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A Class of Optimal Eighth-Order Steffensen-Type Iterative Methods for Solving Nonlinear Equations and their Basins of Attraction

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Abstract: This article concerned with the issue of solving a nonlinear equation with the help of iterative method where no derivative evaluation is required per iteration. Therefore, this work contributes to a new class of optimal eighth-order Steffensen-type methods. Theoretical proof has been given to reveal the eighth-order convergence. Numerical comparisons have been carried out to show the effectiveness of contributed scheme. Furthermore, we compare the performance of our proposed method by the basins of attraction with some existing eighth-order Steffensen-type methods.

Keywords: Nonlinear equation, order of convergence, Steffensen-type method, efficiency index, basins of attraction.

1 Introduction

During the recent past, a wide collection of iterative methods has been presented in many journals, one can see, [5]-[10] and the references therein. In order to find the solution of a nonlinear equation Newton has provided the following iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

Steffensen [2] was the first who furnished the derivative-free form of Newton's scheme given by :

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \; w_n = x_n + f(x_n), \;\; n = 0, 1, 2... \tag{2}$$

both schemes possess the quadratic rate of convergence and same efficiency index 1.414. Although both the methods have the same order of convergence and efficiency index, but Steffensen method is derivative free. In order to increase the rate of convergence and efficiency index of iterative methods the number of function evaluations may increase. Kung and Traub [1] conjectured that a multipoint iteration without memory consuming n evaluation per full iteration can reach the maximum convergence rate 2^{n-1}. A large collection of research papers is available on the higher-order iterative methods agree with the Kung-Traub conjecture. In order to compare different iterative methods of same order the classical efficiency index of an iterative process in [3] given by $p^{\frac{1}{n}}$, where p is the rate of convergence and n is the total number of functional evaluations per iteration. More recently, many researchers have focused to make existing iterative methods free from derivatives, interested researcher can follow [12]-[18] . In many of the science and engineering problem, the evaluation of derivative is difficult and time consuming. Therefore, the Steffensen-type methods have become very popular in terms of solving nonlinear equations. This study is summarized as follows: Firstly, we provide a brief review of available literature to reveal the development of different derivative-free iterative methods. In the next section, we design a new optimal eight-order Steffensen-type iterative method for finding simple roots of nonlinear equations . In section 4, we employ some numerical examples to compare the performance of the new method with some existing eight-order derivative-free methods. Section 5, reveals the graphical comparison by basins of attraction. Finally, in the last section brief conclusion will be given.

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2 A brief review of the available literature

In this section, we give the overview of some recent derivative-free methods. Soleymani et al. [19] have improved the efficiency index of following method in terms of making it derivative-free

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, \ \ w_n = x_n + \beta f(x_n), \ \beta \in \mathbb{R} \setminus \{0\},\\
z_n &= y_n - \frac{f(y_n) f(w_n)}{(f(w_n) - f(y_n)) f[x_n, y_n]},\\
x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)},
\end{aligned}
$$

where $f[.,.]$ denotes the usual divided difference. This method has the eighth-order convergence and 1.516 as its efficiency index. To improve its efficiency index the authors have established two optimal three-step multipoint derivative-free methods given by

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, w_n = x_n + \beta f(x_n),\\
z_n &= y_n - \frac{f(y_n) f(w_n)}{(f(w_n) - f(y_n)) f[x_n, y_n]},\\
x_{n+1} &= z_n - \frac{f(z_n) f(w_n)}{(f(w_n) - f(y_n)) f[x_n, y_n]}\\
&\quad .\{G(\varphi) \times H(\tau) \times Q(\sigma) \times L(\rho)\},
\end{aligned}
$$

and

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, w_n = x_n - \beta f(x_n),\\
z_n &= y_n - \frac{f(y_n) f(w_n)}{(f(w_n) - f(y_n)) f[x_n, y_n]},\\
x_{n+1} &= z_n - \frac{f(z_n) f(w_n)}{(f(w_n) - f(y_n)) f[x_n, y_n]}\\
&\quad .\{G(\varphi) \times H(\tau) \times Q(\sigma) \times L(\rho)\},
\end{aligned}
$$

where $\beta \in \mathbb{R} \setminus \{0\}$, $\varphi = \frac{f(z_n)}{f(y_n)}$, $\tau = \frac{f(z_n)}{f(w_n)}$, $\sigma = \frac{f(z_n)}{f(x_n)}$, $\rho = \frac{f(y_n)}{f(w_n)}$. These methods have eighth-order convergence with efficiency index 1.682 under some conditions on the weight functions given in the same paper.
In [13], Soleymani has accelerated the efficiency index of the following eighth-order multipoint structure

$$
y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \ z_n = y_n - \frac{f(y_n)}{f'(y_n)}, \ x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}.
$$

