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Existence of Positive Solutions for Boundary Value Problem of Nonlinear Fractional Differential Equation

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Abstract: In this paper, we study a boundary value problem of nonlinear fractional differential equation. Existence and positivity results of solutions are obtained by using the fixed-point index theorems. Three examples are given to show the effectiveness of our works.

Keywords: positive solution, fractional Caputo derivative, Cone, fixed point index.

1 Introduction

Fractional calculus has played a significant role in engineering, science, economy, and other fields. Recently, a great number of papers and books on fractional calculus, fractional differential equations have appeared. For details, see [3,7,8,13,18,21,22] and references therein. Moreover, lots of works have appeared, in which fractional derivatives have been used for a better description considering material properties, mathematical modelling based on enhanced rheological models, which naturally leads to the differential equations of fractional order and to the necessity of the formulation of initial conditions to these equations.

In fact, the use of cone theoretic techniques in the study of solutions to boundary value problems has a rich and varied history. For example, some authors have used fixed point theorems to show the existence of positive solutions to boundary value problems for ordinary differential equations, difference equations, and dynamic equations on time scales; see for example [5,9,25,27,28]. However, in other papers, [14,15,26], some authors have used fixed point theory to show the existence of solutions to singular boundary value problems.

In [13], the author considers the existence and multiplicity of the positive solutions of nonlinear fractional differential equation boundary-value problem

$$\lambda Lx = -g(t)\, f(t,x)\,, 0 \le t \le 2\pi,$$
$$x(0) = x(2\pi)\,, x'(0) = x'(2\pi)\,,$$

where $\lambda > 0$ is a parameter, $Lx = x'' - \rho^2 x, \rho > 0$ is a constant. In addition, $f \in C \in ([0,\infty[\times[0,\infty[,[0,\infty[)$ and $g \in L^p[0,2\pi]$ for some $1 \le p \le +\infty$.

Bing Liu [2] has studied the existence of at least one or two positive solutions to the three points boundary value problem

$$y''(t) + a(t)\, f(y(t)) = 0, 0 < t < 1,$$
$$y(0) = 0, y(1) = \beta y(\eta)\,,$$

where $0 < \eta < 1, 0 < \beta < \frac{1}{\eta}$.

In the present paper, we apply topological degree theory combined with partially ordered structure of space to establish the existence and multiplicity of positive solutions to the boundary value problem (P_1)

$$^{c}D^{q}_{0^+} u(t) + a(t)\, f(u(t)) = 0, 0 < t < 1,$$

$$u(0) = u''(0) = 0, u(1) = 0.$$

where $f \in C([0,\infty[,[0,\infty[)$, $a(t) \in C([0,1],[0,\infty[)$ and $2 < q < 3$.

Thus, this work is organized in the following fashion: Section 2 provides some necessary background. In

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particular, it introduces some lemmas and definitions associated with topological degree theory and partially ordered structure of space. Section 3 states and proves main results. Finally, in the last section we give some examples to illustrate the previous results.

2 Preliminaries

In this section, to achieve completeness, we demonstrate and study the definitions and as some basic facts of Caputo's derivatives of fractional order which can be found in [18].

Definition 2.1. [18] The Riemann -Liouville fractional integral of order $\alpha > 0$ of a function $g \in C([a,b]$ is defined by

$$I_{a^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}} ds.$$

Definition 2.2. [18] The caputo fractional derivative of order $\alpha > 0$ of
$g \in AC^n[a,b]$ is defined by

$$^cD_{a^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$ ($[\alpha]$ is the entire part of α).

Lemma 2.3. [18] Let $\alpha, \beta > 0$ and $n = [\alpha] + 1$, then the following relations hold:

$$^cD_{a^+}^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}, \beta > n,$$

and

$$^cD_{a^+}^{\alpha} t^k = 0, k = 0, 1, 2, ..., n-1.$$

Lemma 2.4. [18] Assume that $u \in C^n[a,b]$.Then

$$I_{a^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 + c_2 t + c_3 t^2 + ... + c_n t^{n-1},$$

where, $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n$, and $n = [\alpha] + 1$.

