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Complexity of the Integration on Hölder-Nikolskii Classes with Mixed Smoothness

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Abstract: We study the information complexity of the numerical integration on the Hölder-Nikolskii classes MH_p^r in the randomized setting. We adopt classical Monte Carlo method to approximate this integration and derive the corresponding convergence rate. Comparing our results with the previous known results in the deterministic setting, we see that the randomized algorithms have faster convergence rates.

Keywords: Information complexity, Randomized algorithm, Hölder-Nikolskii class, Mixed smoothness, Convergence rate

1 Introduction

The calculation of d-dimensional integrals occurs in numerous applications including physics, chemistry, finance, and the computational sciences, where the d may be in the hundreds or even in the thousands, see [1,6,7,8,9]. For most integrands we could not compute the integral utilizing the fundamental theorem of the calculus since there is no closed form expression of the antiderivatives. We have to approximate the integral numerically. Algorithms for solving this problem are given by using function values on finite points. The information complexity is the minimal number of the function values, needed to solve the problem to within a threshold ε. It is a lower bound of computational complexity which is defined as the minimal number of information operations and combinatory operations to obtain a solution to within ε. However it is proved that for the integration problem the information complexity is proportional to the computational complexity, cf. [15]. Thus as the computational complexity, the information complexity is a fundamental invariance of computer science. In this paper we concentrate on the information complexity of integration on the classes of functions with mixed smoothness.

A central issue of information complexity theory is to investigate how the information complexity of a given problem depends on ε^{-1} and d. If it depends exponentially on ε^{-1} or d, we say the problem is intractable. This intractability is also called the curse of dimensionality, cf. [1,4,6,9]. It is well-known that the integration problem defined on the usual Sobolev classes of functions suffers from the curse of dimensionality in the deterministic setting. It is also known that the randomization can break the intractability, that is, the complexity in the randomized setting depends polynomially on ε^{-1} and d, cf. [5,6,7, 9]. This shows the advantage of the randomized methods. In this paper we study the efficiency of randomized method in the computation of the high dimensional integration. We consider Hölder-Nikolskii class of functions with mixed smoothness. cf. [3,5,14]. This class plays an important role in the study of the complexity of many numerical problems, such as integration and function approximation, since the bounds of deterministic complexity of these problems depend weakly on the dimension, cf. [2,4,10,11,12,13]. In what follows, we will use randomized method to approximate the integration on this class and derive the corresponding convergence rates. Our results show the randomized algorithms have faster convergence rates than the deterministic ones.

The remaining part of this paper is organized as follows. In Section 2 we formulate the problem of integration in the framework of information-based complexity theory and present the main results. In Section 3 we prove our main results. Finally, in Section 4 we

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illustrate the significance of our results and discuss their potential applications in physics and finance.

2 Preliminary and main results

In this section, we first formulate the problem of the numerical integration in the framework of information-based complexity theory, then we present our main results.

Let $\mathbf{T}^d := [0,2\pi)^d$ be the d-dimensional torus. It is the product of d copies of the interval $[0,2\pi]$ with the identification of the end points. Let $C(\mathbf{T}^d)$ be the space of continuous functions on $\mathbf{T}^d$ and $F \subset C(\mathbf{T}^d)$. For $f \in F$, we would like to approximate the integral

$$Int(f) := \int_{\mathbf{T}^d} f(t)d\lambda^d(t)$$

by some quadrature formula, where λ^d denotes the normalized Lebesgue measure on $\mathbf{T}^d$. When F is specified, we write down the integration problem as (F, Int).