He has approximated the derivatives by replacing $f'(y_n) \approx f[x_n, w_n]$, $f'(z_n) \approx f[x_n, w_n]$ and used the concept of weight functions to make it optimal as well as derivative-free. He proposed the following iterative formula

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, \ \ w_n = x_n + \beta f(x_n),\\
z_n &= y_n - \frac{f(y_n)}{f[x_n, w_n]}[G(A) \times H(B)],\\
x_{n+1} &= z_n - \frac{f(z_n)}{f[x_n, w_n]}[K(\Gamma) \times L(\Delta) \times P(E) \times Q(B) \times J(A)],
\end{aligned}
$$

wherein $\beta \in \mathbb{R} \setminus \{0\}$, $A = \frac{f(y_n)}{f(x_n)}$, $B = \frac{f(y_n)}{f(w_n)}$, $\Gamma = \frac{f(z_n)}{f(x_n)}$, $\Delta = \frac{f(z_n)}{f(w_n)}$, $E = \frac{f(z_n)}{f(y_n)}$. This method has the eighth-order convergence and efficiency index 1.682. Inspired from all these papers we also improve the order of convergence as well as the efficiency index of one existing seventh-order method in the next section.

3 Main method and convergence analysis

First we give some definitions which we will use later.

Definition 3.1 : Let $f(x)$ be a real valued function with a simple root α and let x_n be a sequence of real numbers that converge towards α. The order of convergence m is given by

$$
\lim_{n \to \infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^m} = \zeta \neq 0, \tag{3}
$$

where ζ is the asymptotic error constant and $m \in R^+$.
Definition 3.2 : Let n be the number of function evaluations of the new method. The efficiency of the new method is measured by the concept of efficiency index [[3], [4]] and defined as

$$
m^{1/n}, \tag{4}
$$

where m is the order of convergence of the new method.
Consider the following seventh-order method established by soleymani et al. [5] to build a new eighth-order method:

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)},\\
z_n &= y_n - \frac{f(y_n)}{f[x_n, y_n]}.G(t_n),\\
x_{n+1} &= z_n - \frac{f(z_n)}{f[y_n, z_n]}.H(t_n),
\end{aligned} \tag{5}
$$

where $t_n = \frac{f(y_n)}{f(x_n)}$ and $G(0) = G'(0) = 1$, $|G''(0)| < +\infty$; $H(0) = 1, H'(0) = 0, H''(0) = 2$, $\left|H^{(3)}(0)\right| < +\infty$.
Now our aim is to develop derivative-free version of the method (5). For this we approx $f'(x_n) \approx f[z_n, x_n]$, where $z_n{=}x_n + f(x_n)$ in (5). Here we see that the method (5) under this approximation of $f'(x_n)$ has fifth-order convergence and its error expression is given by

$$
e_{n+1} = \frac{\{(1+c_1)^2 c_2^4\} e_n^5}{c_1^2} + O(e_n^6),
$$

where $c_i = \frac{f^{(i)}(\alpha)}{i!}$, $i = 1, 2, 3\ldots$.
Now to improve its order of convergence without using any new evaluation, we approx $f'(x_n) \approx f[z_n, x_n]$, where $z_n = x_n + f(x_n)^2$, then its error expression becomes

$$
\begin{aligned}
e_{n+1} = &\frac{c_2^2 \left(2c_1^3 c_2 + 2c_1 c_3 + c_2^2(-6 + G''(0))\right)\left(A\right) e_n^7}{12 c_1^6}\\
&+ O(e_n^8),
\end{aligned}
$$

Table 1: Nonlinear functions and their roots.

