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Optimal Approximate Solution for $(\alpha,\beta)_\psi$-Contraction Mappings in Metric Spaces with Applications

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Abstract: In this paper, we extend the concept of generalized proximal contractions of the first kind to proximal $(\alpha,\beta)_\psi$-contractions types A and B. We show with examples that our new classes of mappings is real generalization of many known classes of non-self and self mappings in literature. We also introduce some condition for proving the best proximity point theorems in such classes. Applying our new concepts, we obtain best proximity point results on metric spaces endowed with an arbitrary binary relation and metric spaces endowed with graph.

Keywords: Best proximity points, $(\alpha,\beta)_\psi$-contractions types A and B, α-proximal admissible mappings, β_0-proximal subadmissible mappings.

1 Introduction

It is well-known that many problems in the real word can be formulated as equations of the form $Tx = x$, where T is a self-mapping in some suitable framework. From the fact that fixed point theory find into the existence of a solution to such generic equations and brings out the iterative algorithms to compute a solution to such equations. However, in the case of T is non-self mapping; the aforementioned equation does not necessarily has a solution. In such case, it is worthy to determine an approximate solution x such that the error between a point x and Tx is minimum. This is the idea behind best approximation theory. A classical best approximation theorem was introduced by Fan [2]. Afterward, several authors, including Prolla [12], Reich [14], Sehgal and Singh [21,22], have derived extensions of Fan's Theorem in many directions. Moreover, for a detailed account of global optimization and the existence of a best proximity point, one can refer to [10,11,19,6,17,20,15,7,23,5,8].

In 2012, Samet et al. [18] introduced the concept of α-admissible self mapping and proved the existence and uniqueness theorems of fixed point by using the idea of α-admissible mapping. Afterward, Jleli and Samet [3] extended this concept to non-self version which so called α-proximal admissible mapping. They also give the existence theorems of best proximity points.

From mentioned above, we introduce new classes of β_0-proximal subadmissible mappings and $(\alpha,\beta)_\psi$-contraction mappings which is a generalization of class of generalized proximal contractions of the first kind due to Sadiq Basha [15]. We also give some example to show the real generality of class of $(\alpha,\beta)_\psi$-contraction mappings and obtain new best proximity point theorems for such mappings. Our result improve and complementary several results in literatures. As an application of our results, best proximity point results on metric space endowed with an arbitrary binary relation and metric space endowed with graph are also derived from our results.

2 Preliminaries

In this section, we review some basic concepts and results which will be used later. Throughout this paper, unless otherwise specified, A and B are nonempty subsets of a metric space (X,d). We recall the following notations and

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notions that will be used in what follows:

$$d(A,B) := \inf\{d(x,y) : x \in A \text{ and } y \in B\},$$

$$A_0 := \{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\},$$
$$B_0 := \{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\}.$$

Remark 1*Two sets A_0 and B_0 are nonempty provide that $A \cap B \neq \emptyset$. Further, if A and B are closed subsets of a normed linear space such that $d(A,B) > 0$, then A_0 and B_0 are contained in the boundaries of A and B respectively (see [16]).*

Definition 1*A subset B of X is said to be* approximatively compact *with respect to A if every sequence $\{y_n\}$ in B satisfies the condition that $d(x,y_n) \to d(x,B)$ as $n \to \infty$ for some $x \in A$ has a convergent subsequence.*

Remark 2*It is easy to see that every set is approximatively compact with respect to itself and every compact set is approximatively compact. Moreover, A_0 and B_0 are nonempty set if A is compact and B is approximatively compact with respect to A.*

Definition 2*A point $x \in A$ is said to be a* best proximity point *of the mapping $T : A \to B$, if it satisfies the following condition:*
$$d(x,Tx) = d(A,B).$$

Throughout this paper, we use $Best(T)$ stands for the set of all best proximity point of mapping $T : A \to B$.

Remark 3*It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.*

Definition 3 ([1,13])*A mapping $T : A \to A$ is said to be* weak contraction, *if for each $x,y \in A$,*

$$d(Tx,Ty) \leq d(x,y) - \psi(d(x,y)), \tag{2.1}$$

where $\psi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. If A is bounded, then the infinity condition can be omitted (see [1,13]).

Definition 4 ([15])*A mapping $T : A \to B$ is said to be a* generalized proximal contraction of the first kind *if for each $u,v,x,y \in A$, the following condition holds:*

$$\left.\begin{array}{l} d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{array}\right\} \Longrightarrow d(u,v) \leq d(x,y) - \psi(d(x,y)),$$

where $\psi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing such that $\psi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. If A and B are bounded, then the infinity condition can be dropped.

Definition 5 ([18])*A self mapping $T : X \to X$ is said to be α-admissible, where $\alpha : X \times X \to [0,\infty)$, if*

$$x,y \in X,\ \alpha(x,y) \geq 1 \Longrightarrow \alpha(Tx,Ty) \geq 1.$$

Definition 6 ([4])*Let $T : A \to B$ and $\alpha : A \times A \to [0,\infty)$ be two mappings. We say that T is α-proximal admissible, if*

$$\left.\begin{array}{l} \alpha(x,y) \geq 1, \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{array}\right\} \Longrightarrow \alpha(u,v) \geq 1 \tag{2.2}$$

for all $x,y,u,v \in A$.

Clearly, for self-mapping, T is α-proximal admissible implies that T is α-admissible.

Definition 7 ([9])*Let X be a nonempty set and $\alpha : X \times X \to [0,\infty)$ be a mapping.*

1. *α is said to be* forward transitive *if for each $x,y,z \in X$ for which $\alpha(x,y) \geq 1$ and $\alpha(y,z) \geq 1$, we have $\alpha(x,z) \geq 1$;*
2. *α is said to be* 0-backward transitive *if for each $x,y,z \in X$ for which $0 < \alpha(x,y) \leq 1$ and $0 < \alpha(y,z) \leq 1$, we have $0 < \alpha(x,z) \leq 1$.*

3 Main results

In this section, we introduce the new classes of β_0-proximal subadmissible mappings and proximal $(\alpha,\beta)_\psi$-contraction mappings which is a generalization of class of generalized proximal contraction of the first kind mappings. We give some illustrative example for support real generalization of class of proximal $(\alpha,\beta)_\psi$-contraction mappings. Also, we establish the existence theorems of best proximity points.

