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On the Solvability of a Nonlinear Optimization Problem for Thermal Processes Described by Fredholm Integro-Differential Equations with External and Boundary Controls

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Abstract: In the present paper we studied the problem of nonlinear optimal control of the thermal processes described by Fredholm integro-differential equations when the control parameters are nonlinearly included into the equation as well as into the boundary condition. The concept of weak generalized solution of the boundary value problem is introduced and the algorithm for its construction is indicated. It is established that optimal control is defined as the solution of the system of nonlinear integral equations which contain unknown functions under and out of the integral and satisfy the additional condition in the form of the system of inequalities. Sufficient conditions for the existence of a unique solution of the problem of nonlinear optimization are given, and algorithm of its construction has been developed.

Keywords: Boundary value problem, weak generalized solution, optimal control problem, functional, maximum principle, system of nonlinear integral equations, convergence

1 Introduction

It is well-known that basis of the optimal control theory of processes described by ordinary differential equations was laid in the 50th years of the 20*th* century in the works of L.S. Pontryagin and his colleagues [13] and basis of the optimal control theory of processes described by partial derivatives differential equations was laid in the 60th years of the 20*th* century in the works of A.G. Butkovskiy [12], A.I. Egorov [6].

Moreover, several processes described by ordinary and partial differential equations have been studied extensively by many researchers (see, [16,17,18,19,20, 21] and the references therein). However, such problems were not well-investigated in general.

One of the main research method of optimal control problems is Pontryagin's maximum (or minimum) principle which is used in optimal control theory to find the best possible control for taking a dynamical system from one state to another, especially in the presence of constraints for the state or input controls.

Note that the maximum principle was formulated for systems with lumped parameters, and it is applicable not always in the case for systems with distributed parameters [6].

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The problem of control processes described by integro-differential equations with partial derivatives is often encountered in applications and it has been studied in papers [6,7,8,9,10]. For example, in [15] investigated the problem with taking into account the only external control parameters. When we study of thermal processes, in practice it is necessary to consider the thermal flow passing as well as across the border.

In this article, we investigated the questions of unique solvability of the optimization problem for the thermal processes described by Fredholm integral-differential equations when the controlling external forces as well as boundary control are operated to object , i.e. object is controlled by two control forces. Such problems have not yet been studied in control theory. The quality control is estimated by the quadratic functional. Based on the maximum principle the conditions of control optimality for systems with distributed parameters [6] are obtained in the form of a nonlinear integral equation and differential inequality. The solvability of the nonlinear integral equation is studied according to the method of book [4]. For optimization problems we obtained the sufficient conditions of the unique solvability and we indicated an algorithm for constructing solutions of nonlinear optimization problems with arbitrary precision in the form of the triple $\left((u^0(t),\vartheta^0(t)),v^0(t,x),J[u^0(t),\vartheta^0(t)]\right)$, where $(u^0(t),\vartheta^0(t))$ is vector optimal control, $v^0(t,x)$ is optimal process, and $J[u^0(t),\vartheta^0(t)]$ is the minimum value of the functional.

2 Boundary value problem of the controlled process

Suppose that the state of a thermal process is described by the scalar function $v(t,x)$, which satisfies the integro-differential equation [1,2,3]

$$v_t = v_{xx} + \lambda\int_0^T K(t,\tau)v(\tau,x)d\tau + g(t,x)f[t,u(t)] \quad (1)$$

in the region $Q=\{0<x<1,\quad 0<t<T\}$, and on the boundary of Q it satisfies the initial condition

$$v(0,x)=\psi(x), 0\le x\le 1 \quad (2)$$

and boundary conditions

$$v_x(t,0)=0, v_x(t,1)+\alpha v(t,1)=p[t,\vartheta(t)](0\le t\le T), \quad (3)$$

where $K(t,\tau)$ is a given function defined in the region $D=\{0<t<T,\quad 0<\tau<T\}$ and satisfies the condition

$$\int_0^T\int_0^T K^2(t,\tau)d\tau dt = K_0<\infty, \quad (4)$$

i.e., $K(t,\tau)\in H(D)$; $\psi(x)\in H(0,1)$, $g(t,x)\in H(Q)$ are given functions; $f[t,u(t)]\in H(0,T)$, $p[t,\vartheta(t)]\in H(0,T)$ are functions of external sources which nonlinearly depend from the control functions $u(t)\in H(0,T)$, $\vartheta(t)\in H(0,T)$ and satisfy the conditions

$$f_u[t,u(t)]\ne 0,\quad p_\vartheta[t,\vartheta(t)]\ne 0,\quad \forall t\in(0,T); \quad (5)$$

λ is a parameter; $\alpha>0$ is a constant, T is a fixed moment of time. The Hilbert space of functions defined on the set Y is denoted by $H(Y)$.

In real-world applications, generalized solutions of boundary value problems are used. For the boundary value problem (1)-(3) we will use the following concept of *weak* generalized solution.