Nonlinear function $(f(x))$	*Root* (α)
$f_1(x) = 10\,x\,e^{-x^2} - 1$	$\alpha_1 = 1.67963...$
$f_2(x) = x^2 e^x - sin(x)$	$\alpha_2 = 0$
$f_3(x) = sin(3x) + x\,cos(x)$	$\alpha_3 = 1.19776...$
$f_4(x) = log(x) - x^3 + 2sin(x)$	$\alpha_4 = 1.29799...$
$f_5(x) = cos(x) + sin(2x)\sqrt{1-x^2} + sin(x^2) + x^{14} + x^3 + \frac{1}{2x}$	$\alpha_5 = -0.92577...$
$f_6(x) = e^{-x} + sin(x) - 1$	$\alpha_6 = 2.07683...$
$f_7(x) = (1+x^3)cos(\frac{\pi}{2}) + \sqrt{1-x^2} - \frac{2(9\sqrt{2}+7\sqrt{3})}{27}$	$\alpha_7 = 0.33333...$

where

$A = 6c_1^3 c_2 + 12c_1c_3 + c_2^2(-24 + H^{(3)}(0))$.

In fact, if we approx $f'(x_n) \approx f[z_n, x_n]$, where $z_n = x_n + f(x_n)^n$, $n \geq 2$, then its order of convergence is seven. Clearly, here we use four function evaluations. So, according to Kung-Traub conjecture its maximum (optimal) possible order should be eight. To do this, we consider the following iterative formula

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f[z_n, x_n]}, \\ w_n &= y_n - G(t_1).\frac{f(y_n)}{f[x_n, y_n]}, \\ x_{n+1} &= w_n - H(t_2).\frac{f(w_n)}{f[w_n, y_n]}, \end{aligned} \qquad (6)$$

where $t_1 = \frac{f(y_n)}{f(x_n)}$, $t_2 = \frac{f[w_n, y_n]}{f[w_n, x_n]}$ and $z_n = x_n + f(x_n)^3$. The following theorem shows that the conditions on weight functions under which proposed scheme has eighth-order convergence.

Theorem 3.1. Let us consider $\alpha \in D$ be a simple root of a sufficiently differentiable function $f : D \subseteq \mathbb{R} \to \mathbb{R}$. If x_0 is sufficiently close to the root α. Then the method (6) has eighth-order convergence, when the weight functions $G(t_1)$, $H(t_2)$ satisfy the following conditions:

$$\begin{aligned} &G(0) = 1,\ G'(0) = 1,\ \left|G^{(3)}(0)\right| < +\infty, \\ &H(1) = 1,\ H'(1) = 0,\ H''(1) = 2\ ,\ H^{(3)}(1) = -12, \\ &\left|H^{(4)}(1)\right| < +\infty. \end{aligned} \qquad (7)$$

Proof. With help of Taylor series and symbolic computation we find the error expression of method (6). Furthermore, by Taylor expansion around the simple root α in the n^{th} iteration and by considering $e_n = x_n - \alpha$, $f(\alpha) = 0$, we obtain

$$f(x_n) = c_1e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + ... + O(e_n^{10}), \qquad (8)$$

and

$$\begin{aligned} z_n = &\ \alpha + e_n + c_1^3e_n^3 + 3c_1^2c_2e_n^4 + 3c_1(c_2^2 + c_1c_3)e_n^5 \\ &+ ... + O(e_n^9). \end{aligned} \qquad (9)$$

Subsequently, we obtain

$$\begin{aligned} f(z_n) &= c_1e_n + c_2e_n^2 + (c_1^4 + c_3)e_n^3 \\ &+ (5c_1^3c_2 + c_4)e_n^4 + ... + O(e_n^9). \end{aligned} \qquad (10)$$

With the help of equations (8)-(10), we obtain the Taylor's series expansion of $f[z_n, x_n] = \frac{f(z_n) - f(x_n)}{z_n - x_n}$ as follows:

$$\begin{aligned} f[z_n, x_n] &= c_1 + 2c_2e_n + 3c_3e_n^2 + (c_1^3c_2 + 4c_4)e_n^3 \\ &+ (3c_1^2(c_2^2 + c_1c_3) + 5c_5)e_n^4 \\ &+ ... + O(e_n^9). \end{aligned} \qquad (11)$$

By putting the values of equations (8) and (11) in the first step of equation (6), we attain

$$\begin{aligned} y_n &= \alpha + \frac{c_2}{c_1}e_n^2 - \frac{2(c_2^2 - c_1c_3)}{c_1^2}e_n^3 + \\ &\left(\frac{c_1^5c_2 + 4c_2^3 - 7c_1c_2c_3}{c_1^3} + \frac{3c_4}{c_1}\right)e_n^4 + ... + O(e_n^9). \end{aligned} \qquad (12)$$