Definition 8*Let $T : A \to B$ and $\beta : A \times A \to [0,\infty)$ be two mappings. We say that T is β_0-proximal subadmissible, if*

$$\left.\begin{array}{l} 0 < \beta(x,y) \leq 1, \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{array}\right\} \Longrightarrow 0 < \beta(u,v) \leq 1 \tag{3.1}$$

for all $x,y,u,v \in A$.

Remark 4*If T is self mapping, then the concept of β_0-proximal subadmissible reduces to β_0-subadmissible due to Latif et al. [9].*

Definition 9*A mapping $T : A \to B$ is said to be a* proximal $(\alpha,\beta)_\psi-$contraction type A *if there exists $\alpha,\beta : A \times A \to [0,\infty)$ satisfies the following condition:*

$$\left.\begin{array}{l} d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{array}\right\}$$

$$\Downarrow$$

$$d(u,v) \le \beta(x,y)d(x,y) - \alpha(x,y)\psi(d(x,y)),$$

for all $u,v,x,y \in A$*, where* $\psi : [0,\infty) \to [0,\infty)$ *is continuous and nondecreasing such that* $\psi(t) = 0$ *if and only if* $t = 0$ *and* $\lim_{t\to\infty} \psi(t) = \infty$*.*

Definition 10*A mapping* $T : A \to B$ *is said to be a* proximal $(\alpha,\beta)_\psi-$contraction type B *if there exists* $\alpha,\beta : A\times A \to [0,\infty)$ *satisfies the following condition:*

$$\left.\begin{array}{l} d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{array}\right\}$$

$$\Downarrow$$

$$\alpha(x,y)d(u,v) \le \beta(x,y)d(x,y) - \psi(d(x,y)),$$

for all $u,v,x,y \in A$*, where* $\psi : [0,\infty) \to [0,\infty)$ *is continuous and nondecreasing such that* $\psi(t) = 0$ *if and only if* $t = 0$ *and* $\lim_{t\to\infty} \psi(t) = \infty$*.*

Remark 5*If A and B are bounded, then the infinity condition in Definitions 9 and 10 can be dropped.*

Remark 6*If we take,* $\alpha(x,y) = \beta(x,y) = 1$*, then proximal* $(\alpha,\beta)_\psi$*-contraction mapping types A and B become to a generalized proximal contraction mapping of the first kind (see Definition 4). Moreover, it is easy to see that a self-mapping, proximal* $(\alpha,\beta)_\psi$*-contraction mapping reduces to a weak contraction mapping.*

Next, we give some example to show the real generality of classes of $(\alpha,\beta)_\psi$-contraction mappings.

Example 1*Consider the complete metric space* $X = \mathbb{R}^2$ *with the metric* $d : X\times X \to [0,\infty)$ *defined by*

$$d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

for all $(x_1,x_2),(y_1,y_2) \in X$*. Let*

$$A = \{(0,y) : 0 \le y \le 1\}, \quad B = \{(1,y) : 0 \le y \le 1\}.$$

Then $d(A,B) = 1$*,* $A_0 = A$*,* $B_0 = B$*. Define the mappings* $T : A \to B$ *as follows:*

$$T((0,y)) = \begin{cases} \left(1, \dfrac{y}{1+y}\right), & y \in [0,1/2], \\ (1,y^2), & \text{otherwise} \end{cases}$$

for all $(0,y) \in A$*.*

Now, we show that T is proximal $(\alpha,\beta)_\psi$*-contraction type B with the function* $\psi : [0,\infty) \to [0,\infty)$ *and* $\alpha,\beta : A\times A \to [0,\infty)$ *defined by*

$$\psi(t) = \frac{t^2}{1+t} \quad \text{for all } t \in [0,\infty)$$

and

$$\alpha((0,x),(0,y)) = \begin{cases} 1, & x,y \in [0,1/2], \\ 0, & \text{otherwise,} \end{cases}$$

$$\beta((0,x),(0,y)) = \begin{cases} 1+x+y, & x,y \in [0,1/2], \\ \dfrac{|x-y|}{1+|x-y|}, & \text{otherwise} \end{cases}$$

for all $(0,x),(0,y) \in A$*.*

Let $(0,x_1),(0,x_2),(0,a_1)$ *and* $(0,a_2)$ *be elements in A satisfying*

$$d((0,x_1),T(0,a_1)) = d(A,B) = 1 \tag{3.2}$$

and

$$d((0,x_2),T(0,a_2)) = d(A,B) = 1. \tag{3.3}$$

Case 1: *Suppose that* $a_1,a_2 \in [0,1/2]$*. From (3.2) and (3.3), we get*

$$x_i = \frac{a_i}{1+a_i} \quad \text{for all } i = 1,2.$$

Without loss of generality, we may assume that $a_1 - a_2 \ge 0$*. Then we have*

$$\begin{aligned}
&\alpha((0,x_1),(0,x_2))d((0,x_1),(0,x_2)) \\
&= d\Big((0,\frac{a_1}{1+a_1}),(0,\frac{a_2}{1+a_2})\Big) \\
&= \left|\frac{a_1}{1+a_1} - \frac{a_2}{1+a_2}\right| \\
&= \frac{a_1-a_2}{(1+a_2)(1+a_2)} \\
&\le \frac{a_1-a_2}{1+|a_1-a_2|} \\
&= \frac{a_1-a_2+(a_1-a_2)^2-(a_1-a_2)^2}{1+|a_1-a_2|} \\
&= (a_1-a_2) - \frac{(a_1-a_2)^2}{1+|a_1-a_2|} \\
&\le (1+a_1+a_2)(a_1-a_2) - \frac{(a_1-a_2)^2}{1+|a_1-a_2|} \\
&= \beta\big((0,a_1),(0,a_2)\big)d\big((0,a_1),(0,a_2)\big) \\
&\quad -\psi\big(d\big((0,a_1),(0,a_2)\big)\big).
\end{aligned}$$

Case 2: *Suppose that* $a_1 \notin [0,1/2]$ *or* $a_2 \notin [0,1/2]$*. Then we have*

$$\begin{aligned}
&\alpha((0,x_1),(0,x_2))d((0,x_1),(0,x_2)) \\
&= 0 \\
&\le \beta\big((0,a_1),(0,a_2)\big)d\big((0,a_1),(0,a_2)\big) \\
&\quad -\psi\big(d\big((0,a_1),(0,a_2)\big)\big).
\end{aligned}$$