Denote by $L^1([0,1], \mathbb{R})$ the Banach space of lesbegue integrable functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|Y\|_{L^1} = \int_0^1 |Y(t)| dt$.

Lemma 2.5. [18] Let $p, q \geq 0$, $f \in L_1([a,b]$. Then

$$I_{0^+}^p I_{0^+}^q f(t) = I_{0^+}^{p+q} f(t) = I_{0^+}^q I_{0^+}^p f(t),$$

and

$$^cD_{a^+}^q I_{0^+}^q f(t) = f(t), \forall t \in [a,b].$$

Lemma 2.6. [18] Let $\beta > \alpha > 0$, $f \in L_1([a,b]$. Then for all $t \in [a,b]$ we have

$$^cD_{a^+}^{\alpha} I_{0^+}^{\beta} f(t) = I_{0^+}^{\beta-\alpha} f(t).$$

Lemma 2.7. Given $y \in C([0,1])$ and $2 < q < 3$, the unique solution of the fractional problem (P_0)

$$\begin{cases} ^cD_{0^+}^q u(t) + y(t) = 0, \ 0 < t < 1 \\ u(0) = u''(0) = 0, u(1) = 0, \end{cases}$$

is given by

$$u(t) = \frac{1}{\Gamma(q)} \int_0^1 G(t,s) y(s) ds,$$

where

$$G(t,s) = \begin{cases} t(1-s)^{q-1} - (t-s)^{q-1}, 0 \leq s \leq t, \\ t(1-s)^{q-1}, t \leq s \leq 1. \end{cases} \tag{1}$$

Proof. Using Lemmas 2.3 and 2.4 we have

$$u(t) = -I_{0^+}^q y(t) + C + Bt + At^2,$$

from the conditions $u(0) = u''(0) = 0$, we obtain $C = A = 0$, and the condition $u(1) = 0$ implies

$$B = \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} y(s) ds,$$

so $u(t)$ can be written as

$$u(t) = -I_{0^+}^q y(t) + \frac{1}{\Gamma(q)} \int_0^1 t(1-s)^{q-1} y(s) ds,$$

where G is defined by (1). The proof is completed.□.

Lemma 2.8. For all $s, t \in [0,1]$, the Green function $G(t,s)$ is non negative, continuous and satisfies

$i) G(t,s) \leq (1-s)^{q-1}$,

$ii) \min_{t \in [\tau_1, \tau_2]} G(t,s) \geq \tau_1 \left(1 - (\tau_2)^{q-2}\right)(1-s)^{q-1}$ where $0 < \tau_1 < \tau_2 < 1$.

Proof. From the expression of G(t, s), it is evident that $G(t,s) \in C([0,1] \times [0,1))$ and $G(t,s) \geq 0$ for $s, t \in [0,1]$. Next, we prove (ii). From the definition of G(t, s), we known that, for a given $s \in [0,1]$, G(t, s) is increasing with respect to t for $t \leq s$, then, let

$$g_1(t,s) = t(1-s)^{q-1} - (t-s)^{q-1}, s \leq t,$$

$$g_2(t,s) = t(1-s)^{q-1}, t \leq s.$$

That is, $g_1(t,s)$ is a continuous function for $\tau_1 \leq t \leq \tau_2$, and $g_2(t,s)$ is increasing with respect to t. Hence, we have

$$g_1(t,s) \geq \tau_1 \left(1 - (\tau_2)^{q-2}\right)(1-s)^{q-1}, t \in [\tau_1, \tau_2]$$

$$\min_{t \in [\tau_1, \tau_2]} g_2(t,s) = \tau_1 (1-s)^{q-1} \geq \tau_1 \left(1 - (\tau_2)^{q-2}\right)(1-s)^{q-1}.$$

Therefore,

$$\min_{t\in[\tau_1,\tau_2]} G(t,s) \geq \tau_1 \left(1-(\tau_2)^{q-2}\right)(1-s)^{q-1}.$$

The proof is completed.$\square$.

