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A General Dynamic Inequality of Opial Type

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Abstract: We present a new general dynamic inequality of Opial type. This inequality is new even in both the continuous and discrete cases. The inequality is proved by making use of a recently introduced new technique for Opial dynamic inequalities, the time scales integration by parts formula, the time scales chain rule, and classical as well as time scales versions of Hölder's inequality.

Keywords: Opial's inequality, Hölder's inequality, time scales.

1 Introduction

In 1960, Olech [8] extended an inequality of Opial [9] and proved that if $f \in \mathrm{C}^1([0,h],\mathbb{R})$ with $h>0$ satisfies $f(0)=0$, then

$$\int_0^h \left|f(t)f'(t)\right|\mathrm{d}t \le \frac{h}{2}\int_0^h \left|f'(t)\right|^2\mathrm{d}t. \tag{1}$$

This inequality created a lot of research activity, which was summarized in the monograph [2], both for the continuous and the discrete cases. In [3] (see also [6, Theorem 6.23]), the authors extended (1) to an arbitrary time scale $\mathbb{T}$ and proved that if $f \in \mathrm{C}^1_{\mathrm{rd}}([0,h]_{\mathbb{T}},\mathbb{R})$ with $h>0$ satisfies $f(0)=0$, then

$$\int_0^h \left|\left(f^2\right)^{\Delta}(t)\right|\Delta t \le h\int_0^h \left(f^{\Delta}(t)\right)^2\Delta t. \tag{2}$$

For extensions and generalizations of (2), we refer the reader to the monograph [1]. Over the last sixty years, the study of Opial inequalities (continuous and discrete) or related Hardy operators focused on the investigations of new inequalities or operators with weighted functions. These inequalities have natural applications in applied mathematics, especially in the theory of differential equations in elasticity (ordinary or partial) and led to many interesting questions and connections between different areas of mathematical analysis. For example, Hardy operators are closely related to quasiadditivity properties of capacities and were recently used with Opial-type inequalities to find the gaps between zeros of differential equations that appear in the binding of beams [10].

Here we will not give an introduction to time scales calculus but instead refer the reader to [6, 7]. We only remark that the delta derivative is the usual derivative if $\mathbb{T}=\mathbb{R}$ and the forward difference if $\mathbb{T}=\mathbb{Z}$, and the delta integral is the usual integral if $\mathbb{T}=\mathbb{R}$ and a sum if $\mathbb{T}=\mathbb{Z}$, and that the theory can be applied to any nonempty closed set $\mathbb{T}\subset\mathbb{R}$, the so-called underlying time scale. We note that plugging $\mathbb{T}=\mathbb{R}$ in (2) results in (1).

Using a novel technique in [4], the following generalization of (2) was established, involving two different weight functions s and r, see [4, Theorem 5.2].

Theorem 1. *Assume that $a\in\mathbb{T}$, $b\in(a,\infty)_{\mathbb{T}}$,*

$$r,s\in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}},(0,\infty)), \quad \textit{and} \quad f\in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R}).$$

If $f(a)=0$, then

$$\int_a^b s(t)\left|(f^2)^{\Delta}(t)\right|\Delta t \le K\int_a^b r(t)\left(f^{\Delta}(t)\right)^2\Delta t,$$

where

$$K=\sqrt{\int_a^b s^2(t)(R^2)^{\Delta}(t)\Delta t} \quad \textit{with} \quad R(t)=\int_a^t \frac{\Delta\tau}{r(\tau)}.$$

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We note that plugging $a=0$, $b=h$, and $r=s=1$ in Theorem 1 results in (2).

Refining the technique from [4], the same authors proved in [5] the following generalization of Theorem 1.