Therefore, T is proximal $(\alpha,\beta)_\psi-$*contraction type B.*

Remark 7*From Example 1, we can see that T is not a generalized proximal contraction of the first kind. Indeed, putting*

$$(0,x_1) = (0,25/49),$$
$$(0,x_2) = (0,36/47),$$
$$(0,a_1) = (0,5/7)$$

and

$$(0,a_2) = (0,6/7)$$

are elements in A. Then we get

$$\begin{aligned} d((0,x_1),T(0,a_1)) &= d((0,25/49),(1,25/49)) \\ &= 1 \\ &= d(A,B) \end{aligned}$$

and

$$\begin{aligned} d((0,x_2),T(0,a_2)) &= d((0,36/49),(1,36/49)) \\ &= 1 \\ &= d(A,B) \end{aligned}$$

but

$$\begin{aligned} d((0,x_1),(0,x_2)) &= d((0,25/49),(0,36/49)) \\ &= 11/49 \\ &> 1/8 \\ &= 1/7 - \frac{(1/7)^2}{1+1/7} \\ &= d((0,a_1),(0,a_2)) \\ &\quad -\psi(d((0,a_1),(0,a_2))). \end{aligned}$$

Here, we give the best proximity point theorem for proximal $(\alpha,\beta)_\psi$-contraction non-self mapping type A.

Theorem 1*Let (X,d) be a complete metric space and A and B be nonempty closed subsets of X such that A_0 or B_0 is nonempty set. Suppose that $T : A \to B$ satisfy the following conditions:*

(a)*T is continuous;*
(b)*T is a proximal $(\alpha,\beta)_\psi$-contraction type A;*
(c)*T is α-proximal admissible and β_0-proximal subadmissible;*
(d)*there exist element x_0 and x_1 in A_0 such that*

$$d(x_1,Tx_0) = d(A,B) \text{ and}$$
$$0 < \beta(x_0,x_1) \le 1 \le \alpha(x_0,x_1);$$

(e)*$T(A_0) \subseteq B_0$;*
(f)*α is forward transitive and β is 0-backward transitive.*

Then T has a best proximity point, that is, there exists a point $x^* \in A$ such that

$$d(x^*,Tx^*) = d(A,B).$$

Moreover, if $0 < \beta(x,y) \le 1 \le \alpha(x,y)$ for all $x,y \in Best(T)$, then x^* is a unique best proximity point of T.

Proof. By the hypothesis (d), there exist element x_0 and x_1 in A_0 such that

$$d(x_1,Tx_0) = d(A,B) \tag{3.4}$$

and

$$0 < \beta(x_0,x_1) \le 1 \le \alpha(x_0,x_1). \tag{3.5}$$

In view of the fact that $T(A_0) \subseteq B_0$, it is ascertained that there exists an element $x_2 \in A_0$ such that

$$d(x_2,Tx_1) = d(A,B). \tag{3.6}$$

From (3.4), (3.5) and (3.6), using (2.2) and (3.1), we get

$$0 < \beta(x_1,x_2) \le 1 \le \alpha(x_1,x_2).$$

Since $T(A_0) \subseteq B_0$, we can find an element $x_3 \in A_0$ such that

$$d(x_3,Tx_2) = d(A,B). \tag{3.7}$$

Again, by (3.6) and (3.7) and the conditions (2.2) and (3.1), we have

$$0 < \beta(x_2,x_3) \le 1 \le \alpha(x_2,x_3).$$

By similar fashion, we can construct the sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1},Tx_n) = d(A,B)$$

and

$$0 < \beta(x_n,x_{n+1}) \le 1 \le \alpha(x_n,x_{n+1}) \tag{3.8}$$

for all $n \in \mathbb{N}$. Using proximal $(\alpha,\beta)_\psi$-contractive condition type A, we have

$$\begin{aligned} d(x_n,x_{n+1}) &\le \beta(x_{n-1},x_n)d(x_{n-1},x_n) \\ &\quad -\alpha(x_{n-1},x_n)\psi(d(x_{n-1},x_n)) \\ &\le d(x_{n-1},x_n) - \psi(d(x_{n-1},x_n)) \\ &\le d(x_{n-1},x_n) \end{aligned} \tag{3.9}$$

for all $n \in \mathbb{N}$. Putting $d_n := d(x_{n-1},x_n)$ for each $n \in \mathbb{N}$. So $0 \le d_{n+1} \le d_n - \psi(d_n) \le d_n$ for all $n \in \mathbb{N}$. Therefore $\{d_n\}$ is a nonincreasing sequence and bounded below, then there exists $r \ge 0$ such that

$$\lim_{n\to\infty} d_n = r.$$

Now, let us claim that $r = 0$. Suppose that $r > 0$, from the fact that ψ is an increasing, we obtain that

$$\psi(d_n) \ge \psi(r) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Then for any $n \in \mathbb{N}$, we have

$$\begin{aligned} d_{n+1} &\le \beta(x_{n-1},x_n)d_n - \alpha(x_{n-1},x_n)\psi(d_n) \\ &\le d_n - \psi(d_n) \\ &\le d_n - \psi(r). \end{aligned}$$

Hence, we can deduce that

$$d_{n+k} \le d_n - k\psi(d_n)$$

which is a contradiction for k large enough. Therefore, we have $r=0$ and thus

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0. \tag{3.10}$$

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. Assume that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon>0$ such that

$$d(x_{m_k},x_{n_k})\geq\varepsilon \tag{3.11}$$

for all $n_k>m_k\geq k$, where $k\in\mathbb{N}$. Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k>m_k\geq k$ satisfying (3.11). Then we have

$$d(x_{m_k},x_{n_k})\geq\varepsilon \text{ and } d(x_{m_k},x_{n_k-1})<\varepsilon. \tag{3.12}$$

By using (3.12) and triangular inequality, we get

$$\begin{aligned}\varepsilon &\leq d(x_{m_k},x_{n_k})\\ &\leq d(x_{m_k},x_{n_k-1})+d(x_{n_k-1},x_{n_k})\\ &\leq \varepsilon+d(x_{n_k-1},x_{n_k}).\end{aligned} \tag{3.13}$$

