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Determination of the Unknown Source Function in Time Fractional Parabolic Equation with Dirichlet Boundary Conditions

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Abstract: This article deals with the mathematical analysis of the inverse problem of identifying the distinguishability of input-output mappings in the linear time fractional inhomogeneous parabolic equation $D_t^{\alpha}u(x,t) = (k(x)u_x)_x + r(t)F(x,t) \quad 0 < \alpha \leq 1$, with Dirichlet boundary conditions $u(0,t) = \psi_0(t)$, $u(1,t) = \psi_1(t)$. By defining the input-output mappings $\Phi[\cdot] : \mathscr{K} \to C^1[0,T]$ and $\Psi[\cdot] : \mathscr{K} \to C^1[0,T]$ the inverse problem is reduced to the problem of their invertibility. Hence, the main purpose of this study is to investigate the distinguishability of the input-output mappings $\Phi[\cdot]$ and $\Psi[\cdot]$. Moreover, the measured output data $f(t)$ and $h(t)$ can be determined analytically by a series representation, which implies that the input-output mappings $\Phi[\cdot] : \mathscr{K} \to C^1[0,T]$ and $\Psi[\cdot] : \mathscr{K} \to C^1[0,T]$ can be described explicitly.

Keywords: Distinguishability, fractional parabolic equation,source function

1 Introduction

The inverse problem of unknown source function in a linear inhomogeneous parabolic equation by using over measured data has generated an increasing amount of interest from engineers and scientist during the last few decades. This kind of problems play a crucial role in engineering, physics and applied mathematics. The problem of recovering unknown source function in the mathematical model of a physical phenomena is frequently encountered. Intensive study has been carried out on this kind of problem, and various inverse problems and many numerical methods developed [1–9, 11–17]. Fractional differential equations are generalizations of ordinary and partial differential equations to an arbitrary fractional order. By linear time-fractional parabolic equation, we mean certain parabolic-like partial differential equation governed by master equations containing fractional derivatives in time. The research areas of fractional differential equations range from the theoretical to the applied aspects.

The main goal of this study is to investigate the distinguishability of the unknown source function via input-output mappings in a one dimensional time fractional inhomogeneous parabolic equation. We first obtain the unique solution of this problem using Fourier method of separation of variables with respect to the eigenfunctions of a corresponding Sturm-Liouville eigenvalue problem under certain conditions [10]. As the next step, the noisy free measured output data $f(t)$ and $h(t)$ are used to introduce the input-output mappings $\Phi[\cdot] : \mathscr{K} \to C^1[0,T]$ and $\Psi[\cdot] : \mathscr{K} \to C^1[0,T]$ where $\mathscr{K}$ represents the set of admissible source functions. Finally we investigate the distinguishability of the unknown source function $r(t)$ via the above input-output mappings $\Phi[\cdot]$ and $\Psi[\cdot]$.

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Consider now the following initial boundary value problem:

$$\begin{cases} D_t^{\alpha}u(x,t) = (k(x)u_x)_x + r(t)F(x,t), \quad 0<\alpha\leq 1, \\ \qquad (x,t)\in\Omega_T, \\ u(x,0) = g(x), \quad 0<x<1, \\ u(0,t) = \psi_0(t),\ u(1,t) = \psi_1(t), \quad 0<t<T, \end{cases} \tag{1}$$

where $\Omega_T = \{(x,t)\in R^2 :\ 0<x<1,\ 0<t\leq T\}$ and the fractional derivative $D_t^{\alpha}u(x,t)$ is defined in the Caputo sense $D_t^{\alpha}u(x,t) = (I^{1-\alpha}u')(t), \quad 0<\alpha\leq 1$, I^{α} being the Riemann-Liouville fractional integral

$$(I^{\alpha}f)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau, & 0<\alpha\leq 1 \\ f(t), & \alpha = 0. \end{cases}$$