Definition 1.*Under a weak generalized solution of the boundary value problem (1)-(3) we mean the function $v(t,x)\in H(Q)$ which satisfies the integral identity*

$$\int_0^1(v\phi)_{t_1}^{t_2}dx = \int_{t_1}^{t_2}\int_0^1[v(\phi_t-\phi_{xx})+\phi(t,x) \quad (6)$$
$$\times\left(\lambda\int_0^T K(t,\tau)v(\tau,x)d\tau+g(t,x)f[t,u(t)]\right)]dxdt$$
$$+\int_{t_1}^{t_2}[\phi(t,1)(-\alpha v(t,1)+p[t,\vartheta(t)])$$
$$-\phi_x(t,1)v(t,1)+\phi_x(t,0)v(t,0)]dt$$

for any t_2 and t_2, $0<t_1\le t\le t_2\le T$, and for any function $\phi(t,x)\in C^{1,2}(Q)$, as well as the initial and boundary conditions in a weak sense, i.e., for any functions $\phi_0(x)\in H(0,1)$ and $\phi_1(t)\in H(0,T)$ the following relations hold

$$\lim_{t\to+0}\int_0^1 v(t,x)\phi_0(x)dx = \int_0^1\psi(x)\phi_0(x)dx,$$
$$\lim_{x\to 1-0}\int_0^T(v_x(t,x)-\alpha v(t,x))\phi_1(t)dt = \int_0^T p[t,\vartheta(t)]\phi_1(t)dt,$$
$$\lim_{x\to+0}\int_0^T v_x(t,x)\phi_1(t)dt = 0, \quad (7)$$

where $C^{1,2}(Q)$ is space of functions which has the first derivative with respect to t and the second order derivative with respect to x.

To construct the solution of boundary value problem (1)-(3) we use the eigenfunctions and eigenvalues of boundary problem [6]

$$z''(x)+\lambda_0^2 z(x)=0,\quad z'(0)=0,\quad z'(1)+\alpha z(1)=0. \quad (8)$$

Eigenfunctions have the form

$$z_n(x)=\sqrt{\frac{2(\lambda_n^2+\alpha^2)}{\lambda_n^2+\alpha^2+\alpha}}\cos\lambda_n x,\quad n\in\{1,2,...\}, \quad (9)$$

and form a complete orthonormal basis in the Hilbert space $H(0,1)$. Corresponding eigenvalues λ_n are determined as a solution of the transcendental equation $\lambda tg\lambda = \alpha$ and satisfies

$$\lambda_n \le \lambda_{n+1}, \quad \forall n \in \{1,2,\dots\}, \quad \lim_{n\to\infty} \lambda_n = \infty,$$

and

$$(n-1)\pi < \lambda_n < \frac{\pi}{2}(2n-1). \tag{10}$$

We are looking for the solution of boundary problem (1)-(3) in the form

$$v(t,x) = \sum_{n=1}^{\infty} v_n(t) z_n(x), \tag{11}$$

where

$$v_n(t) = \langle v(t,x), z_n(x) \rangle = \int_0^1 v(t,x) z_n(x) dx \tag{12}$$

are the Fourier coefficients of the function $v(t,x)$. The symbol $< \cdot,\cdot >$ is used for the scalar product in the Hilbert space $H(0,1)$. We also use the expansions

$$g(t,x) = \sum_{n=1}^{\infty} g_n(t) z_n(x), \tag{13}$$

$$g_n(t) = \langle g(t,x), z_n(x) \rangle = \int_0^1 g(t,x) z_n(x) dx,$$

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n z_n(x),$$

$$\psi_n = \langle \psi(x), z_n(x) \rangle = \int_0^1 \psi(x) z_n(x) dx.$$

According to the method [7], the formal solution of the boundary problem (1)-(3) is found by using the integral identity (6). By the arbitrariness of function $\phi(t,x)$ in the integral identity (6) we assume that $\phi(t,x) = z_n(x)$. After some calculations, the integral identity (6) takes the form

$$\int_{t_1}^{t_2} \Big\{ \frac{\partial}{\partial t} < v(t,x), z_n(x) > + \lambda_n^2 < v(t,x), z_n(x) > - \lambda \int_0^T K(t,\tau) < v(\tau,x), z_n(x) > d\tau - < g(t,x), z_n(x) > f[t,u(t)] - z_n(1) p[t,\vartheta(t)] \Big\} dt \equiv 0.$$