We now describe the deterministic quadrature formula. For any given $k \in \mathbb{N}$, we choose k points $t_1, \dots, t_k \in \mathbf{T}^d$ and weights $c_1, \dots, c_k \in \mathbf{R}$ to compose a mapping q via

$$q(f) := \sum_{j=1}^{k} c_j f(t_j), \qquad f \in F,$$

where F is some class of continuous functions. The number k is called the cardinality of the quadrature formula. Denote the set of all k-point quadrature rules, by Q_k. We introduced the classes of admissible quadrature formulas

$$\mathscr{Q}^n(F,\mathbf{R}) := \bigcup_{k \le n} Q_k,$$

and

$$\mathscr{Q}(F,\mathbf{R}) := \bigcup_{n \in \mathbf{N}} \mathscr{Q}^n(F,\mathbf{R}) = \bigcup_{k \ge 0} Q_k.$$

The error of a quadrature rule $q \in \mathscr{Q}(F,\mathbf{R})$ on the class F is defined as

$$e(F, Int, q) := \sup\{|Int(f) - q(f)|, \quad f \in F\}.$$

Having fixed the cardinality n and minimizing the above expression by a good choice of $q \in \mathscr{Q}^{n-1}(F,\mathbf{R})$, we obtain the n-th minimal deterministic error

$$e_n^{det}(F, Int) := \inf\{e(F, Int, q), \quad q \in \mathscr{Q}^{n-1}(F,\mathbf{R})\}. \quad (1)$$

For a given threshold ε, the information complexity of the problem (F, Int) is defined as

$$comp(\varepsilon, Int) := \inf\{n : e_n(F, Int) \le \varepsilon\}.$$

That is, the minimal number needed to solve this integration problem to within ε.

Then we turn to the randomized setting. One can view randomized quadrature formulas as random variables taking values in $\mathscr{Q}(F,\mathbf{R})$. To be precise, we recall the following definition.

Definition 1. A triple $\mathscr{M} := ([\Omega, \mathscr{F}, P], q, k)$ is called a randomized quadrature rule, if

(1) $[\Omega, \mathscr{F}, P]$ is a probability measure space.

(2) $q : \Omega \to \mathscr{Q}(F,\mathbf{R})$ is a mapping, such that for all $f \in F$ the mapping

$$\omega \in \Omega \to q_\omega(f) := (q(\omega))(f) \in \mathbf{R}$$

is a real random variable.

(3) $k : \Omega \to \mathbb{N}$ is a measurable natural number, such that we have

$$q_\omega \in \mathscr{Q}^{k(\omega)}(F,\mathbf{R}), \quad \omega \in \Omega.$$

The error of a randomized algorithm $\mathscr{M}$ on the class F is defined through the integral

$$e(F, Int, \mathscr{M}) := \sup\left\{\int_\Omega |Int(f) - q_\omega(f)| dP(\omega), \quad f \in F\right\}.$$

The cardinality of the algorithm $\mathscr{M}$ is defined by

$$\text{MC-card}(\mathscr{M}) := \int_\Omega k(\omega) dP(\omega).$$

Thus the n-th minimal randomized error for the problem (F, Int) is

$$e_n^{mc}(F, Int) := \inf\{e(F, Int, \mathscr{M}), \quad \text{MC-card}(\mathscr{M}) \le n-1\}, \quad (2)$$

for any $n \in \mathbb{N}$. Similarly one can define the information complexity of (F, Int) in the randomized setting.

Now we introduce the Hölder-Nikolskii classes of functions. Denote by $L_p(\mathbf{T}^d)$ the space of Lebesgue measurable functions defined on the d-dimensional torus $\mathbf{T}^d$ satisfying

$$\|f\|_p = \left((2\pi)^{-d} \int_{\mathbb{T}^d} |f(x)|^p dx\right)^{1/p} < \infty, \quad \text{for } 1 \le p < \infty,$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in \mathbb{T}^d} |f(x)| < \infty.$$

Let e be a subset of natural numbers in $[1, d]$. Denote by

$$\triangle_{\mathbf{t}}^l(e) = \prod_{j \in e} \triangle_{t_j}^l, \quad \triangle_{\mathbf{t}}^l(\emptyset) = I.$$

the mixed l-th difference operators with step t_j in the variable x_j for $j \in e$. Then we define the class MH_p^r as the set of $f \in L_p$ such that for any e

$$\|\triangle_{\mathbf{t}}^l(e) f(\mathbf{x})\|_p \le \prod_{j \in e} |t_j|^r,$$

where $l > r$.