The left and right boundary value functions $\psi_0(t)$ and $\psi_1(t)$ belong to $C[0,T]$. The functions $0<c_0\leq k(x)<c_1$ and $g(x)$ satisfy the following conditions:
(C1) $k(x)\in C^1[0,1]$
(C2) $g(x)\in C^2[0,1]$, $g(0)=\psi_0(0),\ \ g(1)=\psi_1(0)$.
Under these conditions, the initial boundary value problem (1) has the unique solution $u(x,t)$ defined in the domain $\overline{\Omega}_T = \{(x,t)\in R^2 :\ 0\leq x\leq 1,\ 0\leq t\leq T\}$ which belongs to the space $C(\overline{\Omega}_T)\cap W_t^1(0,T]\cap C_x^2(0,1)$. Moreover, it satisfies the equation, initial and boundary conditions. The space $W_t^1(0,T]$ contains the functions $f\in C^1(0,T]$ such that $f'(x)\in L(0,T)$.
This kind of problems play a crucial role in engineering, physics and applied mathematics since it is used successfully to model complex phenomenian various fields such as fluid mechanics, viscoelasticity, physics, chemistry and engineering.

Consider the inverse problem of determining the distinguishability of the unknown function $r(t)$ from the Dirichlet type of measured output data at the boundaries $x=0$ and $x=1$:

$$\Phi[r] = k(x)u_x(x,t;r)|_{x=0}, \quad r\in\mathscr{K}\subseteq C^1(\Omega_T)$$

$$\Psi[r] = k(x)u_x(x,t;r)|_{x=1}, \quad r\in\mathscr{K}\subseteq C^1(\Omega_T).$$

Then, the inverse problem with the measured output data $f(t)$ and $h(t)$ can be formulated as the following operator equations:

$$\Phi[r] = f, \quad f\in C^1(0,T]$$

$$\Psi[r] = h, \quad h\in C^1(0,T].$$

These formulations reduce the inverse problem of determining unknown function $r(t)$ to the problem of invertibility of the input-output mappings $\Phi[\cdot]$ and $\Psi[\cdot]$. Hence this leads us to study the distinguishability of the source function via the above input-output mappings. We say that the mappings $\Phi[\cdot] : \mathscr{K}\to C^1[0,T]$ and $\Psi[\cdot] : \mathscr{K}\to C^1[0,T]$ have the distinguishability property if $\Phi[r_1]\neq\Phi[r_2]$ implies $r_1(t)\neq r_2(t)$ and $\Psi[r_1]\neq\Psi[r_2]$ implies $r_1(t)\neq r_2(t)$. This, in particular, means injectivity of the inverse mappings Φ^{-1} and Ψ^{-1}. In this paper, measured output data of Neumann type at the boundaries $x=0$ and $x=1$ are used respectively in the determination of the distinguishability of the unknown function $r(t)$. In addition, in the determination of this analytical results for input-output data are obtained.

The paper is organized as follows. In section 2, an analysis of the inverse problem with the single measured output data $f(t)$ at the boundary $x=0$ is given. An analysis of the inverse problem with the single measured output data $h(t)$ at the boundary $x=1$ is considered in section 3. Numerical procedure is given in section 4. Finally, some concluding remarks are given in the last section.

2 An analysis of the inverse problem with given measured data $f(t)$

Consider now the inverse problem with one measured output data $f(t)$ at $x=0$. In order to formulate the solution of the parabolic problem (1) by using Fourier method of the separation of variables, let us first introduce an auxiliary function $v(x,t)$ as follows:

$$v(x,t) = u(x,t) - (1-x)\psi_0(t) - \psi_1(t)x,\ \ x\in[0,1],$$

by which we transform the problem (1) into a problem with homogeneous boundary conditions. Hence the initial boundary value problem (1) can be rewritten in terms of $v(x,t)$ in the following form:

$$\begin{cases} D_t^{\alpha}v(x,t) - v_{xx}(x,t) = ((k(x)-1)v_x(x,t))_x - xD_t^{\alpha}\psi_1(t) \\ \qquad - D_t^{\alpha}\psi_0(t) + xD_t^{\alpha}\psi_0(t) - k'(x)\psi_0(t) \\ \qquad + k'(x)\psi_1(t) + r(t)F(x,t), \\ v(x,0) = g(x) - (1-x)\psi_0(0) - \psi_1(0)x, \\ \qquad 0<x<1, \\ v(0,t) = 0,\ v(1,t) = 0, \quad 0<t<T. \end{cases} \tag{2}$$

The unique solution of the initial-boundary value problem can be represented in the following form [10]:

$$v(x,t) = \sum_{n=1}^{\infty} <\zeta(\theta),\phi_n(\theta)> E_{\alpha,1}(-\lambda_n t^{\alpha})\phi_n(x)$$

$$+\sum_{n=1}^{\infty}\left(\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^{\alpha})(<\xi(\theta,t-s),\phi_n(\theta)> \right.$$

$$\left. + <r(t-s)F(\theta,t-s),\phi_n(\theta)>)ds\right)\phi_n(x)$$

where

$$\zeta(x) = g(x) - (1-x)\psi_0(0) - \psi_1(0)x,$$
$$\xi(x,t) = ((k(x)-1)v_x(x,t))_x - xD_t^\alpha\psi_1(t) - D_t^\alpha\psi_0(t) + xD_t^\alpha\psi_0(t) - k'(x)\psi_0(t) + k'(x)\psi_1(t),$$

Moreover $<\zeta(\theta),\phi_n(\theta)>=\int_0^1\phi_n(\theta)\zeta(\theta)d\theta$, $E_{\alpha,\beta}$ being the generalized Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\beta n+\alpha)}.$$

Assume that $\phi_n(x)$ is the solution of the following Sturm-Liouville problem:

$$\begin{cases} -\phi_{xx}(x) = \lambda\phi(x), & 0<x<1, \\ \phi(0)=0, \ \phi(1)=0, & 0<t<T, \end{cases}$$

The Neumann type of measured output data at the boundary $x=0$ in terms of $v(x,t)$ can be written in the following form:

$$k(0)(v_x(0,t)+\psi_1(t)) = f(t), \quad t\in(0,T] \tag{3}$$

In order to arrange the above solution, let us define the followings:

$$z_n(t) =<\zeta(\theta),\phi_n(\theta)> E_{\alpha,1}(-\lambda_n t^\alpha), \tag{4}$$
$$w_n(t) = \int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^\alpha) <\xi(\theta,t-s),\phi_n(\theta)> ds, \tag{5}$$
$$y_n(t) = \int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^\alpha) < r(t-s)F(\theta,t-s), \tag{6}$$
$$\phi_n(\theta) > ds. \tag{7}$$

The solution in terms of $z_n(t)$, $w_n(t)$ and $y_n(t)$ can then be rewritten in the following form:

$$v(x,t) = \sum_{n=1}^{\infty} z_n(t)\phi_n(x) + \sum_{n=1}^{\infty} w_n(t)\phi_n(x) + \sum_{n=1}^{\infty} y_n(t)\phi_n(x).$$

Differentiating both sides of the above identity with respect to x and substituting $x=0$ yields:

$$v_x(0,t) = \sum_{n=1}^{\infty} z_n(t)\phi_n'(0) + \sum_{n=1}^{\infty} w_n(t)\phi_n'(0) + \sum_{n=1}^{\infty} y_n(t)\phi_n'(0).$$