From (3.10) and (3.13), we have

$$\lim_{k\to\infty} d(x_{m_k},x_{n_k})=\varepsilon. \tag{3.14}$$

Again, by the triangular inequality, we get

$$\begin{aligned}d(x_{m_k},x_{n_k}) &\leq d(x_{m_k+1},x_{n_k+1})+d(x_{m_k},x_{m_k+1})\\ &\quad +d(x_{n_k+1},x_{n_k})\\ &\leq d(x_{m_k+1},x_{m_k})+d(x_{m_k},x_{n_k})+d(x_{n_k},x_{n_k+1})\\ &\quad +d(x_{m_k},x_{m_k+1})+d(x_{n_k+1},x_{n_k})\\ &= 2d(x_{m_k+1},x_{m_k})+d(x_{m_k},x_{n_k})\\ &\quad +2d(x_{n_k},x_{n_k+1}).\end{aligned} \tag{3.15}$$

Using (3.10), (3.14) and (3.15), we get

$$\lim_{k\to\infty} d(x_{m_k+1},x_{n_k+1})=\varepsilon. \tag{3.16}$$

Again, by the triangular inequality, we get

$$\begin{aligned}d(x_{m_k},x_{n_k}) &\leq d(x_{m_k},x_{n_k+1})+d(x_{n_k+1},x_{n_k})\\ &\leq d(x_{m_k},x_{n_k})+d(x_{n_k},x_{n_k+1})+d(x_{n_k+1},x_{n_k})\end{aligned} \tag{3.17}$$

Using (3.10) and (3.17), we obtain that

$$\lim_{k\to\infty} d(x_{m_k},x_{n_k+1})=\varepsilon. \tag{3.18}$$

Similarly, we can prove that

$$\lim_{k\to\infty} d(x_{n_k},x_{m_k+1})=\varepsilon. \tag{3.19}$$

By construction of the sequence $\{x_n\}$, we can conclude that

$$d(x_{m_k+1},Tx_{m_k})=d(A,B)$$

and

$$d(x_{n_k+1},Tx_{n_k})=d(A,B).$$

Since α is forward transitive, β is 0-backward transitive and $n_k>m_k$, we have

$$0<\beta(x_{m_k},x_{n_k})\leq 1\leq\alpha(x_{m_k},x_{n_k}).$$

Using the proximal $(\alpha,\beta)_\psi-$contractive condition type A of T, we get

$$\begin{aligned}d(x_{m_k+1},x_{n_k+1}) &\leq \beta(x_{m_k},x_{n_k})d(x_{m_k},x_{n_k})\\ &\quad -\alpha(x_{m_k},x_{n_k})\psi(d(x_{m_k},x_{n_k}))\\ &\leq d(x_{m_k},x_{n_k})-\psi(d(x_{m_k},x_{n_k}))\end{aligned}$$

for all $k\in\mathbb{N}$. Letting $k\to\infty$ in above inequality, by using (3.14), (3.16), (3.18) and (3.19), we obtain that

$$\varepsilon\leq\varepsilon-\psi(\varepsilon)<\varepsilon$$

which is a contradiction. Then, we deduce that $\{x_n\}$ is a Cauchy sequence. Since A is closed subset of complete metric spaces X, then there exists $x^*\in A$ such that $x_n\to x^*$ as $n\to\infty$. By the continuity of T, we get $Tx_n\to Tx^*$ as $n\to\infty$. Hence

$$d(x^*,Tx^*)=\lim_{k\to\infty} d(x_{n_k+1},Tx_{n_k})=d(A,B),$$

that is x^* is a best proximity point of T.

Finally, we prove that x^* is a unique best proximity point of T. Suppose that y^* is another best proximity point of T. By the assumption, we get $0<\beta(x^*,y^*)\leq 1\leq\alpha(x^*,y^*)$. Then, by the property of ψ, we get

$$\begin{aligned}d(x^*,y^*) &\leq \beta(x^*,y^*)d(x^*,y^*)-\alpha(x^*,y^*)\psi(d(x^*,y^*))\\ &< \beta(x^*,y^*)d(x^*,y^*)\\ &\leq d(x^*,y^*),\end{aligned}$$

which is a contradiction and thus $x^*=y^*$. Therefore x^* is an unique best proximity point of T. □

Now, we introduce new condition in stead the continuity of T for prove the new best proximity point theorem, by assuming the following condition for set A:

$(\mathscr{H}')$: If $\{x_n\}$ is a sequence in A such that

$$0<\beta(x_n,x_{n+1})\leq 1\leq\alpha(x_n,x_{n+1})$$

for all n and $x_n\to x$ as $n\to\infty$ for some $x\in A$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$0<\beta(x_{n_k},x)\leq 1\leq\alpha(x_{n_k},x)$$

for all $k\in\mathbb{N}$.

Theorem 2*Let (X,d) be a complete metric space and A and B be nonempty closed subsets of X such that B is approximatively compact with respect to A and A_0 or B_0 is nonempty set. Suppose that $T:A\to B$ satisfy the following conditions:*

(a)*A satisfy condition* (H')*;*
$(b)$$T$ *is a proximal* $(\alpha,\beta)_\psi$*-contraction type A;*
$(c)$$T$ *is* α*-proximal admissible and* β_0*-proximal subadmissible;*
(d)*there exist element* x_0 *and* x_1 *in* A_0 *such that*

$$d(x_1,Tx_0)=d(A,B) \text{ and } 0<\beta(x_0,x_1)\leq 1\leq \alpha(x_0,x_1);$$

$(e)$$T(A_0)\subseteq B_0$*;*
$(f)$$\alpha$ *is forward transitive and* β *is* 0*-backward transitive.*

Then T *has a best proximity point, that is, there exists a point* $x^*\in A$ *such that*

$$d(x^*,Tx^*)=d(A,B).$$

Moreover, if $0<\beta(x,y)\leq 1\leq \alpha(x,y)$ *for all* $x,y\in Best(T)$*, then* x^* *is a unique best proximity point of* T*.*