On the other hand, we find

$$\begin{aligned} f(y_n) &= c_2e_n^2 - 2\left(\frac{c_2^2}{c_1} - c_3\right)e_n^3 \\ &+ \left(\frac{c_2(c_1^5 + 5c_2^2 - 7c_1c_3)}{c_1^2} + 3c_4\right)e_n^4 + ... + O(e_n^9). \end{aligned} \qquad (13)$$

Furthermore, we obtain

$$\begin{aligned} &f[x_n, y_n] \\ &= c_1 + c_2e_n + \left(\frac{c_2^2}{c_1} + c_3\right)e_n^2 \\ &+ \left(\frac{-2c_2^3 + 3c_2c_1c_3}{c_1^2} + c_4\right)e_n^3 \\ &+ \left(\frac{c_1^5c_2^2 + 4c_2^4 - 8c_1c_2^2c_3 + 2c_1^2(c_3^2 + 2c_2c_4)}{c_1^3} + c_5\right)e_n^4 \\ &+ ... + O(e_n^9), \end{aligned} \qquad (14)$$

and

$$\begin{aligned} \frac{f(y_n)}{f[x_n, y_n]} &= \frac{c_2}{c_1}e_n^2 + \left(\frac{-3c_2^2 + 2c_1c_3}{c_1^2}\right)e_n^3 \\ &+ \left(\frac{c_2(c_1^5 + 7c_2^2 - 10c_1c_3)}{c_1^3} + \frac{3c_4}{c_1}\right)e_n^4 \\ &+ ... + O(e_n^9). \end{aligned} \qquad (15)$$

Table 2: Numerical comparison of different derivative-free methods.