Taking into account the over-measured data $k(0)(v_x(0,t)+\psi_1(t)) = f(t)$

$$f(t) = k(0)\left(\psi_1(t) + \sum_{n=1}^{\infty} z_n(t)\phi_n'(0) + \sum_{n=1}^{\infty} w_n(t)\phi_n'(0) + \sum_{n=1}^{\infty} y_n(t)\phi_n'(0)\right), \tag{8}$$

is obtained, which implies that $f(t)$ can be determined analytically. The right-hand side of identity (8) defines the input-output mappings $\Phi[r]$ on the set of admissible source function $\mathcal{K}$:

$$\Phi[r](t) := k(0)\left(\psi_1(t) + \sum_{n=1}^{\infty} z_n(t)\phi_n'(0) + \sum_{n=1}^{\infty} w_n(t)\phi_n'(0)\right. \tag{9}$$
$$\left. + \sum_{n=1}^{\infty} y_n(t)\phi_n'(0)\right), \quad \forall t\in[0,T]. \tag{10}$$

The following lemma implies the relation between the source functions $r_1(t), r_2(t) \in \mathcal{K}$ at $x=0$ and the corresponding outputs $f_j(t) := k(0)u_x(0,t;r_j)$, $j=1,2$.

Lemma 1.1. Let $\upsilon_1(x,t) = \upsilon(x,t;r_1)$ and $\upsilon_2(x,t) = \upsilon(x,t;r_2)$ be the solutions of the direct problem (2), corresponding to the admissible parameters $r_1(t), r_2(t) \in \mathcal{K}$. If $f_j(t) = k(0)(v_x(0,t;r_j)+\psi_1(t))$, $j=1,2$, are the corresponding outputs. The outputs $f_j(t)$, $j=1,2$, satisfy the following series identity:

$$\Delta f(t) = k(0)\left(\sum_{n=1}^{\infty}\Delta w_n(t)\phi_n'(0) + \sum_{n=1}^{\infty}\Delta y_n(t)\phi_n'(0)\right),$$

for each $t\in(0,T]$ where $\Delta f(t) = f_1(t)-f_2(t)$, $\Delta w_n(t) = w_n^1(t) - w_n^2(t)$, $\Delta r(t) = r_1(t)-r_2(t)$ and $\Delta y_n(t) = y_n^1(t) - y_n^2(t) = \int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^\alpha) < [\Delta r(t-s)]F(t-s), \phi_n(\theta) > ds$.

Proof. By using identity (8), the measured output data $f_j(t) := k(0)(v_x(0,t)+\psi_1(t))$, $j=1,2$ can be written as follows:

$$f_1(t) = k(0)\left(\psi_1(t) + \sum_{n=1}^{\infty} z_n^1(t)\phi_n'(0) + \sum_{n=1}^{\infty} w_n^1(t)\phi_n'(0) + \sum_{n=1}^{\infty} y_n^1(t)\phi_n'(0)\right),$$
$$f_2(t) = k(0)\left(\psi_1(t) + \sum_{n=1}^{\infty} z_n^2(t)\phi_n'(0) + \sum_{n=1}^{\infty} w_n^2(t)\phi_n'(0) + \sum_{n=1}^{\infty} y_n^2(t)\phi_n'(0)\right)$$

respectively. Hence the difference of these formulas implies the desired result.

The lemma and the definitions enable us to reach the following conclusion:

Corollary 1.1. Let the conditions of Lemma 1.1

$$< r_1(t) - r_2(t), \phi_n(x) >= 0,$$

$\forall t \in (0,T]$, $\forall n = 0,1,\ldots$ holds, then $f_1(t) = f_2(t), \forall t\in[0,T]$.

Proof. Note that $r_1(t) \neq r_2(t)$ implies $< r_1(t) - r_2(t), \phi_n(x) > \neq 0$ and $\Delta y_n(t) \neq 0$, for some $n \in \mathscr{N}$. Hence by Lemma 1.1 we conclude that $f_1(t) \neq f_2(t)$ $\forall t \in (0,T]$. Moreover, it leads us to the following important consequence that the input-output mapping $\Phi[r]$ is distinguishable, i.e., $r_1(t) \neq r_2(t)$ implies $\Phi[r_1] \neq \Phi[r_2]$.