In this identity by supposing $t_2 = t$ and differentiating with respect to t, we obtain the integro-differential equation

$$\frac{\partial}{\partial t} < v(t,x), z_n(x) > + \lambda_n^2 < v(t,x), z_n(x) > = \lambda \int_0^T K(t,\tau) < v(\tau,x), z_n(x) > d\tau + < g(t,x), z_n(x) > f[t,u(t)] + z_n(1) p[t,\vartheta(t)], \tag{14}$$

which we solve with the initial condition

$$< v(t,x), z_n(x) > |_{t=t_1} = < v(t_1,x), z_n(x) > \tag{15}$$

for each fixed $n \in \{1,2,\dots\}$. Considering the right side of the equation as absolute term, we solve the Cauchy problem (14)-(15) by the formula

$$< v(t,x), z_n(x) > = e^{-\lambda_n^2 (t-t_1)} < v(t_1,x), z_n(x) > + \int_{t_1}^{t} e^{-\lambda_n^2 (t-\tau)} \Big(\lambda \int_0^T K(\tau,s) < v(s,x), z_n(x) > ds + < g(\tau,x), z_n(x) > f[\tau,u(\tau)] + z_n(1) p[\tau,\vartheta(\tau)] \Big) d\tau.$$

Tending t_1 to zero and taking account of (7), (13) we obtain the relation

$$v_n(t) = e^{-\lambda_n^2 t} \psi_n + \int_0^t e^{-\lambda_n^2 (t-\tau)} \Big(\lambda \int_0^T K(\tau,s) v_n(s) ds + g_n(\tau) f[\tau,u(\tau)] + z_n(1) p[\tau,\vartheta(\tau)] \Big) d\tau \tag{16}$$

which is the linear integral equation.

It is easy to see that there is an initial condition

$$v_n(0) = \psi_n. \tag{17}$$

We will rewrite equation (16) as

$$v_n(t) = \lambda \int_0^T K_n(t,s) v_n(s) ds + a_n(t), \tag{18}$$

where

$$K_n(t,s) = \int_0^t e^{-\lambda_n^2 (t-\tau)} K(\tau,s) d\tau, \tag{19}$$

$$a_n(t) = e^{-\lambda_n^2 t} \psi_n + \int_0^t e^{-\lambda_n^2 (t-\tau)} \times (g_n(\tau) f[\tau,u(\tau)] + z_n(1) p[\tau,\vartheta(\tau)]) d\tau. \tag{20}$$

We solve integral equation (18) using the following formula [8,9]

$$v_n(t) = \lambda \int_0^T R_n(t,s,\lambda) a_n(s) ds + a_n(t), \tag{21}$$

where

$$R_n(t,s,\lambda) = \sum_{i=1}^{\infty} \lambda^{i-1} K_{n,i}(t,s), \quad n \in \{1,2,\dots\} \tag{22}$$

is the resolvent of the kernel $K_n(t,s) \equiv K_{n,1}(t,s)$, the iterated kernels $K_{n,i}(t,s)$ are defined by the formula [8,9]

$$K_{n,i+1}(t,s) = \int_0^T K_n(t,\eta) K_{n,i}(\eta,s) d\eta, i \in \{1,2,\dots\}, \tag{23}$$

for each $n \in \{1,2,\dots\}$.

Further, as in [15], we have set the radius of convergence concerning resolvent for any $n \in \{1,2,\dots\}$, as well as proved that the solution of the problem (1)-(3) which defined by (11), (21) is an element of the Hilbert space, i.e. $v(t,x) \in H(Q)$ for any external control $u(t)$ and boundary control $\vartheta(t)$.

3 Formulation of optimal control problem and conditions of optimality

Consider the optimization problem in which it is required to minimize the quadratic integral functional

$$J[u(t),\vartheta(t)] = \int_0^1 [v(T,x)-\xi(x)]^2 dx + \beta\int_0^T [u^2(t)+\vartheta^2(t)]dt \quad (24)$$

for $\beta>0$,where $\xi(x)\in H(0,1)$ is given function on the set of solutions of problem (1)-(3), i.e. we need to find the controls $u^0(t)\in H(0,T)$ and $\vartheta^0(t)\in H(0,T)$ which, together with the corresponding solution $v^0(t,x)$ of boundary value problem (1)-(3), gives the smallest possible value of functional (24). In this case $u^0(t)$ and $\vartheta^0(t)$ are called the optimal controls, and $v^0(t,x)$ is the optimal process.

Since, according to (5) each vector control $(u^0(t),\vartheta^0(t))$ uniquely defines the controlled process $v^0(t,x)$, then the solution of boundary value problem (1)-(3) of the form $v(t,x)+\Delta v(t,x)$ correspond to the controls $u(t)+\Delta u(t)$ and $\vartheta(t)+\Delta\vartheta(t)$, where is the increment that corresponds to the increments $\Delta\vartheta(t)$ and $\Delta u(t)$. According to the procedure of application of the maximum principle [6,10,11], the increment of functional (24) can be written as

$$\Delta J[u,\vartheta] = J[u+\Delta u,\vartheta+\Delta\vartheta]-J[u,\vartheta] = -\int_0^T \Delta\Pi[t,v,\omega,u,\vartheta]dt + \int_0^1 \Delta v^2(T,x)dx, \quad (25)$$

where

$$\Delta\Pi(t,v,\omega,u,\vartheta) = \Pi(t,v,\omega,u+\Delta u(t),\vartheta+\Delta\vartheta) - \Pi(t,v,\omega,u,\vartheta),$$
$$\Pi(t,v,\omega,u,\vartheta) = \omega(t,1)p[t,\vartheta(t)]+\beta(u^2(t)+\vartheta^2(t)) + \int_0^1 g(t,x)\omega(t,x)f[t,u(t)]dx, \quad (26)$$