	$DF_{8,1}$	$DF_{8,2}$	$DF_{8,3}$	$DF_{8,4}$	$DF_{8,5}$	$DF_{8,6}$	$OM8$
$x_0 = 1.72$							
IT	2	2	2	2	2	2	2
TNE	8	8	8	8	8	8	8
$\|f_1\|$	0.7e-53	0.3e-53	0.5e-58	0.8e-63	0.5e-61	0.5e-61	0.4e-80
$x_0 = 1.5$							
IT	3	3	3	3	3	3	3
TNE	12	12	12	12	12	12	12
$\|f_1\|$	0.4e-252	0.1e-243	0.2e-238	0.3e-269	0.3e-329	0.3e-329	0.6e-315
$x_0 = 1.7$							
IT	2	2	2	2	2	2	2
TNE	8	8	8	8	8	8	8
$\|f_1\|$	0.8e-79	0.5e-79	0.9e-78	0.3e-82	0.9e-82	0.9e-82	0.1e-100
$x_0 = 0.1$							
IT	3	3	3	3	3	3	2
TNE	12	12	12	12	12	12	8
$\|f_2\|$	0.2e-107	0.1e-114	0.1e-214	0.1e-238	0.1e-231	0.2e-231	0.1e-52
$x_0 = -0.1$							
IT	3	3	3	3	2	2	2
TNE	12	12	12	12	8	8	8
$\|f_2\|$	0.4e-369	0.1e-363	0.2e-362	0.2e-384	0.1e-52	0.1e-52	0.1e-74
$x_0 = -0.5$							
IT	3	3	3	3	3	3	3
TNE	12	12	12	12	12	12	12
$\|f_2\|$	0.1e-178	0.1e-167	0.5e-162	0.8e-189	0.2e-182	0.2e-182	0.4e-259
$x_0 = 1.0$							
IT	-	-	-	-	-	-	2
TNE	-	-	-	-	-	-	8
$\|f_3\|$	DIV.	NC	NC	DIV.	NC	DIV.	0.1e-58
$x_0 = 0.8$							
IT	4	4	-	-	4	4	3
TNE	16	16	-	-	16	16	12
$\|f_3\|$	0.2e-139	0.7e-194	NC	NC	0.3e-310	0.4e-315	0.1e-64
$x_0 = 1.8$							
IT	-	-	3	3	3	3	3
TNE	-	-	12	12	12	12	12
$\|f_3\|$	DIV.	DIV.	0.5e-85	0.2e-83	0.1e-75	0.3e-75	0.1e-107
$x_0 = 1.4$							
IT	3	3	3	3	3	3	3
TNE	12	12	12	12	12	12	12
$\|f_4\|$	0.4e-214	0.9e-211	0.8e-237	0.1e-253	0.4e-334	0.4e-334	0.4e-333
$x_0 = 1.15$							
IT	-	-	-	-	-	-	3
TNE	-	-	-	-	-	-	12
$\|f_4\|$	I	I	NC	NC	NC	NC	0.2e-284
$x_0 = 1.3$							
IT	2	2	2	2	2	2	2
TNE	8	8	8	8	8	8	8
$\|f_4\|$	0.3e-153	0.3e-153	0.1e-129	0.5e-133	0.2e-136	0.2e-136	0.1e-161
$x_0 = -0.92$							
IT	3	3	2	2	2	2	2
TNE	12	12	8	8	8	8	8
$\|f_5\|$	0.4e-295	0.2e-301	0.4e-57	0.2e-74	0.6e-59	0.6e-59	0.8e-97
$x_0 = -0.93$							
IT	2	2	2	2	2	2	2
TNE	8	8	8	8	8	8	8
$\|f_5\|$	0.3e-61	0.6e-61	0.1e-65	0.5e-70	0.1e-70	0.1e-70	0.4e-98
$x_0 = -0.9$							
IT	-	-	3	3	3	3	3
TNE	-	-	12	12	12	12	12
$\|f_5\|$	I	I	0.1e-108	0.2e-123	0.8e-104	0.1e-103	0.2e-361
$x_0 = 1.9$							
IT	3	3	3	3	3	3	2
TNE	12	12	12	12	12	12	8
$\|f_6\|$	0.4e-231	0.6e-236	0.5e-298	0.2e-328	0.1e-301	0.4e-334	0.3e-59
$x_0 = 2.3$							
IT	3	3	3	3	2	3	2
TNE	12	12	12	12	8	12	8
$\|f_6\|$	0.2e-354	0.2e-351	0.3e-336	0.1e-357	0.6e-54	0.2e-301	0.4e-60
$x_0 = 1.8$							
IT	4	4	3	3	3	3	3
TNE	16	16	12	12	12	12	12
$\|f_6\|$	0.1e-200	0.1e-244	0.8e-172	0.6e-192	0.9e-153	0.2e-152	0.7e-352
$x_0 = 0.8$							
IT	3	3	3	3	3	3	3
TNE	12	12	12	12	12	12	12
$\|f_7\|$	0.2e-57	0.1e-55	0.2e-72	0.2e-82	0.3e-98	0.7e-202	0.8e-219
$x_0 = 0.6$							
IT	3	3	3	3	3	3	3
TNE	12	12	12	12	12	12	12
$\|f_7\|$	0.1e-161	0.2e-162	0.1e-159	0.1e-172	0.6e-202	0.2e-301	0.2e-378
$x_0 = 0.4$							
IT	2	2	2	2	2	2	2
TNE	8	8	8	8	8	8	8
$\|f_7\|$	0.5e-60	0.6e-60	0.1e-52	0.7e-55	0.2e-62	0.2e-62	0.1e-70

Here : x_0 = Initial guess, TNE = Total number of evaluations and IT = Number of iterations.

By using the equations (15), (13) and (8) in the second step of equation (6), we attain

$$w_n = \alpha + \Big(\frac{c_2 - G(0)c_2}{c_1}\Big)e_n^2 + \frac{1}{c_1^2}\Big(-2c_2^2 + 3G(0)c_2^2 + 2c_1c_3 - 2G(0)c_1c_3 - c_2^2G'(0)\Big)e_n^3 + \ldots + O(e_n^9). \tag{16}$$

By virtue of the above equation, and considering $G(0) = 1$, $G'(0) = 1$, we acquire

$$f(w_n) = \frac{1}{2c_1^2}\Big(6c_2^3 - 2c_1c_2c_3 - c_2^3G''(0)\Big)e_n^4 + \frac{1}{6c_1^3}\Big(-6c_1^5c_2^2 - 108c_2^4 + 120c_1c_2^2c_3 - 12c_1^2c_3^2 - 12c_1^2c_2c_4 + 27c_2^4G''(0) - 18c_1c_2^2c_3G''(0) - c_2^4G^{(3)}(0)\Big)e_n^5 + \ldots + O(e_n^9). \tag{17}$$