Lemma 2.9. The solution of problem (P_1) satisfies

$$\min_{t\in[\tau_1,\tau_2]} u(t) \geq \tau_1 \left(1-(\tau_2)^{q-2}\right)\|u\|.$$

Proof. From Lemma 2.7, u can be expressed by

$$u(t) = \frac{1}{\Gamma(q)}\int_0^1 G(t,s)a(s)f(u(s))ds$$
$$\leq \frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}a(s)f(u(s))ds$$

then

$$\|u\| = \max_{0\leq t\leq 1}|u(t)| = \max_{0\leq t\leq 1}\frac{1}{\Gamma(q)}\int_0^1 G(t,s)a(s)f(u(s))ds$$
$$\leq \frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}a(s)f(u(s))ds.$$

Also, we have

$$u(t) \geq \frac{t\left(1-t^{q-2}\right)}{\Gamma(q)}\int_0^1 (1-s)^{q-1}a(s)f(u(s))ds$$
$$\geq t\left(1-t^{q-2}\right)\|u\|,$$

therefore

$$\min_{t\in[\tau_1,\tau_2]} u(t) \geq \tau_1 \left(1-(\tau_2)^{q-2}\right)\|u\|.$$

$\square$.

3 Main results

Let $E = C([0,1],\mathbb{R})$ be the Banach space of all continuous real functions on $[0,1]$ endowed with the norm

$$\|u\| = \max_{t\in[0,1]}|u(t)|.$$

Define the integral operator $T : E \to E$ by

$$Tu(t) = \frac{1}{\Gamma(q)}\int_0^1 G(t,s)a(s)f(u(s))ds, \forall t\in[0,1].$$

We define some important constants:

$f_0 = \lim_{u\to 0^+}\frac{f(u)}{u}$, $f_\infty = \lim_{u\to\infty}\frac{f(u)}{u}$, and $\delta = \tau_1\left(1-(\tau_2)^{q-2}\right)$. Denote

$$P = \left\{u\in E, u(t)\geq 0, 0\leq t\leq 1, \min_{t\in[\tau_1,\tau_2]} u(t)\geq \delta\|u\|\right\}.$$

It is obvious that P is a cone. Moreover, from Lemma 2.9, $T(P)\subset P$, it is also easy to see that $T : E\to E$ is completely continuous.

In what follows, for the sake of convenience, set

$$\alpha = \frac{\Gamma(q)}{\int_0^1 a(s)(1-s)^{q-1}ds}, \beta = \frac{\Gamma(q)}{\delta^2\int_{\tau_1}^{\tau_2} a(s)(1-s)^{q-1}ds}.$$

Theorem 3.1. Let K be a closed convex set in a Banach space X and let D be a bounded open set such that $D_K = D\cap K \neq \emptyset$.Let $T : \overline{D}_k \to K$ be a compact map.

Suppose that $x \neq T(x)$ for all $x\in\partial D_k$.

(P1) (Solution property) If $i(T,D_k)\neq 0$, then T has a fixed point in D_k.

(P2) (Normality) If $u\in D_k$,then $i(\overline{u},D_k) = 1$, where $\overline{u}(x) = u$ for $x\in\overline{D}_k$.

(P3) (Additivity) If V_1, V_2 are disjoint relatively open subsets of D_k such that $x\neq T(x)$ for $x\in\overline{D}_k\setminus(V_1\cup V_2)$, then

$$i(T,D_k) = i(T,V_1) + i(T,V_2).$$

From these properties, one can have the following consequences.