In what follows, we use the notations $\ll$ and $\asymp$. For two sequences $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ of positive real numbers, the order inequality $a_n \ll b_n$ means that there is a number $c > 0$ such that, for all n, we have $a_n \leq cb_n$; and the relation $a_n \asymp b_n$ means $a_n \ll b_n$ and $b_n \ll a_n$.

Now we recall the results about the deterministic complexity of the integration on the classes MH_p^r.

Theorem 1[14] Let $r > 1$ be a real number, $1 < p \leq \infty$. Then

$$e_n^{det}(MH_p^r, Int) \asymp n^{-r}(\log^{(d-1)} n).$$

Our main results are the following two theorems.

Theorem 2. Let $r > 1$ be a real number, $1 < p < \infty$. Then

$$e_n^{mc}(MH_p^r, Int) \ll \begin{cases} n^{-r-1/2}(\log^{(d-1)} n)^{r+1}, & 2 \leq p < \infty; \\ n^{-r-1+1/p}(\log^{(d-1)} n)^{r+1}, & 1 < p < 2. \end{cases}$$

Theorem 3. Let $r > 1$ be a real number, $1 < p < \infty$. Then

$$e_n^{mc}(MH_p^r, Int) \gg \begin{cases} n^{-r-1/2}, & 2 \leq p < \infty; \\ n^{-r-1+1/p}, & 1 < p < 2. \end{cases}$$

We see from Theorem 2 and 3 that modulo a power of logarithm, the sharp bound of $e_n^{mc}(MH_p^r, Int)$ has been determined. Comparing Theorem 1 with Theorem 2 and 3, one can see that the randomized method provides a better convergence rate than the deterministic one. Quantitatively, the improvement amounts to the factor $n^{-1/2}$ if $p \geq 2$ and $n^{-1+1/p}$ if $1 < p < 2$.

3 Proofs of main results

In this section, we shall prove our main results. To this end, we need some auxiliary lemmas. First we invoke a result of approximation of functions from MH_p^r by trigonometric polynomials.

For $m \in \mathbb{N}$, we define the de la Vallee-Poussin kernel $\mathscr{V}_m(x)$ by

$$\mathscr{V}_m(x) = 1 + 2\sum_{k=1}^{m} \cos kx + 2\sum_{k=m+1}^{2m} \left(\frac{2m-k}{m}\right) \cos kx,$$

where $x \in \mathbf{R}$. Denote $x(l) = \frac{\pi l}{2m}$, $l = 1, \ldots, 2m$, we define the linear operator

$$R_m(f,x) = (4m)^{-1} \sum_{l=1}^{4m} f(x(l)) \mathscr{V}_m(x - x(l))$$

and

$$\Delta_n(f,x) = R_{2^n}(f,x) - R_{2^{n-1}}(f,x), \quad n \geq 1,$$

where $\Delta_0(f,x) = R_1(f,x)$. Then we define the operator $\Delta_{\mathbf{s}}(f,\mathbf{x})$ as the composition of the one-dimensional operators

$$\Delta_{\mathbf{s}}(f,\mathbf{x}) = \Delta_{s_d}(\Delta_{s_{d-1}} \ldots \Delta_{s_1}(f, x_1) \ldots, x_d),$$

where Δ_{s_j} acts as a one-dimensional operator on a function depending on the variable x_j. Now the approximation operator T_{Q_n} is defined as follows:

$$T_{Q_n} = \sum_{\|\mathbf{s}\|_1 \leq n} \Delta_{\mathbf{s}}(f,\mathbf{x}),$$

where Q_n denotes the number of function values used in T_{Q_n}. It is clear that $Q_n \asymp 2^n n^{d-1}$.

