left[\sum_{i=0}^{n}s^i\cdot\psi^i(D(y_{n-i+1},T(y_{n-i})))+s^n\cdot\psi^{n+1}(d(y_0,x^*))\right]\\
&\le s\cdot\sum_{i=0}^{n+1}s^i\cdot\psi^i(D(y_{n-i+1},T(y_{n-i}))).
\end{aligned}$$

By Lemma 7, the right side of the inequality above tends to zero as $n\to\infty$. Thus we find that $d(x^*,y_{n+1})\to 0$, as $n\to\infty$. Since the multivalued operator is a MWP operator (1), we obtain that there exists a sequence of successive approximations of T starting from $(x_0,x_1)\in Graph(T)$ which converges to $x^*\in Fix(T)$. From the uniqueness of the fixed point we get that $d(x_n,x^*)\to 0$, as $n\to\infty$. Thus we derive that

$$d(x_n,y_n)\le s\cdot[d(x_n,x^*)+d(x^*,y_n)]\to 0, \text{as } n\to\infty.$$

Hence the condition follows. □

4 Examples of a fixed point inclusions

Example 1
Let us consider the following initial value problem:

$$\begin{cases} x'(t)\in F(x(t)), \text{ for } t\in[a,b]\\ x(t)=x_0 \text{ for } t=a,\end{cases} \tag{10}$$

where F is a lower semicontinuous mutivalued operator and $[a,b]$ a real interval.

Then we can say that there exists a selection $f:\mathbb{R}\to\mathbb{R}$ such that $f(u)\in F(u)$ for all $u\in\mathbb{R}$.

It is clear that any solution of the following problem

$$\begin{cases} x'(t)=f(x(t)), \text{ for } t\in[a,b]\\ x(t)=x_0, \text{ for } t=a\end{cases} \tag{11}$$

is a solution for the problem (10).
The problem (11) is equivalent to

$$x(t)=x_0+\int_a^t f(x(s))ds, \text{ for } t\in[a,b] \tag{12}$$

Let us suppose that f satisfies the following conditions:

(C1) $\int_a^t f(x(s))ds=0$, for all $t\in[a,b]$ if and only if $x\equiv x_0$ on $[a,b]$.
(C2) $\|f(u)-f(v)\|\le L_f\|u-v\|$, for all $u,v\in\mathbb{R}$

We define the operator $T:X\to X$, where $X:=C[a,b]$, with $x\mapsto Tx$, by the formula

$$Tx(t)=x_0+\int_a^t f(x(s))ds, \text{ for } t\in[a,b]. \tag{13}$$

In the second part of the proof we need the functions α, d, ψ, defined below:
$\alpha: X\times X\to\mathbb{R}_+$, by

$$\alpha(x,y)=\begin{cases} k \text{ for } x\not\equiv x_0 \text{ and } y\not\equiv x_0, \text{ where } k\ge 1\\ 0, \text{ otherwise}\end{cases} \tag{14}$$

$d: X\times X\to\mathbb{R}_+$, by

$$d(x,y):=\max_t\|x(t)-y(t)\|^2, \text{ for } t\in[a,b]$$

$\psi:\mathbb{R}_+\to\mathbb{R}_+$, by

$$\psi(t):=L_f^2(b-a)t.$$

First we want to prove the admissibility of T.
From the definition of admissibility we have that $\alpha(x,y)\ge 1$, which implies that $\alpha(x,y)=k$ in our case. We have $x\not\equiv x_0$ and $y\not\equiv x_0$. From condition (C1) we find that $Tx\not\equiv x_0$ and $Ty\not\equiv x_0$ on $[a,b]$. It follows that $\alpha(Tx,Ty)=k\ge 1$. This proves the admissibility of T.
Now we want to prove that T is an $\alpha-\psi$ contractive operator. We estimate $d(Ty,Tx)$ as follows:

$$\begin{aligned}
d(Ty,Tx)&=\max\|\int_a^t f(x(s))ds-\int_a^t f(y(s))ds\|^2\\
&\le\max\int_a^t\|f(x(s))ds-f(y(s))\|^2ds\le\\
&\le\max\int_a^t L_f^2\|x(s)-y(s)\|^2ds\\
&\le L_f^2\int_a^t\max\|x(s)-y(s)\|^2ds\le L_f^2(b-a)d(x,y).
\end{aligned}$$

By applying Theorem 1 we obtain that the problem (11) has a solution. But, if we have a solution for the selection then we have a solution for the problem (10).