Theorem 3.2. Let K be a cone in a real Banach space X. Let D be an open bounded subset of X with $D_K = D\cap K\neq\emptyset$, and $\overline{D}_k\neq K$. Assume that $A : \overline{D}_k\to K$ is completely continuous such that $x\neq T(x)$ for $x\in\partial D_k$. Then the following results hold:

(1) If $\|Ax\|\leq\|x\|$, $x\in\partial D_k$, then $i(A,D_k) = 1$.

(2) If there exists $e\in K\setminus\{0\}$ such that $x\neq Ax+\lambda e$ for all $x\in\partial D_k$ and all $\lambda > 0$, then $i(A,D_k) = 0$.

(3) Let U be open in K such that $\overline{U}\subset D_k$. If $i(A,D_k) = 1$ and $i(A,U_k) = 0$, then A has a fixed point in $D_k\setminus\overline{U}_k$. The same results hold if $i(A,D_k) = 0$ and $i(A,U_k) = 1$.

Theorem 3.3. Let E be a Banach space, and let $K\subset E$, be a cone in E. Let $r > 0$, and define $\Omega_r = \{x\in K \mid \|x\| < r\}$. Assume $A : \overline{\Omega_r}\to K$ is a completely continuous operator such that $Ax\neq x$ for $x\in\partial\Omega_r$.

i)If $\|Au\|\leq\|u\|$, $u\in\partial\Omega_r$, then

$$i(A,\Omega_r,K) = 1.$$

ii)If $\|Au\|\geq\|u\|$, $u\in\partial\Omega_r$, then

$$i(A,\Omega_r,K) = 0.$$

Theorem 3.4. Assume that $\int_{\tau_1}^{\tau_2}(1-s)^{q-1}a(s)ds\neq 0$ and the following assumptions are satisfied.

(H_1) $f_0 = f_\infty = \infty$.

(H_2) There exist constants $r > 0$ and $A\in\left]0,\alpha\right[$ such that

$$f(u)\leq Ar, u\in[0,r].$$

Then, the boundary value problem (P_1) has at least two positive solutions y_1 and y_2 such that

$$0 < \|y_1\| < r < \|y_2\|.$$

Proof. At first, in view of $f_0 = \infty$, then for any $A_1\in\left]\beta,\infty\right[$, there exist $r_1\in\left]0,r\right[$ such that

$$f(u)\geq A_1 u, u\in[0,r_1]$$

Let $\Omega_{r_1} = \{u \in P : \|u\| < r_1\}$. Since $u \in \partial\Omega_{r_1} \subset P$, we have $\min\limits_{\tau_1 \le t \le \tau_2} u(t) \ge \delta \|u\|$. Thus, for any $u \in \partial\Omega_{r_1}$, we have

$$\begin{aligned} Tu(t) &= \frac{1}{\Gamma(q)} \int_0^1 G(t,s)a(s) f(u(s))ds \\ &\ge \frac{A_1}{\Gamma(q)} \int_0^1 a(s) G(t,s) u(s) ds \\ &\ge \frac{A_1}{\Gamma(q)} \int_{\tau_1}^{\tau_2} a(s) G(t,s) u(s) ds \\ &\ge \frac{A_1 \delta^2}{\Gamma(q)} \|u\| \int_{\tau_1}^{\tau_2} a(s) (1-s)^{q-1} ds, \end{aligned}$$

which yields

$$Tu(t) \ge \|u\|, u \in \partial\Omega_{r_1}.$$

Hence, Theorem 3.3 implies

$$i(T, \Omega_{r_1}, P) = 0.$$

On the other hand, since $f_\infty = \infty$, we deduce that for any $A_2 \in]\beta, \infty[$, there exists $r_2 > r$ such that