$\omega(t,x)$ is a solution of the conjugate boundary value problem

$$\omega_t+\omega_{xx}+\int_0^T K(\tau,t)\omega(\tau,x)d\tau = 0,\ 0<x<1,\ 0\le t<T,$$
$$\omega(T,x)+2[v(T,x)-\xi(x)]=0,\ 0<x<1,$$
$$\omega_x(t,0)=0,\ \omega_x(t,1)+\alpha\omega(t,1)=0,\ 0\le t<T$$

and has the form [15]

$$\omega(t,x) = -2[v_n(T)-\xi_n]\Big(e^{-\lambda_n^2(T-t)} + \lambda\int_0^T P_n(s,t,\lambda)e^{-\lambda_n^2(T-s)}ds\Big)z_n(x). \quad (27)$$

According to the maximum principle for systems with distributed parameters [6,10,11], the optimal control is determined by the relations

$$\begin{cases} \dfrac{2\beta u(t)}{f_u[t,u(t)]} = \displaystyle\int_0^1 g(t,x)\omega(t,x)dx, \\ \dfrac{2\beta\vartheta(t)}{p_\vartheta[t,\vartheta(t)]} = \omega(t,1), \end{cases} \quad (28)$$

$$f_u[t,u(t)]\left(\frac{u(t)}{f_u[t,u(t)]}\right)_u > 0,$$
$$p_\vartheta[t,\vartheta(t)]\left(\frac{\vartheta(t)}{p_\vartheta[t,\vartheta(t)]}\right)_\vartheta > 0 \quad (29)$$

which are called the optimality conditions. The relations (28) were obtained from the following condition

$$grad\,\Pi(\cdot,u,\vartheta)=0.$$

The relations (29) were obtained from the system of the conditions by elimination of $\omega(t,x)$ and $\omega(t,1)$

$$grad\,\Pi(\cdot,u,\vartheta)=0,$$
$$\Pi_{uu}(\cdot,u,\vartheta)<0,\ \begin{vmatrix}\Pi_{uu}(\cdot,u,\vartheta) & \Pi_{u\vartheta}(\cdot,u,\vartheta)\\ \Pi_{\vartheta u}(\cdot,u,\vartheta) & \Pi_{\vartheta\vartheta}(\cdot,u,\vartheta)\end{vmatrix}>0.$$

4 Nonlinear integral equation of optimal control

In order to find the optimal control, we use optimality conditions (28) and (29). We substitute $\omega(t,x)$ in (28) with the solution of the conjugate boundary value problem defined by (27). First, we calculate the integral

$$\int_0^1 g(t,x)\omega(t,x)dx = \int_0^1 \sum_{n=1}^\infty g_n(t)z_n(x)\sum_{k=1}^\infty \omega_k(t)z_k(x)dx = \sum_{n=1}^\infty g_n(t)\omega_n(t),$$

and rewrite equality (28) in the form

$$\beta u(t)f_u^{-1}[t,u(t)] = -\sum_{n=1}^\infty g_n(t)[v_n(T)-\xi_n] \times\Big(e^{-\lambda_n^2(T-t)}+\lambda\int_0^T P_n(s,t,\lambda)e^{-\lambda_n^2(T-s)}ds\Big),$$
$$\beta\vartheta(t)p_\vartheta^{-1}[t,\vartheta(t)] = -\sum_{n=1}^\infty z_n(1)[v_n(T)-\xi_n] \times\Big(e^{-\lambda_n^2(T-t)}+\lambda\int_0^T P_n(s,t,\lambda)e^{-\lambda_n^2(T-s)}ds\Big).$$

According to (12) we further reduce this equality to the form

$$\beta\begin{pmatrix}\frac{u(t)}{f_u[t,u(t)]}\\ \frac{\vartheta(t)}{p_\vartheta[t,\vartheta(t)]}\end{pmatrix}+\sum_{n=1}^{\infty}\begin{pmatrix}g_n(t)\\ z_n(1)\end{pmatrix}E_n(T,t,\lambda)\times\int_0^T L_n(T,\tau,\lambda)\big(g_n(\tau),z_n(1)\big)\begin{pmatrix}f[\tau,u(\tau)]\\ p[\tau,\vartheta(\tau)]\end{pmatrix}d\tau=\sum_{n=1}^{\infty}\begin{pmatrix}g_n(t)\\ z_n(1)\end{pmatrix}E_n(T,t,\lambda)h_n, \tag{30}$$

where

$$E_n(T,t,\lambda)=e^{-\lambda_n^2(T-t)}+\lambda\int_0^T R_n^*(s,t,\lambda)e^{-\lambda_n^2(T-s)}ds, \tag{31}$$

$$L_n(T,\tau,\lambda)=e^{-\lambda_n^2(T-\tau)}+\lambda\int_0^T R_n^*(T,s,\lambda)e^{-\lambda_n^2(s-\tau)}ds, \tag{32}$$

$$h_n=\xi_n-\psi_n[e^{-\lambda_n^2T}+\lambda\int_0^T R_n(T,s,\lambda)e^{-\lambda_n^2 s}ds]. \tag{33}$$