Theorem 1.1. Let conditions (C1),(C2) hold. Assume that $\Phi[\cdot] : \mathscr{K} \to C^1[0,T]$ is the input-output mapping defined by (9) and corresponding to the measured output $f(t) := k(0)u_x(0,t)$. In this case the mapping $\Phi[r]$ has the distinguishability property in the class of admissible parameters $\mathscr{K}$, i.e.,

$$\Phi[r_1] \neq \Phi[r_2] \ \ \forall r_1, r_2 \in \mathscr{K} \Rightarrow r_1(t) \neq r_2(t).$$

Proof. From the above explanations the proof of the theorem is clear.

3 An analysis of the inverse problem with given measured data $h(t)$

Consider now the inverse problem with one measured output data $h(t)$ at $x = 1$. Taking into account the over-measured data $h(t) = k(1)(v_x(1,t) + \psi_1(t))$

$$h(t) = k(1)\left(\psi_1(t) + \sum_{n=1}^{\infty} z_n(t)\phi_n'(1) + \sum_{n=1}^{\infty} w_n(t)\phi_n'(1) + \sum_{n=1}^{\infty} y_n(t)\phi_n'(1)\right), \tag{11}$$

is obtained which implies that $h(t)$ can be determined analytically. The right-hand side of identity (11) defines the input-output mappings $\Psi[r]$ on the set of admissible source functions $\mathscr{K}$:

$$\Psi[r](t) := k(1)\left(\psi_1(t) + \sum_{n=1}^{\infty} z_n(t)\phi_n'(1) + \sum_{n=1}^{\infty} w_n(t)\phi_n'(1) + \sum_{n=1}^{\infty} y_n(t)\phi_n'(1)\right), \ \ \forall t \in (0,T]. \tag{12}$$

The following lemma implies the relation between the parameters $r_1(t), r_2(t) \in \mathscr{K}$ at $x = 1$ and the corresponding outputs $h_j(t) := k(1)u_x(1,t;r_j)$, $j = 1,2$.

Lemma 2.1. Let $\upsilon_1(x,t) = \upsilon(x,t;r_1)$ and $\upsilon_2(x,t) = \upsilon(x,t;r_2)$ be the solutions of the direct problem (2), corresponding to the admissible parameters $r_1(t), r_2(t) \in \mathscr{K}$. If $h_j(t) = k(1)(v_x(1,t;r_j) + \psi_1(t))$, $j = 1,2$, are the corresponding outputs. The outputs $h_j(t)$, $j = 1,2$, satisfy the following integral identity:

$$\Delta h(t) = k(1)\left(\sum_{n=1}^{\infty} \Delta w_n(t)\phi_n'(1) + \sum_{n=1}^{\infty} \Delta y_n(t)\phi_n'(1)\right),$$

for each $t \in (0,T]$ where $\Delta h(t) = h_1(t) - h_2(t)$, $\Delta w_n(t) = w_n^1(t) - w_n^2(t)$, $\Delta r(t) = r_1(t) - r_2(t)$.

Proof. By using identity (11), the measured output data $h_j(t) := k(1)\,(v_x(1,t) + \psi_1(t))$, $j = 1,2$ can be written as follows:

$$h_1(t) = k(1)\left(\psi_1(t) + \sum_{n=1}^{\infty} z_n^1(t)\phi_n'(1) + \sum_{n=1}^{\infty} w_n^1(t)\phi_n'(1) + \sum_{n=1}^{\infty} y_n^1(t)\phi_n'(1)\right),$$

$$h_2(t) = k(1)\left(\psi_1(t) + \sum_{n=1}^{\infty} z_n^2(t)\phi_n'(1) + \sum_{n=1}^{\infty} w_n^2(t)\phi_n'(1) + \sum_{n=1}^{\infty} y_n^2(t)\phi_n'(1)\right),$$

respectively. Since $z_n^1(t) = z_n^2(t)$ from the definition then the difference of these formulas implies the desired result.

