Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100119

Mixed Vector Equilibrium Problem Involving Multi-Valued Mapping

Mijanur Rahaman[1,∗], *Adem Kılıçman*[2] *and Rais Ahmad*[1]

[1] Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India
[2] Department of Mathematics and Institute for Mathematical Research, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia

Received: 21 Jun. 2015, Revised: 19 Aug. 2015, Accepted: 20 Aug. 2015
Published online: 1 Jan. 2016

Abstract: In this paper, we consider and study a mixed vector equilibrium problem involving multi-valued mapping in a Hausdorff topological vector space. We prove some existence results for mixed vector equilibrium problem involving multi-valued mapping using KKM theorem, the concept of coercing family for multi-valued mappings and core of a set. The problem of this paper is a combination of a vector equilibrium problem and a vector variational inequality problem and is more general than many existing problems available in the literature.

Keywords: Equilibrium problem, Variational inequality, Vector, Generalized KKM theorem, Core, Coercing family.

1 Introduction

The equilibrium problem has been extensively studied, beginning with *Blum* and *Oettli* [5] where they proposed it as a generalization of optimization and variational inequality problem.

Let K be a convex subset of a topological vector space X, and let $f : K \times K \longrightarrow \mathbb{R}$ be a given function with $f(x,x) = 0$ on K. The scalar-valued equilibrium problem deals with the existence of $\bar{x} \in K$ such that

$$f(\bar{x},y) \geq 0,\ \forall y \in K. \quad (1)$$

Its turns out that this problem includes, as special cases many problems such as fixed point problem, complementarity problem, Nash equilibrium problem etc.. For more details, we refer to [2,3,4].

Let Y be an another Hausdorff topological vector space and $C \subseteq Y$ a cone. Given a vector-valued mapping $f : K \times K \longrightarrow Y$. The problem of finding $\bar{x} \in K$ such that

$$f(\bar{x},y) \notin -intC,\ \forall y \in K. \quad (2)$$

Problem (2) is called vector equilibrium problem, see e.g. [8,9,10,2].

Let $T : K \longrightarrow L(X,Y)$ be a mapping, where $L(X,Y)$ denotes the space of all linear bounded mappings from X into Y. The vector variational inequality problem is to find $\bar{x} \in K$ such that

$$\langle T(\bar{x}), y-\bar{x}\rangle \notin -intC,\ \forall y \in K.$$

In this paper, we consider and study a mixed vector equilibrium problem involving multi-valued mapping which is a combination of a vector equilibrium problem and a vector variational inequality problem. We prove some existence results for our problem using different concepts. It is easy to check that mixed vector equilibrium problem involving multi-valued mapping includes vector equilibrium problems, equilibrium problems, variational inequalities, vector variational inequalities etc. as special cases.

2 Preliminaries and Formulation

Throughout this paper, let X and Y be two Hausdorff topological vector spaces. Let K be a nonempty convex closed subset of X and $C \subseteq Y$ a pointed closed convex cone with nonempty interior i.e., $intC \neq \emptyset$. The partial order " $\leq_C$ " on Y induced by C is defined by $x \leq_C y$ if and only if $y - x \in C$. Let $f : K \times K \longrightarrow Y$ and $T : K \longrightarrow 2^{L(X,Y)}$ be two mappings. We consider the following problem:

* Corresponding author e-mail: mrahman96@yahoo.com

Find $x \in K$, $v \in T(x)$ such that for all $y, b \in K$, and $\lambda \in (0,1]$,

$$f(\lambda x + (1-\lambda)b, y) + \langle v, y - x\rangle \notin -intC. \qquad (3)$$

We call problem (3) as mixed vector equilibrium problem involving multi-valued mapping. We prove some existence results for problem (3) in different settings.

The following definitions and concepts are needed to prove the results of this paper.

Definition 1.[7] *Let K be a nonempty convex subset of a topological vector space X. A multi-valued mapping $F : K \longrightarrow 2^X$ is said to be KKM mapping, if for every finite subset $\{x_1, x_2, \cdots, x_n\}$ of K,*

$$Co\{x_1, x_2, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i),$$

where $Co\{x_1, x_2, \cdots, x_n\}$ denotes the convex hull of $\{x_1, x_2, \cdots, x_n\}$.