Thus, the optimal control is defined as the solution of nonlinear integral equation (30), and the condition (29), here, must be satisfied. Condition (29) restricts the class of functions of external actions $f[t,u(t)]$ and $p[t,\vartheta(t)]$. Therefore, we assume that the functions $f[t,u(t)]$ and $p[t,\vartheta(t)]$ satisfy the (29) for each of the controls $u(t)\in H(0,T)$ and $\vartheta(t)\in H(0,T)$.

Nonlinear integral control (30) is solved according to the method [4,5]. Suppose that

$$\frac{u(t)}{f_u[t,u(t)]}=\theta_1(t),\quad \frac{\vartheta(t)}{p_\vartheta[t,\vartheta(t)]}=\theta_2(t). \tag{34}$$

Lemma 1. *The vector function $\theta(t)=(\theta_1(t),\theta_2(t))$ is an element of space $H^2(0,T)=H(0,T)\times H(0,T)$.*

Proof. According to (5), we have the estimate

$$sup|f_u^{-1}[t,u(t)]|\le M_1,$$
$$sup|p_\vartheta^{-1}[t,\vartheta(t)]|\le M_2\quad \forall t\in[0,T].$$

Since $u(t)\in H(0,T)$ and $\vartheta(t)\in H(0,T)$, then the assertion of the lemma comes from the following inequality

$$\int_0^T\theta_1^2(t)dt\le\beta^2\int_0^T|f_u^{-1}[t,u(t)]|^2|u(t)|^2dt\le\beta^2M_1^2\int_0^T u^2(t)dt<\infty,$$
$$\int_0^T\theta_2^2(t)dt\le\beta^2\int_0^T|p_\vartheta^{-1}[t,\vartheta(t)]|^2|\vartheta(t)|^2dt\le\beta^2M_2^2\int_0^T\vartheta^2(t)dt<\infty.$$

According to (29), the optimal controls $u(t)$ and $\vartheta(t)$ are uniquely determined by equality (34), i.e. there are functions φ_1 and φ_2 such that

$$u(t)=\varphi_1(t,\theta_1(t),\beta),\quad \vartheta(t)=\varphi_2(t,\theta_2(t),\beta). \tag{35}$$

Using (34) and (35), we rewrite system of equations (30) in the form

$$\begin{pmatrix}\theta_1(t)\\ \theta_2(t)\end{pmatrix}+\sum_{n=1}^{\infty}\begin{pmatrix}g_n(t)\\ z_n(1)\end{pmatrix}E_n(T,t,\lambda)\int_0^T L_n(T,\tau,\lambda)\times\big(g_n(\tau),z_n(1)\big)\begin{pmatrix}f[\tau,\varphi_1(\tau,\theta_1(\tau),\beta)]\\ p[\tau,\varphi_2(\tau,\theta_2(\tau),\beta)]\end{pmatrix}d\tau=\sum_{n=1}^{\infty}\begin{pmatrix}g_n(t)\\ z_n(1)\end{pmatrix}E_n(T,t,\lambda)h_n. \tag{36}$$

Introducing the notations

$$\theta(t)=\begin{pmatrix}\theta_1(t)\\ \theta_2(t)\end{pmatrix},\quad G_n(t,1)=\begin{pmatrix}g_n(t)\\ z_n(1)\end{pmatrix},$$
$$F(\tau,u(\tau),\vartheta(\tau))=\begin{pmatrix}f[\tau,u(\tau)]\\ p[\tau,\vartheta(\tau)]\end{pmatrix},$$

we rewrite equation (30) in the form

$$\theta(t)+\sum_{n=1}^{\infty}G_n(t,1)E_n(T,t,\lambda)\int_0^T L_n(T,\tau,\lambda)\times G_n^*(\tau,1)F(\tau,\varphi_1[\tau,\theta_1(\tau),\beta],\varphi_2[\tau,\theta_2(\tau),\beta])d\tau=\sum_{n=1}^{\infty}G_n(t,1)E_n(T,t,\lambda)h_n \tag{37}$$

or in the operator form

$$\theta(t)=E[\theta_1(t),\theta_2(t)]+\bar{h}(t), \tag{38}$$

where

$$E[\theta_1(t),\theta_2(t)]=-\sum_{n=1}^{\infty}G_n(t,1)E_n(T,t,\lambda)\times\int_0^T L_n(T,\tau,\lambda)G_n^*(\tau,1)\times F(\tau,\varphi_1[\tau,\theta_1(\tau),\beta],\varphi_2[\tau,\theta_2(\tau),\beta])d\tau,$$
$$\bar{h}(t)=\sum_{n=1}^{\infty}G_n(t,1)E_n(T,t,\lambda)h_n. \tag{39}$$

Now, we investigate the question of unique solvability of the operator equation (38).