Proof. As in the proof of Theorem 1, we can construct the sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1},Tx_n)=d(A,B)$$

and

$$0<\beta(x_n,x_{n+1})\leq 1\leq \alpha(x_n,x_{n+1})$$

for all $n\in\mathbb{N}\cup\{0\}$. Furthermore, we can prove that $\{x_n\}$ is a Cauchy sequence and it converges to some point $x\in A$. Since $T(A_0)\subseteq B_0$ and B is approximatively compact with respect to A, the sequence $\{Tx_n\}$ has a convergent subsequence $\{Tx_{n_k}\}$, that is

$$\lim_{k\to\infty} d(Tx_{n_k},b)=0,$$

for some $b\in B$ and hence

$$d(x,b)=\lim_{k\to\infty} d(x_{n_k+1},Tx_{n_k})=d(A,B)$$

which implies that $x\in A_0$. Thus, $Tx\in B_0$ and then

$$d(u,Tx)=d(A,B) \tag{3.20}$$

for some $u\in A_0$. From, the condition (H') of T, then there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that

$$\beta(x_{n_l},x)\leq 1\leq \alpha(x_{n_l},x)$$

for all $l\in\mathbb{N}$. Using the proximal $(\alpha,\beta)_\psi$-contractive condition type A of T, we get

$$\begin{aligned}d(x_{n_l+1},u)&\leq \beta(x_{n_l},x)d(x_{n_l},x)-\alpha(x_{n_l},x)\psi(d(x_{n_l},x))\\&\leq d(x_{n_l},x)-\psi(d(x_{n_l},x))\end{aligned}$$

for all $l\in\mathbb{N}$. Since ψ is continuous, we get

$$\lim_{l\to\infty} d(x_{n_l+1},u)=0,$$

that is $x_{n_l}\to u$ as $l\to\infty$. By the uniqueness of limit of the sequence $\{x_n\}$, we conclude that $u=x$. From (3.20), we get $d(x,Tx)=d(A,B)$. Therefore, x is a best proximity point of T.

For the uniqueness part of the proof, it follows as in Theorem 1. Then, in order to avoid repetition, the details are omitted. □

Next, we replace the proximal $(\alpha,\beta)_\psi-$contraction mapping type A by proximal $(\alpha,\beta)_\psi-$contraction mapping type B and show that the best proximity point theorems is still hold.

Theorem 3*Let* (X,d) *be a complete metric space and* A *and* B *be nonempty closed subsets of* X *such that* A_0 *or* B_0 *is nonempty set. Suppose that* $T:A\to B$ *satisfy the following conditions:*

$(a)$$T$ *is continuous;*
$(b)$$T$ *is a proximal* $(\alpha,\beta)_\psi$*-contraction type B;*
$(c)$$T$ *is* α*-proximal admissible and* β_0*-proximal subadmissible;*
(d)*there exist element* x_0 *and* x_1 *in* A_0 *such that*

$$d(x_1,Tx_0)=d(A,B) \text{ and } 0<\beta(x_0,x_1)\leq 1\leq \alpha(x_0,x_1);$$

$(e)$$T(A_0)\subseteq B_0$*;*
$(f)$$\alpha$ *is forward transitive and* β *is* 0*-backward transitive.*

Then T *has a best proximity point, that is, there exists a point* $x^*\in A$ *such that*

$$d(x^*,Tx^*)=d(A,B).$$

Moreover, if $0<\beta(x,y)\leq 1\leq \alpha(x,y)$ *for all* $x,y\in Best(T)$*, then* x^* *is a unique best proximity point of* T*.*

Proof. By the same argument as Theorem 1, we can construct a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1},Tx_n)=d(A,B)$$

and

$$0<\beta(x_n,x_{n+1})\leq 1\leq \alpha(x_n,x_{n+1})$$

for all $n\in\mathbb{N}\cup\{0\}$. Sine T is a proximal $(\alpha,\beta)_\psi$-contraction type B, we have

$$\begin{aligned}d(x_n,x_{n+1})&\leq \alpha(x_{n-1},x_n)d(x_n,x_{n+1})\\&\leq \beta(x_{n-1},x_n)d(x_{n-1},x_n)-\psi(d(x_{n-1},x_n))\\&\leq d(x_{n-1},x_n)-\psi(d(x_{n-1},x_n))\\&\leq d(x_{n-1},x_n)\end{aligned}$$

for all $n\in\mathbb{N}$. Putting $d_n:=d(x_{n-1},x_n)$ for all $n\in\mathbb{N}$, then we have

$$0\leq d_{n+1}\leq d_n-\psi(d_n)\leq d_n$$

for all $n\in\mathbb{N}$ and hence $\{d_n\}$ is a nonincreasing sequence and bounded below. Thus, there exists $r\geq 0$ such that

$$\lim_{n\to\infty} d_n=r.$$

Now, let us claim that $r=0$. Suppose $r>0$, from the fact that ψ is an increasing, we obtain that

$$\psi(d_n)\geq\psi(r)>0 \quad \text{for all } n\in\mathbb{N}.$$

Then for any $n\in\mathbb{N}$, we have

$$\begin{aligned} d_{n+1} &\leq \alpha(x_{n-1},x_n)d_{n+1} \\ &\leq \beta(x_{n-1},x_n)d_n-\psi(d_n) \\ &\leq d_n-\psi(d_n) \\ &\leq d_n-\psi(r). \end{aligned}$$

Hence, we can deduce that

$$d_{n+p}\leq d_n-p\psi(d_n)$$

which is a contradiction for positive integer p large enough. Therefore, $r=0$ and thus $\lim_{n\to\infty}d(x_n,x_{n+1})=0$. Next, we will prove that $\{x_n\}$ is a Cauchy sequence. As a same argument in Theorem 1, we have

$$\begin{aligned} \lim_{k\to\infty}d(x_{m_k},x_{n_k}) &= \lim_{k\to\infty}d(x_{m_k},x_{n_k+1}) \\ &= \lim_{k\to\infty}d(x_{n_k},x_{m_k+1}) \\ &= \varepsilon. \end{aligned}$$

and

$$d(x_{m_k},x_{m_k+1})=d(A,B) \text{ and } d(x_{n_k},x_{n_k+1})=d(A,B).$$

Moreover, we get

$$0<\beta(x_{m_k},x_{n_k})\leq 1\leq\alpha(x_{m_k},x_{n_k}).$$

Using the proximal $(\alpha,\beta)_\psi$-contractive condition type B of T, we get

$$\begin{aligned} d(x_{m_k+1},x_{n_k+1}) &\leq \alpha(x_{m_k},x_{n_k})d(x_{m_k+1},x_{n_k+1}) \\ &\leq \beta(x_{m_k},x_{n_k})d(x_{m_k},x_{n_k})-\psi(d(x_{m_k},x_{n_k})) \\ &\leq d(x_{m_k},x_{n_k})-\psi(d(x_{m_k},x_{n_k})). \end{aligned}$$

Taking $k\to\infty$, we obtain that

$$\varepsilon\leq\varepsilon-\psi(\varepsilon)<\varepsilon$$

which is a contradiction. Then, we deduce that $\{x_n\}$ is a Cauchy sequence and converges to some element $x^*\in A$. By the continuity of T, we get $Tx_n\to Tx^*$ as $n\to\infty$. Hence

$$d(x^*,Tx^*)=\lim_{k\to\infty}d(x_{n_k+1},Tx_{n_k})=d(A,B).$$

That is x^* is a best proximity point of T.