Lemma 1.[14]. Let $1 \leq p \leq \infty, r > 1/p$. For $f \in MH_p^r$, we have

$$\|f - T_{Q_n}(f)\|_p \ll 2^{-rn} n^{d-1}.$$

The proof of Theorem 2 is based on variance reduction. To carry out this process, we will use the classical Monte Carlo quadrature formula which is defined as follows. Let $(\xi_i)_{i=1}^n$ be independent, $\mathbf{T}^d$-valued, uniformly distributed over $\mathbf{T}^d$ random variables on some probability measure space (Ω, Σ, μ). For $f \in C(\mathbf{T}^d)$, we put

$$Q_\omega(f) = \frac{1}{n}\sum_{i=1}^{n} f(\xi_i(\omega)), \qquad \omega \in \Omega.$$

Lemma 2.[3,6] Let $1 \leq p \leq \infty$. Then for all $f \in C(\mathbf{T}^d)$,

$$\int_\Omega |Int(f) - Q_\omega(f)| d\mu(\omega) \leq \begin{cases} n^{-1/2}\|f\|_p, & 2 \leq p \leq \infty, \\ c_p n^{1/p-1}\|f\|_p, & 1 \leq p < 2, \end{cases}$$

where $c_p = 2^{2/p-1}$.

Proof of Theorem 2. We first construct a randomized algorithm based on the classical Monte Carlo quadrature rule.

Let (Ω, Σ, μ), $(\xi_i)_{i=1}^m$, and Q_ω be defined as above, where $m = Q_n$. For $f \in MH_p^r$, we set for $\omega \in \Omega$,

$$\begin{aligned} A_\omega(f) &= Q_\omega(f - T_{Q_n}(f)) + Int(T_{Q_n}(f)) \\ &= Q_\omega(f - T_{Q_n}(f)) + q_m(f). \end{aligned}$$

It is easy to see that $\mathscr{M} = ((\Omega, \Sigma, \mu), (A_\omega)_{\omega \in \Omega})$ is a randomized method, where the required measurability follows from the fact that the mapping

$$(f, \mathbf{x}) \longrightarrow f(\mathbf{x}) - (T_{Q_n} f)(\mathbf{x})$$

from $C(\mathbf{T}^d) \times \mathbf{T}^d$ into $\mathbf{R}$ is continuous. Obviously, $\mathscr{M} \in \mathscr{Q}^{2m}(MH_p^r, \mathbf{R})$. Then we derive a upper estimate of the error of A_ω which leads to the required bound. Using Lemma 2, we get for $f \in MH_p^r$

$$\begin{aligned} &\int_\Omega |Int(f) - A_\omega(f)| d\mu(\omega) \\ &= \int_\Omega |Int(f) - Q_\omega(f - T_{Q_n}(f)) - Int(T_{Q_n}(f))| d\mu(\omega) \\ &= \int_\Omega |Int(f - T_{Q_n}(f)) - Q_\omega(f - T_{Q_n}(f))| d\mu(\omega) \\ &\leq m^{-\alpha_p}\|f - T_{Q_n}(f)\|_p, \end{aligned} \tag{3}$$

where $\alpha_p = 1/2$ for $2 \le p < \infty$ and $\alpha_p = 1 - 1/p$ for $1 < p < 2$. Combining the relationship $m = Q_n \asymp 2^n n^{d-1}$, Lemma 1 and (3), we get

$$e_n^{mc}(MH_p^r, Int) \ll n^{-r-1/2+(1/p-1/2)_+}(\log^{(d-1)} n)^{r+1}.$$

The proof of Theorem 2 is complete.

Next we turn to the proof of the lower bounds. We will reduce the lower estimate in the randomized setting to that of the average case setting. For this purpose we recall the definition of the n-th minimal error in the average setting.