With help of equations (13), (16), (17) and (8), we have

$$f[w_n, y_n] = c_1 + \frac{c_2^2}{c_1}e_n^2 + \frac{2c_2(-c_2^2 + c_1c_3)}{c_1^2}e_n^3 + \frac{1}{2c_1^3}\Big(c_2(2c_1^5c_2 - 14c_1c_2c_3 + 6c_1^2c_4 - c_2^3(-14 + G''(0))\Big)e_n^4 + \ldots + O(e_n^9), \tag{18}$$

and

$$f[w_n, x_n] = c_1 + c_2e_n + c_3e_n^2 + c_4e_n^3 - \frac{1}{2c_1^3}\Big(\{2c_1c_2^2c_3 - 2c_1^3c_5 + c_2^4(-6 + G''(0))\}\Big)e_n^4 + \ldots + O(e_n^9). \tag{19}$$

Now, putting the values of equations (18), (19) and (17), in the last step of equation (6), we find

$$e_{n+1} = \Big(\frac{(-1 + H(1))c_2(2c_1c_3 + c_2^2(-6 + G''(0)))}{2c_1^3}\Big)e_n^4 + \frac{1}{6c_1^4}\Big[6(-1 + H(1))c_1^5c_2^2 + 12(-1 + H(1)) c_1^2(c_3^2 + c_2c_4) + 6c_1c_2^2c_3\Big(20 - 20H(1) - H'(1) + 3(-1 + H(1))G''(0)\Big) + c_2^4\Big(18(-6 + 6H(1) + H'(1)) - 3(-9 + 9H(1) + H'(1))G''(0) + (-1 + H(1))G^{(3)}(0)\Big)\Big]e_n^5 + \ldots + O(e_n^9).$$

By putting $H(1) = 1$, $H'(1) = 0$, $H''(1) = 2$, $H^{(3)}(1) = -12$, in the above equation the final error expression is given by

$$e_{n+1} = \frac{1}{48c_1^7}\Big(c_2(2c_1c_3 + c_2^2(-6 + G''(0)))(-24c_1^5c_2^2 + 24c_1^2(c_3^2 - c_2c_4) + c_2^4(-96 + H^{(4)}(1)))\Big)e_n^8 + O(e_n^9). \tag{20}$$

Particular Case:

Let $G(t_1) = \frac{1-2t_1}{1-3t_1}$ and $H(t_2) = 4 - 8t_2 + 7t_2^2 - 2t_2^3$, then the method (6) becomes

$$y_n = x_n - \frac{f(x_n)}{f[z_n, x_n]}, \quad z_n = x_n + f(x_n)^3$$
$$w_n = y_n - \left(\frac{1-2t_1}{1-3t_1}\right) \frac{f(y_n)}{f[x_n, y_n]}$$
$$x_{n+1} = w_n - (4 - 8t_2 + 7t_2^2 - 2t_2^3). \frac{f(w_n)}{f[w_n, y_n]}, \qquad (21)$$

where $t_1 = \frac{f(y_n)}{f(x_n)}$ and $t_2 = \frac{f[w_n, y_n]}{f[w_n, x_n]}$, then its error expression becomes

$$e_{n+1} = -\frac{c_2c_3(c_1^5c_2^2 + 4c_2^4 + c_1^2(-c_3^2 + c_2c_4))}{c_1^6} e_n^8 + O(e_n^9). \qquad (22)$$

Remark 1: By taking different appropriate values of $G(t_1)$ and $H(t_2)$ one may get a number of eight-order derivative-free iterative methods for finding the simple roots.

Remark 2: In order of removing derivatives from iterative methods the number of function evaluation usually increases. But in new scheme we increase the efficiency index without increasing more function evaluations.

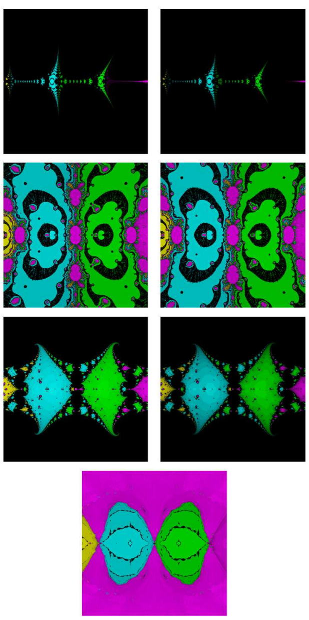

Fig. 1: *Basins of attraction for Example* 5.1. First row: methods ($DF_{8,1}$) (left) and ($DF_{8,2}$) (right). Second row: methods ($DF_{8,3}$) (left) and ($DF_{8,4}$) (right). Third row: methods ($DF_{8,5}$) (left) and ($DF_{8,6}$) (right). Last row: method ($OM8$).