Finally, we prove that x^* is a unique best proximity point of T. Suppose that y^* is another best proximity point of T. By the assumption, we get

$$0<\beta(x^*,y^*)\leq 1\leq\alpha(x^*,y^*). \tag{3.21}$$

By above inequality and the property of ψ, we get

$$\begin{aligned} d(x^*,y^*) &\leq \alpha(x^*,y^*)d(x^*,y^*) \\ &\leq \beta(x^*,y^*)d(x^*,y^*)-\psi(d(x^*,y^*)) \\ &< \beta(x^*,y^*)d(x^*,y^*) \\ &\leq d(x^*,y^*), \end{aligned}$$

which is a contradiction and thus $x^*=y^*$. Therefore x^* is an unique best proximity point of T. □

Theorem 4_Let (X,d) be a complete metric space and A and B be nonempty closed subsets of X such that B is approximatively compact with respect to A and A_0 or B_0 is nonempty set. Suppose that $T:A\to B$ satisfy the following conditions:_

- (a)_A satisfy condition (H');_
- (b)_T is a proximal $(\alpha,\beta)_\psi$-contraction type B;_
- (c)_T is α-proximal admissible and β_0-proximal subadmissible;_
- (d)_there exist element x_0 and x_1 in A_0 such that_

$$\begin{gathered} d(x_1,Tx_0)=d(A,B) \text{ and} \\ 0<\beta(x_0,x_1)\leq 1\leq\alpha(x_0,x_1); \end{gathered}$$

- (e)_$T(A_0)\subseteq B_0$;_
- (f)_α is forward transitive and β is 0-backward transitive._

Then T has a best proximity point, that is, there exists a point $x^*\in A$ such that

$$d(x^*,Tx^*)=d(A,B).$$

Moreover, if $0<\beta(x,y)\leq 1\leq\alpha(x,y)$ for all $x,y\in Best(T)$, then x^* is a unique best proximity point of T.

Proof. As in the proof of Theorem 1, we can construct a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1},Tx_n)=d(A,B)$$

and

$$0<\beta(x_n,x_{n+1})\leq 1\leq\alpha(x_n,x_{n+1})$$

for all $n\in\mathbb{N}\cup\{0\}$. Furthermore, we obtain that $\{x_n\}$ is a Cauchy sequence and converges to some point $x\in A_0$ with

$$d(u,Tx)=d(A,B) \tag{3.22}$$

for some $u\in A_0$ (see the proof of Theorem 3). From, the condition (H') of T, then there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that

$$0<\beta(x_{n_l},x)\leq 1\leq\alpha(x_{n_l},x)$$

for all $l\in\mathbb{N}$. By previous equation and the proximal $(\alpha,\beta)_\psi$-contractive condition type B of T, we get

$$\begin{aligned} d(x_{n_l+1},u) &\leq \alpha(x_{n_l},x)d(x_{n_l+1},u) \\ &\leq \beta(x_{n_l},x)d(x_{n_l},x)-\psi(d(x_{n_l},x)) \\ &\leq d(x_{n_l},x)-\psi(d(x_{n_l},x)) \end{aligned}$$

for all $l \in \mathbb{N}$. By the continuity of ψ, we get

$$\lim_{l\to\infty} d(x_{n_l+1}, u) = 0,$$

that is $x_{n_l} \to u$ as $l \to \infty$. Using the uniqueness of limit of the sequence $\{x_n\}$, we conclude that $u = x$. From (3.22), wet get $d(x, Tx) = d(A, B)$.

For the uniqueness part of the proof, it follows as in Theorem 3. Then, in order to avoid repetition, the details are omitted. □

Putting $\alpha(x,y) = 1$ and $\beta(x,y) = 1$ in Theorem 1 or Theorem 3, we get the following result.

Corollary 1 *Let (X,d) be a complete metric space and A and B be nonempty closed subsets of X such that A_0 or B_0 is nonempty set. Suppose that $T : A \to B$ is a continuous generalized proximal contraction of the first kind and $T(A_0) \subseteq B_0$. Then T has a unique best proximity point.*

4 Consequence

In this section, we give the several best proximity point results which are obtained by our results in Section 3.

4.1 Best proximity point results on metric spaces endowed with an arbitrary binary relation

In this subsection, we give the existence of fixed point theorems on a metric space endowed with an arbitrary binary relation. Before presenting our results, we give the following notions and definition.

Definition 11 *Let A and B be nonempty subsets of metric space (X,d) and $\mathscr{R}$ be a binary relation over A. We say that $T : A \to B$ is a* proximal monotone mapping with respect to $\mathscr{R}$ *if the following condition holds:*

$$\left.\begin{array}{l} x\mathscr{R}y, \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{array}\right\} \Longrightarrow u\mathscr{R}v \tag{4.1}$$

for all $x,y,u,v \in A$.

Definition 12 *Let X be a nonempty set and $\mathscr{R}$ be a binary relation over X. We say that X has a* transitive property with respect to $\mathscr{R}$ *if*

$$x,y,z \in X, \quad x\mathscr{R}y \text{ and } y\mathscr{R}z \Longrightarrow x\mathscr{R}z.$$

Definition 13 *Let A and B be nonempty subsets of metric space (X,d) and $\mathscr{R}$ be a binary relation over A. A mapping $T : A \to B$ is said to be a* generalized proximal contraction of the first kind with respect to $\mathscr{R}$ *if, for each $u,v,x,y \in A$, the following condition holds:*

$$\left.\begin{array}{l} x\mathscr{R}y, \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{array}\right\} \Longrightarrow d(u,v) \le d(x,y) - \psi(d(x,y)),$$

where $\psi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing such that $\psi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$.