Corollary 2.1. Let the conditions of Lemma 2.1. If in addition

$$< r_1(t) - r_2(t), \phi_n(x) >= 0, \forall n = 0,1,...$$

holds, then $h_1(t) = h_2(t), \forall t \in (0,T]$.

Proof. Note that $r_1(t) \neq r_2(t)$ implies $< r_1(t) - r_2(t), \phi_n(x) > \neq 0$ and $\Delta y_n(t) \neq 0$, for some $n \in \mathscr{N}$. Hence by Lemma 2.1 we conclude that $h_1(t) \neq h_2(t)$ $\forall t \in [0,T]$. Moreover, it leads us to the following important consequence that the input-output mapping $\Psi[r]$ is distinguishable, i.e., $r_1(t) \neq r_2(t)$ implies $\Psi[r_1] \neq \Psi[r_2]$.

The lemma and the definitions given above enable us to reach the following conclusion: the input-output mapping $\Psi[r]$ is distinguishable, i.e.,$r_1(x) \neq r_2(x)$ implies $\Psi[r_1] \neq \Psi[r_2]$.

Theorem 2.1. Let conditions (C1),(C2) hold. Assume that $\Psi[\cdot] : \mathscr{K} \to C^1[0,T]$ is the input-output mapping defined by (12) and corresponding to the measured output $h(t) := k(1)u_x(1,t)$. In this case the mapping $\Psi[r]$ has the distinguishability property in the class of admissible parameters $\mathscr{K}$, i.e.,

$$\Psi[r_1] \neq \Psi[r_2] \ \ \forall r_1, r_2 \in \mathscr{K} \Rightarrow \ \ r_1(t) \neq r_2(t).$$

Proof. From the above explanations the proof of the theorem is clear.

4 Numerical procedure

We use finite difference method to the problem (1). We subdivide the intervals $[0,1]$ and $[0,T]$ into M and N subintervals of equal lengths $h=\frac{1}{M}$ and $\tau=\frac{T}{N}$, respectively. We choose the implicit scheme, which is absolutely stable and has a first order accuracy in both h and τ, [7–9]. The implicit scheme of problem (1) for $\alpha=1/2$ is as follows:

$$\frac{1}{\sqrt{\pi}}\sum_{k=1}^{j}\frac{\Gamma(j-k+\frac{1}{2})}{(j-k)!}\left(\frac{u_i^j-u_i^{j-1}}{\tau^{1/2}}\right)= \frac{1}{h}\left(k_{i+1}\frac{u_{i+1}^j-u_i^j}{h}-k_i\frac{u_i^j-u_{i-1}^j}{h}\right)+r^jF_i^j \tag{13}$$

$$u_i^0=g_i \tag{14}$$

$$u_0^j=\psi_0^j \tag{15}$$

$$u_M^j=\psi_1^j, \tag{16}$$

where $1\le i\le M$ and $0\le j\le N$ are the indices for the spatial and time steps respectively, $u_i^j=u(x_i,t_j), r^j=r(t_j)$, $g_i=g(x_i)$, $F_i^j=F(x_i,t_j), \psi_0^j=\psi_0(t_j)$, $\psi_1^j=\psi_1(t_j), x_i=ih$, $t_j=j\tau$. At the $t=0$ level, adjustment should be made according to the initial condition and the compatibility requirements.

Now, let us construct the predicting-correcting mechanism. Firstly, if we use the measured output data is $u(1,t)=h(t)$ we obtain

$$r(t)=\frac{D_t^{1/2}u(1,t)-(k(1)u_x(1,t))_x}{F(1,t)}. \tag{17}$$

The finite difference approximation of $r(t)$ is

$$r^j=\frac{\left[\psi_1^j-\frac{1}{h}\left(H^j-k_M\frac{u_M^j-u_{M-1}^j}{h}\right)\right]}{F_M^j}, \tag{18}$$

where $H^j=D_t^{1/2}h(t_j)$, $j=0,1,...,N$.