4 Numerical results

The prime objective of this section is to demonstrate the performance of the new eighth-order derivative-free method. In order to verify the effectiveness of the proposed iterative method we have considered seven nonlinear test functions. The test non-linear functions and their roots are listed in Table-1. The entire computations reported here have been performed on the programming package *MATHEMATICA* [8] using 1000 digit floating point arithmetic using "*SetAccuraccy*" command. It can be observed from Table 2 that almost in most cases, our proposed derivative-free scheme is superior than other methods. In Table 2 DIV. stands for divergent, NC and I stands for not convergent and indeterminate, respectively. For comparing the number of iterations and total number of function evaluations, we have used the following stopping criterion $|f(x_n)| < 1.E-50$. We have taken three different initial guesses for comparing the convergence rate of each scheme. Here we compare the performance of the proposed methods (21) ($OM8$) with the methods (2.13) ($DF_{8,3}$), (2.15) ($DF_{8,4}$) of [19]; (4.17) ($DF_{8,1}$), (4.19) ($DF_{8,2}$) of [13] and (33) ($DF_{8,5}$) and (35) ($DF_{8,6}$) of [20]. The results of comparison of the test functions are summarized in Table 2. From Table 2 we observe that the new scheme is superior than some existing methods.

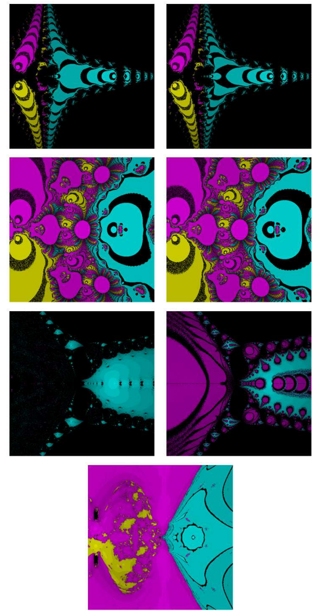

Fig. 2: *Basins of attraction for Example* 5.2. First row: methods ($DF_{8,1}$) (left) and ($DF_{8,2}$) (right). Second row: methods ($DF_{8,3}$) (left) and ($DF_{8,4}$) (right). Third row: methods ($DF_{8,5}$) (left) and ($DF_{8,6}$) (right). Last row: method ($OM8$).

5 Basins of attraction

In this section, we are describing the fractal behavior of eighth-order derivative-free methods used in numerical section. For the dynamical comparisons, we consider a complex rectangle $D = [-3,3] \times [-3,3] \in \mathbb{C}$ and then we assign a color to each complex point $z_0 \in D$ according to the root at which the corresponding method starting from z_0 converges. Here, black color represents the points at which the method do not converge. We have used stopping criteria $|f| < 10^{-4}$ and maximum number of iterations 100 for each method. The following test polynomials have been considered for comparison

Example 5.1 : $z^4 - 10z^2 + 9$. This polynomial, has roots 3, -3, -1 and 1. According to figure 1, we found that $OM8$ is best. Methods $DF_{8,1}$, $DF_{8,2}$, $DF_{8,5}$ and $DF_{8,6}$ have a large number of diverging points. Methods $DF_{8,3}$ and $DF_{8,4}$ show chaotic behavior.

Example 5.2 : $z^3 + 4z^2 - 10$. This polynomial, has roots $-2.68262 + 0.358259I$, $-2.68262 - 0.358259I$ and 1.36523. From figure 2, we observe that again the performance of $OM8$ is best and other methods do not perform well. Based on figures 1 and 2 we conclude that methods $DF_{8,1}$, $DF_{8,2}$, $DF_{8,3}$, $DF_{8,4}$, $DF_{8,5}$ and $DF_{8,6}$ have more diverging points (black area) in comparison with $OM8$. Finally, our method does not show any chaotic behavior and has large basins of attraction. These figures confirm the numerical results illustrated in Table 2.

6 Concluding remarks

In the present study, we have contributed an efficient eighth-order Steffensen-type method. We have also described the dynamical behavior of some eight-order derivative-free methods. New scheme requires four function evaluations per iteration, so its efficiency index is $8^{\frac{1}{4}} \approx 1.682$. Some numerical examples have been carried out to confirm the underlying theory of this study. From numerical and graphical comparisons one can observe that our contributed scheme is superior than some existing methods.

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