Definition 2. *A multi-valued mapping $T : K \longrightarrow 2^{L(X,Y)}$ is called C-monotone, if for any $x, y \in K$*

$$\langle s - t, y - x\rangle \in -C, \ \forall s \in T(x), t \in T(y);$$

or, equivalently

$$\langle s, y - x\rangle \leq_C -\langle t, x - y\rangle, \ \forall s \in T(x), t \in T(y).$$

Definition 3. *Let (Y, C) be an ordered topological vector space. A mapping $T : X \longrightarrow Y$ is said to be C-convex, if for any pair of points $x, y \in X$, and $\lambda \in [0,1]$,*

$$T(\lambda x + (1-\lambda)y) \leq_C \lambda T(x) + (1-\lambda)T(y).$$

Lemma 1.[6] *Let (Y, C) be an ordered topological vector space with a pointed closed convex cone C with $intC \neq \emptyset$. Then for all $x, y \in Y$, we have*

(i) $y - x \in intC$ and $y \notin intC$ imply $x \notin intC$;
(ii) $y - x \in C$ and $y \notin intC$ imply $x \notin intC$;
(iii) $y - x \in -intC$ and $y \notin -intC$ imply $x \notin -intC$;
(iv) $y - x \in -C$ and $y \notin -intC$ imply $x \notin -intC$.

Definition 4.[4] *Consider a subset K of a topological vector space X and a topological space Y. A family $\{(C_i, Z_i)\}_{i\in I}$ of pair of sets is said to be coercing for a mapping $F : K \longrightarrow 2^Y$ if and only if*

(i) for each $i \in I$, C_i is contained in a compact convex subset of K and Z_i is a compact subset of Y;
(ii) for each $i, j \in I$, there exists $k \in I$ such that $C_i \cup C_j \subseteq C_k$;
(iii) for each $i \in I$, there exists $l \in I$ with $\bigcap_{x \in C_l} F(x) \subseteq Z_i$.

Theorem 1.[4] *Let X be a Hausdorff topological vector space, Y a convex subset of X, K a nonempty subset of Y and $F : K \longrightarrow 2^Y$ a KKM mapping with compactly closed values in Y (i.e., for all $x \in K$, $F(x) \cap Z$ is closed for every compact set Z of Y). If F admits a coercing family, then*

$$\bigcap_{x \in K} F(x) \neq \emptyset.$$

Definition 5.[5] *Let K and D be convex subsets of X with $D \subset K$. The core of D relative to K, denoted by $core_K D$, is the set defined by $a \in core_K D$ if and only if $a \in D$ and $D \cap (a, y) \neq \emptyset$, for all $y \in K \setminus D$.*

3 Existence Results

Theorem 2. *Let X and Y be two Hausdorff topological vector spaces and K a nonempty subset of X. Let C be a closed convex pointed cone in Y with $intC \neq \emptyset$ and $W : K \longrightarrow 2^Y$ defined by $W = Y \setminus \{-intC\}$. Let $f : K \times K \longrightarrow Y$ and $T : K \longrightarrow 2^{L(X,Y)}$ be two mappings such that following conditions holds:*

(i) T is C-monotone and hemicontinuous;
(ii) f is continuous in the first argument and C-convex in the second argument;
(iii) $f(\lambda z + (1-\lambda)b, z) = 0$, for all $z, b \in K$ and $\lambda \in (0,1]$;
(iv) W is closed;
(v) there exists a family $\{(C_i, Z_i)\}_{i\in I}$ satisfying conditions (i) and (ii) of Definition 4 and the following condition: For each $i \in I$, there exists $l \in I$ such that

$$\{x \in K \ : \ f(\lambda x + (1-\lambda)b, y) - \langle u, x - y\rangle \notin -intC, \ \forall\, y \in C_l,\, u \in T(y)\} \subseteq Z_i.$$

Then, there exists a point $x \in K$ such that for all $y \in K$, $v \in T(x)$,

$$f(\lambda x + (1-\lambda)b, y) + \langle v, y - x\rangle \notin -intC.$$

For the proof of Theorem 2, we need the following proposition, for which all the assumptions of Theorem 2 are remain same.

Proposition 1. *The following two problems are equivalent:*

(I) Find $x \in K$: $f(\lambda x + (1-\lambda)b, y) - \langle u, x - y\rangle \notin -intC$; $\forall b, y \in K$, $u \in T(y)$;
(II) Find $x \in K$: $f(\lambda x + (1-\lambda)b, y) + \langle v, y - x\rangle \notin -intC$; $\forall b, y \in K$, $v \in T(x)$;

where $\lambda \in (0,1]$.

Proof. Suppose that (II) holds. Then there exists $x \in K$ such that for $v \in T(x)$,

$$f(\lambda x + (1-\lambda)b, y) + \langle v, y - x\rangle \notin -intC.$$

Since T is C-monotone, we have

$$\langle v, y - x\rangle \leq_C -\langle u, x - y\rangle, v \in T(x), u \in T(y).$$

Also

$$f(\lambda x+(1-\lambda)b,y)+\langle v,y-x\rangle \leq_C f(\lambda x+(1-\lambda)b,y) -\langle u,x-y\rangle. \quad (4)$$

Since $f(\lambda x+(1-\lambda)b,y)+\langle v,y-x\rangle \notin -intC$, using (iv) of Lemma 1 and (4), we obtain

$$f(\lambda x+(1-\lambda)b,y)-\langle u,x-y\rangle \notin -intC;$$

i.e., (I) holds.