Finally, we prove that the fixed point inclusion (10) is the generalized Ulam-Hyers stable. We define $\beta(r) := r - sL_f^2(b-a)r$ in relation to $\psi(r) = L_f^2(b-a)r$. Since β is a continuous strictly increasing function, we get $\lim_{r\to 0^+}\beta(r) = 0$ and $\lim_{r\to+\infty}\beta(r) = +\infty$. Thus, we conclude that β is strictly increasing and onto. Consequently, all the hypothesis of Theorem 4 hold, and hence the fixed point inclusion (10) is β^{-1} generalized Ulam-Hyers stable.

Example 2

Let us consider the following problem:

$$\begin{cases} x'(t) \in F(t,x(t),x(t-h)), \text{ for } t\in[a,b] \\ x(t) = x_0, \text{ for } t\in[a-h,a] \end{cases} \tag{15}$$

where F is a lower semicontinuous mutivalued operator and $[a,b]$ a real interval and h a positive real nonzero parameter.

Then we can say that there exists a selection $f:\mathbb{R}\to\mathbb{R}$ such that $f(u)\in F(u)$ for all $u\in\mathbb{R}$.

We can see that any solution of the following problem

$$\begin{cases} x'(t) = f(t,x(t),x(t-h)), \text{ for } t\in[a,b] \\ x(t) = x_0, \text{ for } t\in[a-h,a] \end{cases} \tag{16}$$

is a solution for the problem (15).

The problem (16) is equivalent with

$$\begin{cases} x(t) = x_0 + \int_a^t f(s,x(s),x(s-h))ds, \text{ for } t\in[a,b] \\ x(t) = x_0, \text{ for } t\in[a-h,a] \end{cases} \tag{17}$$

Let us suppose for f the following conditions:

(C1) $\int_a^t f(s,x(s),x(s-h))ds = 0$, for all $t\in[a,b]$ if and only if $x\equiv x_0$ on $[a,b]$.

(C2) $\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \le L_f(\|u_1-u_2\|+\|v_1-v_2\|)$, for all $u_1,v_1,u_2,v_2\in\mathbb{R}$

We define the following operator $T: X\to X$, where $X := C[a-h,b]\times\mathbb{R}\times\mathbb{R}$, with $x\mapsto Tx$, by the formula

$$Tx(t) = \begin{cases} x_0 + \int_a^t f(s,x(s),x(s-h))ds, \text{ for } t\in[a,b] \\ x(t) = x_0, \text{ for } t\in[a-h,a] \end{cases} \tag{18}$$

In the second part of the proof we need the functions α, d, ψ, defined below:

$\alpha: X\times X\to\mathbb{R}_+$, by

$$\alpha(x,y) = \begin{cases} k \text{ for } x\not\equiv x_0 \text{ and } y\not\equiv x_0, \text{ where } k\ge 1 \\ 0, \text{ otherwise} \end{cases} \tag{19}$$

$d: X\times X\to\mathbb{R}_+$, by

$$d(x,y) := \max_t \|x(t)-y(t)\|^2, \text{ for } t\in[a-h,b]$$

$\psi:\mathbb{R}_+\to\mathbb{R}_+$, by

$$\psi(t) := 4L_f^2(b-a)t.$$

First we want to prove the admissibility for T.

From the definition of admissibility we have that $\alpha(x,y)\ge 1$. Which in our case implies that $\alpha(x,y) = k$. We have that $x\not\equiv x_0$ and $y\not\equiv x_0$. From condition (C1) we have that $Tx\not\equiv x_0$ and $Ty\not\equiv x_0$ on $[a,b]$. It follows that $\alpha(Tx,Ty) = k\ge 1$. This proves the admissibility of T.