Theorem 5 *Let (X,d) be a complete metric space and A and B be nonempty closed subsets of X such that A_0 or B_0 is nonempty set. Suppose that $\mathscr{R}$ is a binary relation over A and $T : A \to B$ satisfy the following conditions:*

(A) T is continuous;
(B) T is a generalized proximal contraction of the first kind with respect to $\mathscr{R}$;
(C) T is proximal monotone mapping with respect to $\mathscr{R}$;
(D) there exist element x_0 and x_1 in A_0 such that

$$d(x_1, Tx_0) = d(A,B) \text{ and } x_0\mathscr{R}x_1;$$

(E) $T(A_0) \subseteq B_0$;
(F) A has a transitive property with respect to $\mathscr{R}$.

Then T has a best proximity point, that is, there exists a point $x^* \in A$ such that

$$d(x^*, Tx^*) = d(A,B).$$

Moreover, if $x\mathscr{R}y$ for all $x,y \in Best(T)$, then x^* is a unique best proximity point of T.

Proof. Consider two mappings $\alpha, \beta : A \times A \to [0,\infty)$ defined by

$$\alpha(x,y) = \beta(x,y) = \begin{cases} 1 & \text{if } x\mathscr{R}y; \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

for all $x,y \in A$. From condition (D), we get $\alpha(x_0, Tx_0) = \beta(x_0, Tx_0) = 1$. It follows from T is proximal monotone mapping with respect to $\mathscr{R}$ that T is α-proximal admissible and β_0-proximal subadmissible. Yield to A has a transitive property with respect to $\mathscr{R}$, we get α is forward transitive and β is 0-backward transitive. Since T is a generalized proximal contraction of the first kind with respect to $\mathscr{R}$, T is a proximal $(\alpha,\beta)_\psi-$contraction type A and is also type B. Now all the hypotheses of Theorem 1 (or Theorem 3) are satisfied and thus the existence and uniqueness of the best proximity point of T follows from Theorem 1 (or Theorem 3). □

In order to remove the continuity of T, we need the following condition:

Definition 14 *Let $\mathscr{R}$ be a binary relation over nonempty set X. We say that X satisfy condition $(\mathscr{H}'_{\mathscr{R}})$ if $\{x_n\}$ is a sequence in A such that $x_n\mathscr{R}x_{n+1}$ for all n and $x_n \to x$ as $n \to \infty$ for some $x \in A$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\mathscr{R}x$ for all $k \in \mathbb{N}$.*

Theorem 6*Let (X,d) be a complete metric space and A and B be nonempty closed subsets of X such that B is approximatively compact with respect to A and A_0 or B_0 is nonempty set. Suppose that $\mathscr{R}$ is a binary relation over A and $T : A \to B$ satisfy the following conditions:*

(A)*A satisfies condition $(\mathscr{H}'_{\mathscr{R}})$;*
(B)*T is a generalized proximal contraction of the first kind with respect to $\mathscr{R}$;*
(C)*T is proximal monotone mapping with respect to $\mathscr{R}$;*
(D)*there exist element x_0 and x_1 in A_0 such that*

$$d(x_1,Tx_0) = d(A,B) \text{ and } x_0\mathscr{R}x_1;$$

(E)*$T(A_0) \subseteq B_0$;*
(F)*A has a transitive property with respect to $\mathscr{R}$.*

Then T has a best proximity point, that is, there exists a point $x^* \in A$ such that

$$d(x^*,Tx^*) = d(A,B).$$

Moreover, if $x\mathscr{R}y$ for all $x,y \in Best(T)$, then x^* is a unique best proximity point of T.

Proof. The result follows from Theorem 2 (or Theorem 4) by considering the mappings α and β given by (4.2) and by observing that condition $(\mathscr{H}'_{\mathscr{R}})$ implies condition $(\mathscr{H}')$. □

4.2 Best proximity point results on metric spaces endowed with graph

Throughout this subsection, let A be a nonempty closed subset of a metric space (X,d). A set $\{(x,x) : x \in A\}$ is called a diagonal of the Cartesian product $A \times A$ and is denoted by Δ_A. Consider a graph G_A such that the set $V(G_A)$ of its vertices coincides with A and the set $E(G_A)$ of its edges contains all loops, i.e., $\Delta_A \subseteq E(G_A)$. We assume G_A has no parallel edges, so we can identify G_A with the pair $(V(G_A),E(G_A))$. Moreover, we may treat G_A as a weighted graph by assigning to each edge the distance between its vertices. A graph G_A is connected if there is a path between any two vertices.

In this subsection, we give the existence of best proximity point theorems on a metric space endowed with graph. Before presenting our results, we give the following notions and definitions.

Definition 15*Let (X,d) be a metric space and A and B be two nonempty closed subsets of X endowed with a graph G_A and G_B, respectively, and $T : A \to B$ be mapping. We say that T* proximal preserves edges *if*

$$\left.\begin{array}{l}(x,y) \in E(G_A),\\ d(u,Tx) = d(A,B),\\ d(v,Ty) = d(A,B)\end{array}\right\} \Longrightarrow (Tx,Ty) \in E(G_B) \qquad (4.3)$$

for all $x,y,u,v \in A$.

Definition 16*Let (X,d) be a metric space and A be nonempty closed subset of X endowed with a graph G_A. We say that A has a* transitive property with respect to graph *G_A if*

$$x,y,z \in A, (x,y) \in E(G_A) \text{ and } (y,z) \in E(G_A) \Longrightarrow (x,z) \in E(G_A).$$

Remark 8*It is easy to see that if G_A is connected graph, then A has a transitive property with respect to graph G_A.*

Definition 17*Let (X,d) be a metric space and A and B be two nonempty closed subsets of X endowed with a graph G_A and G_B, respectively. A mapping $T : A \to B$ is said to be a* generalized proximal contraction of the first kind with respect to G_A *if, for each $u,v,x,y \in A$, the following condition holds:*

$$\left.\begin{array}{l}(x,y) \in E(G_A),\\ d(u,Tx) = d(A,B),\\ d(v,Ty) = d(A,B)\end{array}\right\} \Longrightarrow d(u,v) \le d(x,y) - \psi(d(x,y)),$$

where $\psi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing such that $\psi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$.