Lemma 2. *The function $\bar{h}(t)$ is an element of space $H^2(0,T)$.*

Proof. By the straightforward calculations, we obtain the inequality

$$\int_0^T \|\overline{h}(t)\|_{R^2}^2 dt = \int_0^T \left(h_1^2(t)+h_2^2(t)\right)dt$$
$$= \int_0^T \left\{ \left(\sum_{n=1}^{\infty} g_n(t)E_n(T,t,\lambda)h_n \right)^2 + \left(\sum_{n=1}^{\infty} z_n(1)E_n(T,t,\lambda)h_n \right)^2 \right\} dt$$
$$\le 2\|g(t,x)\|_H^2 \left(1+\frac{\lambda^2 K_0 T}{\left(\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0 T}\right)^2}\right) \times \left(\frac{1}{\lambda_2^1}+\frac{1}{6}\right) 2\left\{ \|\xi(x)\|_H^2 + 2\|\psi(x)\|_H^2 \times \left(1+\frac{\lambda^2 K_0 T}{\left(\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0 T}\right)^2}\frac{1}{2\lambda_1^2}\right)\right\}$$
$$+\left(1+\frac{\lambda^2 K_0 T}{\left(\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0 T}\right)^2}\right)\left(\frac{1}{\lambda_2^1}+\frac{1}{6}\right)2\left\{\|\xi(x)\|_H^2 +2\|\psi(x)\|_H^2 \left(1+\frac{\lambda^2 K_0 T}{\left(\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0 T}\right)^2}\frac{1}{2\lambda_1^2}\right)\right\}$$
$$\le \left(2\|g(t,x)\|_H^2+1\right)\left(1+\frac{\lambda^2 K_0 T}{\left(\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0 T}\right)^2}\right) \times \left(\frac{1}{\lambda_2^1}+\frac{1}{6}\right) 2\left\{ \|\xi(x)\|_H^2 + 2\|\psi(x)\|_H^2 \times \left(1+\frac{\lambda^2 K_0 T}{\left(\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0 T}\right)^2}\frac{1}{2\lambda_1^2}\right)\right\} < \infty, \tag{40}$$

from which the assertion of lemma is implied.

Lemma 3. *The operator $= E[\theta_1(t),\theta_2(t)]$ maps the space $H^2(0,T)$ into itself, i.e. is an element of the space $H^2(0,T)$.*

Proof. By the straightforward calculations, we obtain the inequality

$$\int_0^T E^2[\theta_1(t),\theta_2(t)]dt$$
$$= \int_0^T \left(\left\| \sum_{n=1}^{\infty} G_n(t,1)E_n(T,t,\lambda) \int_0^T L_n(T,\tau,\lambda)G_n^*(\tau,1) \times F(\tau,\varphi_1[\tau,\theta_1(\tau),\beta],\varphi_2[\tau,\theta_2(\tau),\beta])d\tau \right\| \right)^2 dt$$
$$\le \int_0^T \left(\sum_{n=1}^{\infty} \|G_n(t,1)\|_{R^2} |E_n(T,t,\lambda)| \times \int_0^T |L_n(T,\tau,\lambda)| \|G_n^*(\tau,1)\|_{R^2} \times \left\|F(\tau,\varphi_1[\tau,\theta_1(\tau),\beta],\varphi_2[\tau,\theta_2(\tau),\beta])\right\|_{H^2} d\tau\right)^2 dt$$
$$\le \int_0^T \sum_{n=1}^{\infty} \|G_n(t,1)\|_{R^2}^2 |E_n(T,t,\lambda)|^2 dt \times \sum_{n=1}^{\infty} \int_0^T |L_n(T,\tau,\lambda)|^2 \|G_n^*(\tau,1)\|_{R^2}^2 d\tau \times \int_0^T \left\|F(\tau,\varphi_1[\tau,\theta_1(\tau),\beta],\varphi_2[\tau,\theta_2(\tau),\beta])\right\|_{H^2}^2 d\tau$$
$$\le \left(\|g(t,x)\|_H^2+2\right)^2 \times \left(1+\frac{\lambda^2 K_0 T}{\left(\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0 T}\right)^2}\right)^2 \left(\frac{1}{\lambda_2^1}+\frac{1}{6}\right)^2 \times \left(\left\| f\left[t,\varphi_1[t,\theta_1(t),\beta]\right]\right\|_H^2 + \left\| p\left[t,\varphi_2[t,\theta_2(t),\beta]\right]\right\|_H^2 \right) < \infty, \tag{41}$$

from which the assertion of lemma is implied.