Theorem 2. *Assume that $a\in\mathbb{T}$, $b\in(a,\infty)_{\mathbb{T}}$,*

$$r,s\in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}},(0,\infty)),\quad \textit{and}\quad f\in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R}).$$

Let $\alpha>1$ and $\beta\geq 0$. If $f(a)=0$, then

$$\int_a^b s(t)\left|(f^\alpha)^\Delta(t)(f^\Delta(t))^\beta\right|\Delta t\leq K\int_a^b r(t)\left|f^\Delta(t)\right|^{\alpha+\beta}\Delta t,$$

where

$$K=\frac{\alpha(\beta+1)^{\frac{\beta+1}{\alpha+\beta}}}{\alpha+\beta}\left\{\int_a^b\frac{(s(t))^{\frac{\alpha+\beta}{\alpha-1}}(R^{\alpha+\beta})^\Delta(t)}{(r(t))^{\frac{\beta(\alpha+\beta)}{(\alpha-1)(\alpha+\beta-1)}}}\Delta t\right\}^{\frac{\alpha-1}{\alpha+\beta}}$$

with

$$R(t)=\int_a^t\frac{\Delta\tau}{(r(\tau))^{\frac{1}{\alpha+\beta-1}}}.$$

We note that plugging $\alpha=2$ and $\beta=0$ in Theorem 2 results in Theorem 1.

The purpose of this paper is to apply the new technique that was developed in [4,5] in order to prove the following generalization of Theorem 2.

Theorem 3. *Assume that $a\in\mathbb{T}$, $b\in(a,\infty)_{\mathbb{T}}$,*

$$r,s\in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}},(0,\infty)),\quad \textit{and}\quad f\in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R}).$$

Let $\alpha\geq 1$, $\beta\geq 0$, and $k>\beta+1$. If $f(a)=0$, then

$$\int_a^b s(t)\left|(f^\alpha)^\Delta(t)(f^\Delta(t))^\beta\right|\Delta t \leq K\left\{\int_a^b r(t)\left|f^\Delta(t)\right|^k\Delta t\right\}^{\frac{\alpha+\beta}{k}},$$

where

$$K=c\left\{\int_a^b(s(t))^{\frac{k}{k-\beta-1}}\frac{\left(R^{\frac{k\alpha-\alpha-\beta}{k-\beta-1}}\right)^\Delta(t)}{(r(t))^{\frac{k\beta}{(k-1)(k-\beta-1)}}}\Delta t\right\}^{\frac{k-\beta-1}{k}}$$

with

$$c=\alpha\left(\frac{k-\beta-1}{k\alpha-\alpha-\beta}\right)^{\frac{k-\beta-1}{k}}\left(\frac{\beta+1}{\alpha+\beta}\right)^{\frac{\beta+1}{k}}$$

and

$$R(t)=\int_a^t\frac{\Delta\tau}{(r(\tau))^{\frac{1}{k-1}}}.$$

We note that plugging $k=\alpha+\beta$ in Theorem 3 results in Theorem 2.

The paper is organized as follows: In Section 2, we present the basic definitions of time scales calculus that will be used in the sequel. In Section 3, we prove Theorem 3 and give some remarks. We prove our main result by using the time scales chain rule, the time scales integration by parts formula, and classical continuous and discrete as well as time scales versions of Hölder's inequality.

2 Time Scales Preliminaries

In this section, we briefly present some basic definitions and results concerning the delta calculus on time scales that we will use in this article. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. We define the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ by $\sigma(t):=\inf\{s\in\mathbb{T}: s>t\}$ for $t\in\mathbb{T}$. For any function $f:\mathbb{T}\to\mathbb{R}$, we put $f^\sigma=f\circ\sigma$. A function $f:\mathbb{T}\to\mathbb{R}$ is called rd-continuous, denoted by $f\in \mathrm{C}_{\mathrm{rd}}$, if it is continuous at each right-dense point (i.e., $\sigma(t)=t$) and there exists a finite left-sided limit at all left-dense points (i.e., $\rho(t)=t$, where the backward jump ρ is defined in a similar way as the forward jump σ). For the definition of the delta derivative and the delta integral, we refer to [6,7]. If $f\in \mathrm{C}^1(\mathbb{R},\mathbb{R})$ and $g:\mathbb{T}\to\mathbb{R}$ is delta differentiable, then the *time scales chain rule*, see [6, Theorem 1.90], states that

$$(f\circ g)^\Delta=g^\Delta\int_0^1 f'(hg^\sigma+(1-h)g^\Delta)\mathrm{d}h,$$

and a special case, which we will use in this paper, is given by

$$(f^\gamma)^\Delta=\gamma f^\Delta\int_0^1(hf^\sigma+(1-h)f)^{\gamma-1}\mathrm{d}h\quad\text{for}\quad\gamma\in\mathbb{R}.$$