Conversely, suppose that (I) holds. Then

$$f(\lambda x+(1-\lambda)b,y)-\langle u,x-y\rangle \notin -intC, u\in T(y).$$

Let for all $y\in K$, $x_\alpha=\alpha y+(1-\alpha)x$, $0\leq\alpha\leq 1$. Then $x_\alpha\in K$ and hence we have

$$f(\lambda x+(1-\lambda)b,x_\alpha)-\langle u',x-x_\alpha\rangle \notin -intC, u'\in T(x_\alpha),$$

and therefore

$$(1-\alpha)f(\lambda x+(1-\lambda)b,x_\alpha)-(1-\alpha)\langle u',x-x_\alpha\rangle \notin -intC, \quad (5)$$

for $u'\in T(x_\alpha)$.

Since $\langle u',x-x_\alpha\rangle=\alpha\langle u',x-y\rangle$, therefore (5) can be written as

$$(1-\alpha)f(\lambda x+(1-\lambda)b,x_\alpha)-\alpha(1-\alpha)\langle u',x-y\rangle \notin -intC, \quad (6)$$

for $u'\in T(x_\alpha)$.

As f is C-convex in the second argument and $f(\lambda x+(1-\lambda)b,x)=0$, for all $x\in K$, we have for $u'\in T(x_\alpha)$

$$(1-\alpha)f(\lambda x+(1-\lambda)b,x_\alpha)-\alpha(1-\alpha)\langle u',x-y\rangle \leq_C \alpha(1-\alpha)f(\lambda x+(1-\lambda)b,y)+\alpha(1-\alpha)\langle u',y-x\rangle. \quad (7)$$

Hence by (6) and (iv) of Lemma 1, (7) implies that

$$\alpha(1-\alpha)f(\lambda x+(1-\lambda)b,y)+\alpha(1-\alpha)\langle u',y-x\rangle \notin -intC, \quad (8)$$

for $u'\in T(x_\alpha)$.

Dividing by $\alpha(1-\alpha)$, we have

$$f(\lambda x+(1-\lambda)b,y)+\langle u',y-x\rangle \notin -intC, u'\in T(x_\alpha). \quad (9)$$

Since T is hemicontinuous and W is closed, from (9) we have

$$f(\lambda x+(1-\lambda)b,y)+\langle v,y-x\rangle \in W, v\in T(x),$$

and thus

$$f(\lambda x+(1-\lambda)b,y)+\langle v,y-x\rangle \notin -intC, v\in T(x),$$

i.e., (II) holds. □

Proof of Theorem 2. For each $y\in K$, consider the set

$$F(y)=\{x\in K:\ f(\lambda x+(1-\lambda)b,y)-\langle u,x-y\rangle \notin -intC; u\in T(y)\}.$$

We claim that F is a KKM mapping. If F is not a KKM mapping, then there exists a finite subset $\{y_1,y_2,\cdots,y_n\}$ of K and $t_i\geq 0$, $i=1,2,\cdots,n$ with $\sum_{i=1}^n t_i=1$ such that

$$z=\sum_{i=1}^n t_iy_i\notin\bigcup_{i=1}^n F(y_i).$$

Then

$$f(\lambda z+(1-\lambda)b,y_i)-\langle u,z-y_i\rangle \in -intC. \quad (10)$$

It follows that

$$\sum_{i=1}^n t_if(\lambda z+(1-\lambda)b,y_i)-\sum_{i=1}^n t_i\langle u,z-y_i\rangle \in -intC. \quad (11)$$

From the conditions imposed on f, we have

$$0=f(\lambda z+(1-\lambda)b,z)\leq_C \sum_{i=1}^n t_if(\lambda z+(1-\lambda)b,y_i). \quad (12)$$

Also, since

$$\begin{aligned}0&=\langle u,z-z\rangle\\ &=\left\langle u,\sum_{i=1}^n t_iz-\sum_{i=1}^n t_iy_i\right\rangle\\ &=\sum_{i=1}^n t_i\langle u,z-y_i\rangle, \quad (13)\end{aligned}$$

therefore, combining (12) and (13), we have

$$\sum_{i=1}^n t_if(\lambda z+(1-\lambda)b,y_i)-\sum_{i=1}^n t_i\langle u,z-y_i\rangle \in C,$$

which contradicts (11). Hence F is a KKM mapping.