Lemma 4. *Suppose that the conditions*

$$\left\|f[t,u(t)]-f[t,\overline{u}(t)]\right\|_{H(0,T)} \le f_0\left\|u(t)-\overline{u}(t)\right\|_{H(0,T)}, \quad f_0>0, \tag{42}$$

$$\left\|p[t,\vartheta(t)]-p[t,\overline{\vartheta}(t)]\right\|_{H(0,T)} \le p_0\left\|\vartheta(t)-\overline{\vartheta}(t)\right\|_{H(0,T)}, \quad p_0>0, \tag{43}$$

$$\left\|\varphi_i[t,\theta_i(t),\beta]-\varphi_i[t,\overline{\theta}_i(t),\beta]\right\|_{H(0,T)} \le \varphi_{i0}\left\|\theta_i(t)-\overline{\theta}_i(t)\right\|_{H(0,T)}, \quad \varphi_{0i}(\beta)>0, \quad i=1,2 \tag{44}$$

are satisfied. When the condition

$$\gamma = \left(\|g(t,x)\|_H^2+2\right)\left(\frac{1}{\lambda_2^1}+\frac{1}{6}\right) \times \left(1+\frac{a_0^2 K_0}{\left(\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0 T}\right)^2}\right) \times B\left(f_0,p_0,\varphi_{10}(\beta),\varphi_{10}(\beta)\right) \tag{45}$$

is met, the operator $E[\theta]$ is contractive.

*Proof.*By the straightforward calculations, we obtain the inequality

$$\left\|E[\theta]-\overline{E}[\theta]\right\|_{H^2}^2=\int_0^T\left(\sum_{n=1}^{\infty}G_n(t,1)E_n(T,t,\lambda)\right.$$
$$\times\int_0^T L_n(T,\tau,\lambda)G_n^*(\tau,1)\begin{pmatrix} f\left[\tau,\varphi_1[\tau,\theta_1(\tau),\beta]\right] \\ p\left[\tau,\varphi_2[\tau,\theta_2(\tau),\beta]\right]\end{pmatrix}d\tau$$
$$-\sum_{n=1}^{\infty}G_n(t,1)E_n(T,t,\lambda)\int_0^T L_n(T,\tau,\lambda)G_n^*(\tau,1)$$
$$\left.\times\begin{pmatrix} f\left[\tau,\varphi_1[\tau,\overline{\theta}_1(\tau),\beta]\right] \\ p\left[\tau,\varphi_2[\tau,\overline{\theta}_2(\tau),\beta]\right]\end{pmatrix}d\tau\right)^2 dt$$
$$\leq\int_0^T\left(\sum_{n=1}^{\infty}G_n(t,1)E_n(T,t,\lambda)\int_0^T L_n(T,\tau,\lambda)G_n^*(\tau,1)\right.$$
$$\left.\times\begin{pmatrix} f\left[\tau,\varphi_1[\tau,\theta_1(\tau),\beta]\right]-f\left[\tau,\varphi_1[\tau,\overline{\theta}_1(\tau),\beta]\right] \\ p\left[\tau,\varphi_2[\tau,\theta_2(\tau),\beta]\right]-p\left[\tau,\varphi_2[\tau,\overline{\theta}_2(\tau),\beta]\right]\end{pmatrix}d\tau\right)^2 dt$$
$$\leq\left(\|g(t,x)\|_H^2+2\right)^2\left(\frac{1}{\lambda_2^1}+\frac{1}{6}\right)^2$$
$$\times\left(1+\frac{a_0^2K_0}{\left(\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0T}\right)^2}\right)^2$$
$$\times\left(\left\|f\left[t,\varphi_1[t,\theta_1(t),\beta]\right]-f\left[t,\varphi_1[t,\overline{\theta}_1(t),\beta]\right]\right\|_H^2\right.$$
$$\left.+\left\|p\left[t,\varphi_2[t,\theta_2(t),\beta]\right]-p\left[t,\varphi_2[t,\overline{\theta}_2(t),\beta]\right]\right\|_H^2\right)$$
$$\leq\left(\|g(t,x)\|_H^2+2\right)^2\left(\frac{1}{\lambda_2^1}+\frac{1}{6}\right)^2$$
$$\times\left(1+\frac{a_0^2K_0}{\left(\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0T}\right)^2}\right)^2$$
$$\times\left(f_0^2\varphi_{10}^2(\beta)\left\|\theta_1(t)-\overline{\theta_1}(t)\right\|_H^2\right.$$
$$\left.+p_0^2\varphi_{20}^2(\beta)\left\|\theta_2(t)-\overline{\theta_2}(t)\right\|_H^2\right)$$

$$\leq\left(\|g(t,x)\|_H^2+2\right)^2\left(\frac{1}{\lambda_2^1}+\frac{1}{6}\right)^2$$
$$\times\left(1+\frac{a_0^2K_0}{\left(\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0T}\right)^2}\right)^2$$
$$\left.\times B^2\left(f_0,p_0,\varphi_{10}(\beta),\varphi_{20}(\beta)\right)\left\|\theta(t)-\overline{\theta}(t)\right\|_H^2\right)$$
$$<\infty, \tag{46}$$