$$f(u) \ge A_2 u, u \ge \delta r_2.$$

Let $\Omega_{r_2} = \{u \in P : \|u\| < r_2\}$. Since $u \in \partial\Omega_{r_2} \subset P$, we have $\min\limits_{\tau_1 \le t \le \tau_2} u(t) \ge \delta \|u\| = \delta r_2$, and hence for any $u \in \partial\Omega_{r_2}$, we can obtain

$$\begin{aligned} Tu(t) &= \frac{1}{\Gamma(q)} \int_0^1 G(t,s)a(s) f(u(s))ds \\ &\ge \frac{1}{\Gamma(q)} \int_{\tau_1}^{\tau_2} G(t,s)a(s) f(u(s))ds \\ &\ge \frac{A_2}{\Gamma(q)} \int_{\tau_1}^{\tau_2} a(s) G(t,s) u(s) ds \\ &\ge \frac{A_2 \delta^2}{\Gamma(q)} \|u\| \int_{\tau_1}^{\tau_2} a(s) (1-s)^{q-1} ds, \end{aligned}$$

which implies

$$Tu(t) \ge \|u\|, for u \in \partial\Omega_{r_2}.$$

Thus Theorem 3.3 yields

$$i(T, \Omega_{r_2}, P) = 0.$$

Let $\Omega_r = \{u \in P : \|u\| < r\}$. For any $u \in \partial\Omega_r$, we have

$$\begin{aligned} Tu(t) &\le \frac{Ar}{\Gamma(q)} \int_0^1 a(s) G(t,s) ds \\ &\le \frac{Ar}{\Gamma(q)} \int_0^1 a(s) (1-s)^{q-1} ds \\ &\le r = \|u\|, \end{aligned}$$

therefore, $\|Tu\| \le \|u\|$ for any $u \in \partial\Omega_r$.
From Theorem 3.3, we obtain

$$i(T, \Omega_r, P) = 1.$$

Hence, since $r_1 < r < r_2$ it follows from the additivity of the fixed point index that

$$\begin{aligned} &i\left(T, \Omega_r \setminus \overline{\Omega}_{r_1}, P\right) = 1, \\ &i\left(T, \Omega_{r_2} \setminus \overline{\Omega}_r, P\right) = -1. \end{aligned}$$

Therefore, T has a fixed point y_1 in $\Omega_r / \overline{\Omega}_{r_1}$ and a fixed point y_2 in $\Omega_{r_2} / \overline{\Omega}_r$ and $0 < \|y_1\| < r < \|y_2\|$.The proof is complete.□.

Theorem 3.5. Assumes that $\int_{\tau_1}^{\tau_2} (1-s)^{q-1} a(s) ds \ne 0$ and the following assumptions are satisfied.

(H_3) $f_0 = 0$.

(H_4) There exist constants $\rho > 0$ and $B \in]\beta, \infty[$ such that

$$f(u) \ge B\rho, u \in [\delta\rho, \rho].$$

Then, the boundary value problem (P_1) has at least one positive solutions y_1

$$0 < \|y_1\| < \rho.$$

Proof. At first, in view of $f_0 = 0$, then for any $\varepsilon \in]0, \alpha[$, there exists $r_1 \in]0, \rho[$ such that

$$f(u) \le \varepsilon u, u \in [0, r_1].$$

Letting $\Omega_{r_1} = \{u \in P : \|u\| < r_1\}$. For any $u \in \partial\Omega_{r_1}$ we have

$$\begin{aligned} Tu(t) &= \frac{1}{\Gamma(q)} \int_0^1 G(t,s)a(s) f(u(s))ds \\ &\le \frac{\varepsilon}{\Gamma(q)} \int_0^1 a(s) G(t,s) u(s) ds \\ &\le \frac{\varepsilon}{\Gamma(q)} \|u\| \int_0^1 a(s) (1-s)^{q-1} ds \\ &\le \|u\|, \end{aligned}$$