In numerical computation, since the time step is very small, we can take $r^{j(0)}=r^{j-1}$, $u_i^{j(0)}=u_i^{j-1}$, $j=0,1,2,....N$, $i=1,2,...,M$. At each s-th iteration step we first determine $r^{j(s)}$ from the formula

$$r^{j(s)}=\frac{\left[\psi_1^j-\frac{1}{h}\left(H^j-k_M\frac{u_M^{j(s)}-u_{M-1}^{j(s)}}{h}\right)\right]}{F_M^j},. \tag{19}$$

Then from (13)-(15) we obtain

$$\frac{1}{\sqrt{\pi}}\sum_{k=1}^{j}\frac{\Gamma(j-k+\frac{1}{2})}{(j-k)!}\left(\frac{u_i^{j(s)}-u_i^{j(s-1)}}{\tau^{1/2}}\right)= \frac{1}{h}\left(k_{i+1}\frac{u_{i+1}^{j(s)}-u_i^{j(s)}}{h}-k_i\frac{u_i^{j(s)}-u_{i-1}^{j(s)}}{h}\right)+r^jF_i^j \tag{20}$$

$$u_i^0=g_i \tag{21}$$

$$u_0^{j(s)}=\psi_0^j \tag{22}$$

$$u_M^{j(s)}=\psi_1^j, s=0,1,2,.... \tag{23}$$

The system of equations (20)-(22) can be solved by the Gauss elimination method and $u_i^{j(s)}$ is determined. If the difference of values between two iterations reaches the prescribed tolerance, the iteration is stopped and we accept the corresponding values $r^{j(s)}$, $u_i^{j(s)}(i=1,2,...,N_x)$ as r^j, $u_i^j(i=1,2,...,N_x)$, on the (j)-th time step, respectively. In virtue of this iteration, we can move from level j to level $j+1$.

*Example 1.*Consider the following problem:

$$D_t^{1/2}u(x,t)=(e^xu_x)_x+r(t)\left[\left(\frac{16}{5\sqrt{\pi}}\sqrt{t}-\pi^2te^x\right)\sin\pi x-\pi te^x\cos\pi x\right], \tag{24}$$

$$u(x,0)=0, \tag{25}$$

$$u(0,t)=0, \tag{26}$$

$$u(1,t)=0, \tag{27}$$

and the measured output data is $h(t)=-\pi t^3$.

The exact solution of this problem is $\{r(t),u(x,t)\}=\{t^2,t^3\sin\pi x\}$.

5 Conclusion

The aim of this study was to investigate the distinguishability properties of the input-output mappings $\Phi[\cdot]:\mathscr{K}\to C[0,T]$ and $\Psi[\cdot]:\mathscr{K}\to C^1[0,T]$, which are determined by the measured output data at $x=0$ and $x=1$, respectively. In this study, we conclude that the distinguishability of the input-output mappings hold which implies the injectivity of the inverse mappings Φ^{-1} and Ψ^{-1}. The measured output data $f(t)$ and $h(t)$ are obtained analytically by a series representation, which leads to the explicit form of the input-output mappings $\Phi[\cdot]$ and $\Psi[\cdot]$. This work advances our understanding of the use of the Fourier method of separation of variables and the input-output mapping in the investigation of inverse problems for fractional parabolic equations. The author plans to consider various fractional inverse problems in future studies, since the method discussed has a wide range of applications.

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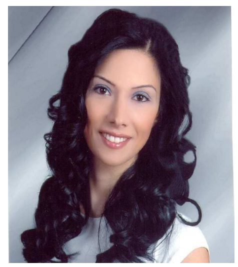

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