where

$B^2\left(f_0,p_0,\varphi_{10}(\beta),\varphi_{10}(\beta)\right)=max\left(f_0^2,p_0^2,\varphi_{10}^2(\beta),\varphi_{10}^2(\beta)\right),$ and from the inequality we find that

$$\left\|E[\theta]-\overline{E}[\theta]-\right\|_{H^2}\leq\left(\|g(t,x)\|_H^2+2\right)$$
$$\times\left(\frac{1}{\lambda_2^1}+\frac{1}{6}\right)\left(1+\frac{a_0^2K_0}{\left(\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0T}\right)^2}\right)$$
$$\times B\left(f_0,p_0,\varphi_{10}(\beta),\varphi_{20}(\beta)\right)\left\|\theta(t)-\overline{\theta}(t)\right\|_H$$
$$=\gamma\left\|\theta(t)-\overline{\theta}(t)\right\|_H<\infty.$$

Theorem 1.*Suppose that conditions (4) - (5), (29), (42) - (45) are satisfied. Then the operator equation (38) has a unique solution in the space $H^2(0,T)$.*

*Proof.*According to *Lemmas 1* and 3, operator equation (38) could be investigated in the space $H^2(0,T)$. According to *Lemma 4*, operator $E[\theta]$ is contractive. Since the Hilbert space $H^2(0,T)$ is a complete metric space, according to contraction mapping theorem [12], the operator $E[\theta]$ has a unique fixed point, i.e. operator equation (38) has a unique solution.

The solution of operator equation (38) can be found by the method of successive approximations, i.e. k^{th} approximation of the solution is found by the formula

$$\theta_k(t)=E[\theta_{k-1}(t)],\quad n\in\{1,2,3,...\},$$

where $\theta_0(t)$ is an arbitrary element of the space $H(0,T)$, and we obtain the estimate

$$\left\|\theta(t)-\theta_k(t)\right\|_{H^2(0,T)}$$
$$\leq\frac{\gamma^k}{1-\gamma}\left\|E[\theta_0(t)]+\overline{h}(t)-\theta_0(t)\right\|_{H^2(0,T)},$$

which, by the arbitrariness of the $\theta_0(t)$ when $\theta_0(t)=\overline{h}(t)$, has the form

$$\left\|\theta(t)-\theta_k(t)\right\|_{H^2(0,T)}\leq\frac{\gamma^k}{1-\gamma}\left\|E[\theta_0(t)]\right\|_{H^2(0,T)}.$$

The exact solution $\theta(t)$ could be found as the limit of the approximate solutions $\theta_k(t)$, i.e.,

$$\overline{\theta}(t)=\lim_{k\to\infty}\theta_k(t).$$

Substituting $\theta_1(t)$ and $\theta_2(t)$ in (35) with this solution, we find the required optimal controls

$$u^0(t)=\varphi_1(t,\overline{\theta}_1(t),\beta),$$
$$\vartheta^0(t)=\varphi_2(t,\overline{\theta}_2(t)\beta). \tag{47}$$

The optimal process $v^0(t,x)$, which is the solution of boundary value problem (1)-(3) that corresponds to the optimal controls $u^0(t)$ and $\vartheta^0(t)$, according to (6), (11)-(12) is found by the formula

$$v^0(t,x) = \sum_{n=1}^{\infty} \left(\lambda \int_0^T R_n(t,s,\lambda) a_n(s) ds + a_n(t) \right) z_n(x)$$
$$= \sum_{n=1}^{\infty} \left(\psi_n \left[e^{-\lambda_n^2 t} \lambda \int_0^T R_n(t,s,\lambda) e^{-\lambda_n^2 s} ds \right] \int_0^T A_n(t,\tau,\lambda) \left(g_n(\tau) f[\tau, u^0(\tau)] + z_n(1) p[\tau, \vartheta^0(\tau)] \right) d\tau \right) z_n(x), \quad (48)$$

where

$$A_n(t,\tau,\lambda) = \begin{cases} e^{-\lambda_n^2 (t-\tau)} + \lambda \int_\tau^T R_n(t,s,\lambda) e^{-\lambda_n^2 (s-\tau)} ds, \\ 0 \le \tau \le t, \\ \lambda \int_\tau^T R_n(t,s,\lambda) e^{-\lambda_n^2 (s-\tau)} ds, \\ t \le \tau \le T. \end{cases}$$

The minimum value of the functional (24) is calculated by the formula

$$J[u^0(t), \vartheta^0(t)] = \int_0^1 [v^0(T,x) - \xi(x)]^2 dx + \beta \int_0^T \left([u^0(t)]^2 + [\vartheta^0(t)]^2 \right) dt. \quad (49)$$

The obtained triple $\left((u^0(t), \vartheta^0(t)), v^0(t,x), J[u^0(t), \vartheta^0(t)] \right)$ is the solution of the nonlinear optimization problem.

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