which yields

$$Tu(t) \le \|u\|, u \in \partial\Omega_{r_1}.$$

Thus, Theorem 3.3 implies

$$i(T, \Omega_{r_1}, P) = 1.$$

Let $\Omega_\rho = \{u \in P : \|u\| < \rho\}$. Since $u \in \partial\Omega_\rho \subset P$, we have $\min\limits_{\tau_1 \le t \le \tau_2} u(t) \ge \delta \|u\| = \delta\rho$, and hence for any $u \in \partial\Omega_\rho$, we obtain

$$\begin{aligned} Tu(t) &= \frac{1}{\Gamma(q)} \int_0^1 G(t,s)a(s) f(u(s))ds \\ &\ge \frac{1}{\Gamma(q)} \int_{\tau_1}^{\tau_2} G(t,s)a(s) f(u(s))ds \\ &\ge \frac{B\rho}{\Gamma(q)} \int_{\tau_1}^{\tau_2} a(s) G(t,s) ds \\ &\ge \frac{B\rho\delta^2}{\Gamma(q)} \int_{\tau_1}^{\tau_2} a(s) (1-s)^{q-1} ds \\ &\ge \rho = \|u\|, \end{aligned}$$

which implies

$$Tu(t) \geq \|u\|, foru \in \partial\Omega_\rho.$$

Thus, Theorem 3.3 yields

$$i\left(T,\Omega_\rho,P\right) = 0.$$

Hence, since $r_1 < \rho$, it follows from the additivity of the fixed point index that

$$i\left(T,\Omega_\rho\diagdown\overline{\Omega}_{r_1},P\right) = -1.$$

Therefore, T has a fixed point y_1 in $\Omega_\rho\diagdown\overline{\Omega}_{r_1}$.□.

Theorem 3.6. Assume that (H_2) and (H_4) hold and that $r \neq \rho$. Then the boundary value problem (P_1) has at least one positive solution y satisfying $r < \|y\| < \rho$ or $\rho < \|y\| < r$.

Proof. Without losing the generality, we may assume that $r < \rho$. Let $\Omega_r = \{u \in P : \|u\| < r\}$. For any $u \in \partial\Omega_r$ and from (H_2), we get

$$\begin{aligned}
Tu(t) &\leq \frac{Ar}{\Gamma(q)}\int_0^1 a(s)G(t,s)ds \\
&\leq \frac{Ar}{\Gamma(q)}\int_0^1 a(s)(1-s)^{q-1}ds \\
&\leq r = \|u\|,
\end{aligned}$$

hence, $\|Tu\| \leq \|u\|$ for any $u \in \partial\Omega_r$.

Thus, an application of Theorem 3.3 again shows that

$$i(T,\Omega_r,P) = 1. \tag{2}$$

Finally, let $\Omega_\rho = \{u \in P : \|u\| < \rho\}$. For $u \in \partial\Omega_\rho$, since $u \in P$, $\min\limits_{\tau_1 \leq t \leq \tau_2} u(t) \geq \delta\|u\| = \delta\rho$, and hence for any $u \in \partial\Omega_\rho$ and from (H_4) we can obtain

$$\begin{aligned}
Tu(t) &= \frac{1}{\Gamma(q)}\int_0^1 G(t,s)a(s)f(u(s))ds \\
&\geq \frac{1}{\Gamma(q)}\int_{\tau_1}^{\tau_2} G(t,s)a(s)f(u(s))ds \\
&\geq \frac{B\rho}{\Gamma(q)}\int_{\tau_1}^{\tau_2} a(s)G(t,s)ds \\
&\geq \frac{B\rho\delta^2}{\Gamma(q)}\int_{\tau_1}^{\tau_2} a(s)(1-s)^{q-1}ds \\
&\geq \rho = \|u\|,
\end{aligned}$$

which implies $Tu(t) \geq \|u\|$, for $u \in \partial\Omega_\rho$. Thus,

$$i\left(T,\Omega_\rho,P\right) = 0. \tag{3}$$

Combining (2) and (3) gives

$$i\left(T,\Omega_\rho\diagdown\overline{\Omega}_r,P\right) = -1.$$

Consequently, T has at least one fixed point on $\Omega_\rho\diagdown\overline{\Omega}_r$. Thus (P_1) has at least one positive solution and

$$r < \|y\| < \rho.$$

This completes the proof.□.