Theorem 7*Let (X,d) be a complete metric space and A and B be two nonempty closed subsets of X endowed with a graph G_A and G_B, respectively, such that A_0 or B_0 is nonempty set. Suppose that $T : A \to B$ satisfy the following conditions:*

(A)*T is continuous;*
(B)*T is a generalized proximal contraction of the first kind with respect to G_A;*
(C)*T proximal preserves edges;*
(D)*there exist element x_0 and x_1 in A_0 such that*

$$d(x_1,Tx_0) = d(A,B) \text{ and } (x_0,x_1) \in E(G_A);$$

(E)*$T(A_0) \subseteq B_0$;*
(F)*A has transitive property with respect to graph G_A.*

Then T has a best proximity point, that is, there exists a point $x^* \in A$ such that

$$d(x^*,Tx^*) = d(A,B).$$

Moreover, if $(x,y) \in E(G_A)$ for all $x,y \in Best(T)$, then x^* is a unique best proximity point of T.

Proof. Consider two mappings $\alpha,\beta : A \times A \to [0,\infty)$ defined by

$$\alpha(x,y) = \beta(x,y) = \begin{cases} 1 & \text{if } (x,y) \in E(G_A);\\ 0 & \text{otherwise .}\end{cases} \qquad (4.4)$$

From condition (D), we get $\alpha(x_0,Tx_0) = \beta(x_0,Tx_0) = 1$. It follows from T proximal preserves edges that T is α-proximal admissible and β_0-proximal subadmissible. A transitive property with respect to graph G_A yield to α is forward transitive property and β is 0-backward transitive. Since T is a generalized proximal contraction of the first kind with respect to G_A, we get T is a proximal $(\alpha,\beta)_\psi$−contraction type A and is also type B. Therefore, all the hypotheses of Theorem 1 (or Theorem 3) are satisfied. Now the existence and uniqueness of the best proximity point of T follows from Theorem 1 (or Theorem 3). □

In order to remove the continuity of T, we need the following condition:

Definition 18*Let A be a closed subset of a metric space (X,d) such that A endowed with a graph G_A. We say that A has G_A*-regular property *if if $\{x_n\}$ is the sequence in A such that $(x_n,x_{n+1}) \in E(G_A)$ for all $n \in \mathbb{N}$ and it converges to the point $x \in X$, then $(x_n,x) \in E(G)$ for all $n \in \mathbb{N}$.*

Definition 19*Let A be a closed subset of a metric space (X,d) such that A endowed with a graph G_A. We say that A has* weakly G_A-regular property *if if $\{x_n\}$ is the sequence in A such that $(x_n,x_{n+1}) \in E(G_A)$ for all $n \in \mathbb{N}$ and it converges to the point $x \in X$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k},x) \in E(G)$ for all $k \in \mathbb{N}$.*

Remark 9*If A has G_A-regular property, then it also has weakly G_A-regular property. Also, if G_A is connected graph, then A has G_A-regular property.*

Theorem 8*Let (X,d) be a complete metric space and A and B be two nonempty closed subsets of X endowed with a graph G_A and G_B, respectively, such that B is approximatively compact with respect to A and A_0 or B_0 is nonempty set. Suppose that $T : A \to B$ satisfy the following conditions:*

(A)A has weakly G_A-regular property;
(B)T is a generalized proximal contraction of the first kind with respect to G_A;
(C)T proximal preserves edges;
(D)there exist element x_0 and x_1 in A_0 such that

$$d(x_1,Tx_0) = d(A,B) \text{ and } (x_0,x_1) \in E(G_A);$$

$(E)$$T(A_0) \subseteq B_0$;
(F)A has transitive property with respect to graph G_A.

Then T has a best proximity point, that is, there exists a point $x^* \in A$ such that

$$d(x^*,Tx^*) = d(A,B).$$

Moreover, if $(x,y) \in E(G_A)$ for all $x,y \in Best(T)$, then x^* is a unique best proximity point of T.

Proof. The result follows from Theorem 2 (or Theorem 4) by considering the mappings α and β given by (4.4) and by observing that weakly G-regular property implies property $(\mathscr{H}')$. □

Using Remark 9, we get the following result:

Corollary 2*Let (X,d) be a complete metric space and A and B be two nonempty closed subsets of X endowed with a graph G_A and G_B, respectively, such that B is approximatively compact with respect to A and A_0 or B_0 is nonempty set. Suppose that $T : A \to B$ satisfy the following conditions:*

(A)A has G_A-regular property;
(B)T is a generalized proximal contraction of the first kind with respect to G_A;
(C)T proximal preserves edges;
(D)there exist element x_0 and x_1 in A_0 such that

$$d(x_1,Tx_0) = d(A,B) \text{ and } (x_0,x_1) \in E(G_A);$$

$(E)$$T(A_0) \subseteq B_0$;
(F)A has transitive property with respect to graph G_A.

Then T has a best proximity point, that is, there exists a point $x^* \in A$ such that

$$d(x^*,Tx^*) = d(A,B).$$

Moreover, if $(x,y) \in E(G_A)$ for all $x,y \in Best(T)$, then x^* is a unique best proximity point of T.

Corollary 3*Let (X,d) be a complete metric space and A and B be two nonempty closed subsets of X endowed with a graph G_A and G_B, respectively, such that B is approximatively compact with respect to A and A_0 or B_0 is nonempty set. Suppose that $T : A \to B$ satisfy the following conditions:*

$(A)$$G_A$ is connected graph;
(B)T is a generalized proximal contraction of the first kind with respect to G_A;
(C)T proximal preserves edges;
(D)there exist element x_0 and x_1 in A_0 such that

$$d(x_1,Tx_0) = d(A,B) \text{ and } (x_0,x_1) \in E(G_A);$$

$(E)$$T(A_0) \subseteq B_0$;
(F)A has transitive property with respect to graph G_A.

Then T has a best proximity point, that is, there exists a point $x^* \in A$ such that

$$d(x^*,Tx^*) = d(A,B).$$

Moreover, if $(x,y) \in E(G_A)$ for all $x,y \in Best(T)$, then x^* is a unique best proximity point of T.

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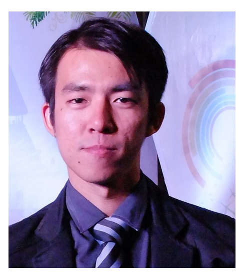

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