Let X be a Banach space and F be a closed bounded subset of X. Assume that the set F is equipped with a Borel field $\mathscr{B}(F)$ which is the σ-algebra containing all open subsets of F. Let μ be a probability measure defined on $(F, \mathscr{B}(F))$. Denote the subset of those $q \in \mathscr{Q}^{n-1}(F, \mathbb{R})$ which are $(\mathscr{B}(F), \mathscr{B}(\mathbb{R}))$ measurable by $\mathscr{Q}_{n-1}^{mes}(F, \mathbb{R})$. Then, for $q \in \mathscr{Q}_{n-1}^{mes}(F, \mathbb{R})$, we define the average error with respect to μ as

$$e(Int, q, \mu) := \int_F |Int(f) - q(f)| d\mu(f).$$

Minimizing the errors with respect to the choice of $q \in \mathscr{Q}_{n-1}^{mes}(F, \mathbb{R})$, we yield the n-th minimal average error with respect to μ

$$e_{n+1}^{avg}(Int, \mu) := \inf_{q \in \mathscr{Q}_n^{mes}(F,\mathbb{R})} e(Int, q, \mu).$$

Lemma 3.[4]. For every probability measure μ on F and for all $n \in \mathbb{N}$,

$$e_n^{mc}(F, Int) \ge e_{2n-1}^{avg}(Int, \mu)/2.$$

Proof of Theorem 3. In the proof of the lower bounds, we separate two cases, $2 \le p \le \infty$ and $1 \le p < 2$.

First, we consider the case $2 \le p \le \infty$. It suffices to consider the case $p = \infty$. We shall define a probability measure on $F = MH^r_{\infty,}$. For this purpose, we need construct $\widetilde{n} := 2n$ functions with mutually disjoint supports. For a given $n \in \mathbb{N}$, we divide the torus $\mathbf{T}^d$ into $2n$ equal subsections $\{G_i\}_{i=1}^{2n}$ with mutually disjoint interiors,

$$G_i = \left\{ \mathbf{x} \in \mathbf{T}^d : x_1 \in \widetilde{\mathbf{T}}, x_j \in \mathbf{T}, \quad j = 2, \ldots, d \right\},$$

where

$$\widetilde{\mathbf{T}} = \left\{ x_1 | \frac{(i-1)\pi}{n} \le x_1 < \frac{(i-1)\pi}{n} + \frac{\pi}{n} \right\},$$

and $i = 1, \ldots, \widetilde{n}$. Then we choose ϕ to be a fixed bump function in $C^\infty(\mathbf{R})$ with support contained in $\mathbf{T}_0$, where $\mathbf{T}_0$ is the interior of $\mathbf{T}$, such that $0 \le \phi(t) \le 1$, $t \in \mathbf{T}$, $\phi(t) = 1$, when $t \in [\pi/2, 3\pi/2]$. The function f_i is defined to have a bump only in the rectangle G_i as follows:

$$f_i(\mathbf{x}) = a_p (2n)^{-r+1/p} \phi(2n(x_1 - (i-1)\pi/n)) \phi(x_2) \cdots \phi(x_d),$$

for $\mathbf{x} \in \mathbf{T}^d$, where $1 \le p \le \infty$. It is easily seen that

$$\int_{\mathbf{T}^d} f_i(\mathbf{x}) d\lambda^d(\mathbf{x}) = \begin{cases} a_p C^d (2n)^{-r-1+1/p}, & 1 \le p < \infty, \\ a_p C^d (2n)^{-r-1}, & p = \infty, \end{cases} \tag{4}$$

where $\int_{\mathbf{T}} \phi(t) dt = C$. Furthermore, let $\{\varepsilon_i\}_{i=1}^{\widetilde{n}}$ be a sequence of independent, $\{-1, 1\}$-valued random variables on some probability space $(\Omega_1, \Sigma_1, \mu_1)$ with

$$\mu_1\{\varepsilon_i = 1\} = \mu_1\{\varepsilon_i = -1\} = 1/2, \qquad i = 1, \ldots, \widetilde{n}.$$

For $p = \infty$, we choose a constant $a_p > 0$ such that $f_i \in MH_\infty^r$ for $i = 1, \ldots, \widetilde{n}$ and put

$$\mu = \text{dist}\left(\sum_{i=1}^{\widetilde{n}} \varepsilon_i f_i \right),$$

where dist means the distribution of the MH_∞^r-valued random variable. For any system of points $\mathbf{x}_1, \ldots, \mathbf{x}_n$, let us define I by

$$I = \{ i : 1 \le i \le \widetilde{n}, \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \cap G_i = \varnothing \}.$$