4 Illustrative Examples

Example 4.1. Let us consider the fractional boundary value problem

$${}^cD_{0^+}^{\frac{11}{4}}u(t) + (1-t)\left(\exp(u) + u^2\right) = 0, 0 < t < 1,$$

$$u(0) = u''(0) = 0, u(1) = 0.$$

here $q = \frac{11}{4}, a(t) = 1-t$,

$$f(u) = \exp(u) + u^2.$$

It is easy to see that $f_0 = f_\infty = \infty$, then (H_1) holds.

We have $(q+1)\Gamma(q) = 9.559$, because $f(u)$ is monotone increasing function for $u \geq 0$, taking $r = \frac{1}{2}, A = 2\exp\left(\frac{1}{2}\right) + \frac{1}{2} \in]0, 9.559[$,then we get for u$\in [0,r]$

$$f(u) \leq f\left(\tfrac{1}{2}\right) = \exp\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{2}\left(2\exp\left(\tfrac{1}{2}\right) + \tfrac{1}{2}\right) = Ar.$$

Hence (H_2) holds. Applying Theorem 3.4, we deduce that there exist at least two positive solutions y_1 and y_2 such that

$$0 < \|y_1\| < \frac{1}{2} < \|y_2\|.$$

Example 4.2. Consider the following fractional boundary value problem

$${}^cD_{0^+}^{\frac{9}{4}}u(t) + 10^{35}u^{\frac{9}{8}}\exp(-u)(1-t) = 0, 0 < t < 1,$$

$$u(0) = u''(0) = 0, u(1) = 0.$$

where $q = \frac{9}{4}, f(u) = 10^{35}u^{\frac{9}{8}}\exp(-u), a(t) = 1-t, \delta = \frac{1}{4}\left(1-\left(\frac{3}{4}\right)^{q-2}\right) = 0.017, \beta = 33394, \Gamma\left(\frac{9}{4}\right) = 1.133.$

It is easy to check that (H_3) holds. Since $f(u)$ is monotone decreasing function for $u \geq \frac{9}{8}$, for $\rho = 70, B = \frac{4732800}{70} \in]33394, \infty[$, we have

$$f(u) \geq f(70) = 4732800 = B\rho, u \in [\delta\rho, \rho],$$

therefore, (H_4) holds. Applying Theorem 3.5 we deduce that there exist at least one positive solutions y_1

$$0 < \|y_1\| < \rho.$$

Example 4.3. Consider the same problem as in example 4.2. Since $f(u)$ is monotone increasing function for $u \in \left[0, \frac{9}{8}\right]$, taking $r = 10^{-290}, A = 0.0562 \in]0, 3.682[$, then we get for $u \in [0,r]$;

$$f(u) \leq f\left(10^{-290}\right) = Ar,$$

hence (H_2) holds.

We have $f(u)$ is monotone decreasing function for $u \geq \frac{9}{8}$, for $\rho = 70, B = \frac{4732800}{70} \in]4732800, \infty[$, we have for $u \in [\delta\rho, \rho]$:

$f(u) \geq f(70) = 4732800 = B\rho,$

hence (H_4) holds. Then, from Theorem 3.6 the boundary value problem (P_1) has at least one positive solution y satisfying $r < \|y\| < \rho$ or $\rho < \|y\| < r$.

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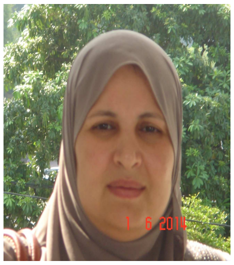

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