Now we want to prove that T is an $\alpha-\psi$ contractive operator. In order to show that we estimate $d(Ty,Tx)$ and we obtain:

$$\begin{aligned} d(Ty,Tx) &= \max\left\|\int_a^t f(s,x(s),x(s-h))ds - \int_a^t f(s,y(s),y(s-h))ds\right\|^2 \\ &\le \max\int_a^t \|f(s,x(s),x(s-h))ds - \int_a^t f(s,y(s),y(s-h))\|^2 ds \\ &\le \max\int_a^t L_f^2\|y(s)-x(s)\|^2 2^2 ds \\ &\le 4L_f^2\int_a^t \max\|x(s)-y(s)\|^2 ds \\ &\le 4L_f^2(b-a)d(x,y). \end{aligned}$$

Applying Theorem 1 we obtain that the problem (16) has a solution. But, if we have a solution for the selection then we have a solution for the problem (15).

In what follows we prove the generalized Ulam-Hyers stability for the fixed point inclusion (15). Due to $\psi(r) = 4L_f^2(b-a)r$, we define $\beta(r) := r - 4sL_f^2(b-a)r$. Since β is a continuous strictly increasing function, we find that $\lim_{r\to 0^+}\beta(r) = 0$ and $\lim_{r\to+\infty}\beta(r) = +\infty$. Hence, β is strictly increasing and onto. All the hypothesis of Theorem 4 hold, so the fixed point inclusion (15) is β^{-1} generalized Ulam-Hyers stable.

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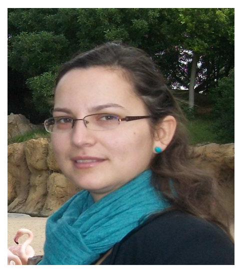
Monica-Felicia Bota received the Ph.D. degree in 2009 in Mathematics at Babeş-Bolyai University, Cluj-Napoca, Romania. Her research interests include Fixed point theory, applied mathematics, operator theory, nonlinear analysis, differential equations, dynamical systems. She is author of more than 20 papers, 15 of them are published in ISI Journals. She is Scientific Secretary at Fixed Point Theory. She is referee for AMS and Zentralblatt and for several international ISI journals such as: Fixed Point Theory, Fixed Point Theory and Applications, Journal of Inequalities and Applications, etc.

Veronica Ilea received the Ph.D. degree, in 2005, in Mathematics at Babeş-Bolyai University, Cluj-Napoca, Romania. Her research interests include: Nonlinear Operators and Differential Equations: Fixed Point Theory, Applied Mathematics, Dynamical Systems, Differential Equations with a Deviation of the Argument, Differential and Integral Equations with Application in Biology. She is the author of over 20 papers. She is referee for several international journals (FPT, Carpathian, FPTA, etc).

Erdal Karapınar is an associated professor at the Department of Mathematics, Atılım University. He received MS degree in Mathematics (Point Set Topology) in 1998 and Ph.D. degree in Mathematics (Functional Analysis) from Middle East Technical University in 2004. He is an author of more than 150 articles most of which are published in international ISI journals. His research activity has been developed in the framework of ”linear topological invariants”, ”Orlicz spaces”,”Fréchet spaces”, ”Köthe spaces”, ”Ulam-Hyers stability results via fixed point theory”, ”best proximity theory” and ”Fixed point theory”. Current research interests of E. Karapınar is centered on ”Fixed point theory and its applications”. He has been acting as an editor in several international ISI journals and referee of a number of scientific international journals of mathematics (http://erdalkarapinar.wordpress.com)



Oana Mleşni̧ is received the Ph.D. degree in Mathematics at Babeş-Bolyai University, Cluj-Napoca, Romania in 2013. Her research interests include fixed point theory, coincidence theory with single-valued and multi-valued operators, Ulam-Hyers stability results, applied mathematics, operator theory. She is author of several papers. She is referee for several international journals.