It is clear that the cardinality of the set I satisfies

$$|I| \ge \widetilde{n} - n = n. \tag{5}$$

Further, $q(f_i) = 0$ for $i \in I$, and we can estimate

$$\begin{aligned} & \int_{MH_\infty^r} |Int(f) - q(f)| d\mu(f) \\ &= \int_{\Omega_1} \left| Int\left(\sum_{i=1}^{\widetilde{n}} \varepsilon_i(\omega) f_i \right) - q\left(\sum_{i=1}^{\widetilde{n}} \varepsilon_i(\omega) f_i \right) \right| d\mu_1(\omega) \\ &:= \int_{\Omega_1} \left| I_1 + I_2 - q\left(\sum_{i \notin I} \varepsilon_i(\omega) f_i \right) \right| d\mu_1(\omega), \end{aligned} \tag{6}$$

where

$$I_1 := \sum_{i \in I} \varepsilon_i(\omega) Int(f_i), \quad I_2 := \sum_{i \notin I} \varepsilon_i(\omega) Int(f_i).$$

The distribution of $\{\varepsilon_i\}_{i=1}^{\widetilde{n}}$ does not change if we replace ε_i by $-\varepsilon_i$ for $i \in I$ and leave it unchanged for $i \notin I$. Consequently, we can continue (6) as follows

$$\begin{aligned} &= \int_{\Omega_1} \left| -I_1 + I_2 - q\left(\sum_{i \notin I} \varepsilon_i(\omega) f_i \right) \right| d\mu_1(\omega) \\ &\ge \int_{\Omega_1} \left| \sum_{i \in I} \varepsilon_i(\omega) Int(f_i) \right| d\mu_1(\omega) \gg \left(\sum_{i \in I} |Int(f_i)|^2 \right)^{1/2} \\ &= |I|^{1/2} |Int(f_1)| \gg n^{-r-1/2}, \end{aligned}$$

which together with Lemma 3 completes the lower estimates for $2 \le p \le \infty$. In the above proof, we used the Khintchine's inequality, and the relations (4), (5). Now let $1 \le p < 2$ and let μ be the equi-distribution on the set

$$\{\pm f_i : i = 1, \ldots, \widetilde{n}\},$$

where this time we choose a constant $a_p > 0$ such that $f_i \in MH_p^r$, for $i = 1, \ldots, \widetilde{n}$. With I as above, using the relations (4) and (5) again, we get

$$\begin{aligned}
&\int_{MH_p^r} |Int(f) - q(f)| d\mu(f) \\
&= \frac{1}{2\widetilde{n}} \sum_{i=1}^{\widetilde{n}} \sum_{\sigma=\pm 1} |\sigma Int(f_i) - q(\sigma f_i)| \\
&\geq \frac{1}{2\widetilde{n}} \sum_{i\in I} \sum_{\sigma=\pm 1} |\sigma Int(f_i)| \\
&\geq \frac{1}{2\widetilde{n}} \sum_{i\in I} |Int(f_i)| \gg n^{-r+1/p-1},
\end{aligned}$$

which together with Lemma 3 again completes the lower estimates for $1 \leq p < 2$. The proof of Theorem 3 is complete.

4 Conclusion:

We determine the information complexity of the integration over the class MH_p^r. Our results show that this problem is tractable in the randomized setting. Moreover if we neglect the logarithmic factor, then we find that the convergence rate does not depend on d. This property again shows the great advantage of the randomized methods. It allows us to use randomized methods to approximate high dimensional integration when the integrand is taken from the class MH_p^r. In particular we can approximate the path integration by randomized algorithms. In this case the dimension d can be arbitrarily large. Since path integration lies at the foundation of quantum mechanics, statistical mechanics and mathematical finance. Our results may have potential applications in these fields.

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