Next, we show that $F(y)$ is closed. Let $\{x_n\}$ be a sequence in $F(y)$ such that $x_n\to x_0$. As f is continuous in the first argument, we have

$$f(\lambda x_n+(1-\lambda)b,y)-\langle u,x_n-y\rangle \longrightarrow f(\lambda x_0+(1-\lambda)b,y) -\langle u,x_0-y\rangle.$$

As W is closed, we have

$$f(\lambda x_0+(1-\lambda)b,y)-\langle u,x_0-y\rangle \in W.$$

It follows that

$$f(\lambda x_0+(1-\lambda)b,y)-\langle u,x_0-y\rangle \notin -intC.$$

It implies that $x_0\in F(y)$, so $F(y)$ is closed. In view of assumption (v), F has compactly closed values.

Assumption (v) implicates that the family $\{(C_i,Z_i)\}_{i\in I}$ satisfies the following condition which is for all $i\in I$, there exists $l\in I$ such that

$$\bigcap_{y\in C_l}F(y)\subseteq Z_i;$$

and therefore it is a coercing family for F.

Hence by applying Theorem 1, we have

$$\bigcap_{y\in K} F(y) \neq \emptyset.$$

Thus, there exists $x \in K$ such that for all $y, b \in K$,

$$f(\lambda x + (1-\lambda)b, y) - \langle u, x-y\rangle \notin -intC,\ u \in T(y).$$

Lastly, we apply Proposition 1 and we obtain

$$f(\lambda x + (1-\lambda)b, y) + \langle v, y-x\rangle \notin -intC,\ v \in T(x).$$

Hence problem (3) admits a solution. This completes the proof. □

Proposition 2.[1] *Assume that $\phi : K \longrightarrow Y$ is C-convex, $x_0 \in core_K D$, $\phi(x_0) \notin intC$ and $\phi(y) \notin -intC$, for all $y \in D$. Then, $\phi(y) \notin -intC$, for all $y \in K$.*

Theorem 3. *Let X, Y, C, W, f and T be same as in Theorem 2 and satisfying conditions $(i)-(iv)$ of Theorem 2. In addition, the following condition is satisfied which is there exists a nonempty convex compact subset D of K such that $x \in D \setminus core_K D$ and $y \in core_K D$. Then there exists $x \in D$ such that for all $y, b \in K$ and $\lambda \in (0,1]$,*

$$f(\lambda x + (1-\lambda)b, y) + \langle v, y-x\rangle \notin -intC,\ v \in T(x).$$

Proof. From Proposition 1, it follows that the following problems are equivalent i.e., find $x \in D$ such that

(I) $f(\lambda x + (1-\lambda)b, y) - \langle u, x-y\rangle \notin -intC; \forall b \in K, y \in D, u \in T(y)$;
(II) $f(\lambda x + (1-\lambda)b, y) + \langle v, y-x\rangle \notin -intC; \forall b \in K, y \in D, v \in T(x)$;

where $\lambda \in (0,1]$.

Set $\phi(y) = f(\lambda x + (1-\lambda)b, y) + \langle v, y-x\rangle$. Clearly $\phi(y)$ is C-convex and $\phi(y) \notin -intC$, for all $y \in D$.

If $x \in core_K D$, then set $x_0 = x$. If $x \in D \setminus core_K D$, then set $x_0 = y$, where y is same as in the hypothesis of the theorem. In both cases, $x_0 \in core_K D$ and $\phi(x_0) \notin intC$. Hence by Proposition 2, it follows that

$$\phi(y) \notin -intC, \forall y \in K.$$

Thus there exists $x \in D$ such that for all $y \in K$,

$$f(\lambda x + (1-\lambda)b, y) + \langle v, y-x\rangle \notin -intC,\ v \in T(x).$$

This completes the proof. □

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Mijanur Rahaman received his M.Phil degree from Aligarh Muslim University in 2013. Currently, he is pursuing his Ph.D degree in Mathematics at Aligarh Muslim University, Aligarh, India. His research interests are in the areas of nonlinear analysis, variational inequalities and optimization.

Adem Kılıçman is a full Professor at the Department of Mathematics, Faculty of Science, University Putra Malaysia. He received his B.Sc. and M.Sc. degrees from Department of Mathematics, Hacettepe University, Turkey and Ph.D from Leicester University, UK. He has joined University Putra Malaysia in 1997, since then working in Faculty of Science. He is also an active member of Institute for Mathematical Research, University Putra Malaysia. His research areas includes Functional Analysis and Topology.

Rais Ahmad received his Ph.D degree in Mathematics at Aligarh Muslim University, Aligarh, India. He is a full Professor in the Department of Mathematics of Aligarh Muslim University, India. His research interests are nonlinear functional analysis and optimization, equilibrium problems, complementarity and fixed point problems. He has visited a number of countries for research purposes. He has published more than 100 research articles in various journals of international repute.