The *time scales Hölder inequality*, see [6, Theorem 6.13], says

$$\int_a^b|f(t)g(t)|\Delta t\leq\left\{\int_a^b|f(t)|^\gamma\Delta t\right\}^{\frac{1}{\gamma}}\left\{\int_a^b|g(t)|^\nu\Delta t\right\}^{\frac{1}{\nu}},$$

where $a,b\in\mathbb{T}$, $f,g\in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R})$, $\gamma>1$, and $\nu=\gamma/(\gamma-1)$.

3 Proof of the Opial Inequality

In this section, we present the proof of our main result, Theorem 3, and give some corollaries and concluding remarks.

Proof. Define

$$g(t):=\int_a^t r(\tau)\left|f^\Delta(\tau)\right|^k\Delta\tau.$$

Then $g(a)=0$,

$$g^{\Delta}=r\left|f^{\Delta}\right|^{k}\quad\text{so that}\quad\left|f^{\Delta}\right|=\left(\frac{g^{\Delta}}{r}\right)^{\frac{1}{k}},$$

and

$$\begin{aligned}|f(t)|&=\left|\int_a^t\frac{1}{(r(\tau))^{\frac1k}}(r(\tau))^{\frac1k}f^{\Delta}(\tau)\Delta\tau\right|\\&\le\int_a^t\frac{1}{(r(\tau))^{\frac1k}}(r(\tau))^{\frac1k}\left|f^{\Delta}(\tau)\right|\Delta\tau\\&\le\left\{\int_a^t\frac{\Delta\tau}{(r(\tau))^{\frac{1}{k-1}}}\right\}^{\frac{k-1}{k}}\left\{\int_a^t r(\tau)\left|f^{\Delta}(\tau)\right|^k\Delta\tau\right\}^{\frac1k}\\&=(R(t))^{\frac{k-1}{k}}(g(t))^{\frac1k},\end{aligned}$$

where we have used the time scales Hölder inequality with conjugate exponents $\frac{k}{k-1}$ and $k>1$. Thus, for $h\in[0,1]$, we obtain

$$\begin{aligned}&|hf^{\sigma}+(1-h)f|\le h|f^{\sigma}|+(1-h)|f|\\&\le h(R^{\sigma})^{\frac{k-1}{k}}(g^{\sigma})^{\frac1k}+(1-h)R^{\frac{k-1}{k}}g^{\frac1k}\\&=(hR^{\sigma})^{\frac{k-1}{k}}(hg^{\sigma})^{\frac1k}+((1-h)R)^{\frac{k-1}{k}}((1-h)g)^{\frac1k}\\&\le(hR^{\sigma}+(1-h)R)^{\frac{k-1}{k}}(hg^{\sigma}+(1-h)g)^{\frac1k},\end{aligned}$$

where we have used the classical Hölder inequality for sums with conjugate exponents $\frac{k}{k-1}$ and $k>1$. Hence

$$\begin{aligned}&\left|\int_0^1(hf^{\sigma}+(1-h)f)^{\alpha-1}\mathrm{d}h\right|\\&\le\int_0^1|hf^{\sigma}+(1-h)f|^{\alpha-1}\mathrm{d}h\\&\le\int_0^1(hR^{\sigma}+(1-h)R)^{\frac{(k-1)(\alpha-1)}{k}}(hg^{\sigma}+(1-h)g)^{\frac{\alpha-1}{k}}\mathrm{d}h\\&\le\left\{\int_0^1(hR^{\sigma}+(1-h)R)^{\frac{(k-1)(\alpha-1)}{k-\beta-1}}\mathrm{d}h\right\}^{\frac{k-\beta-1}{k}}\\&\quad\times\left\{\int_0^1(hg^{\sigma}+(1-h)g)^{\frac{\alpha-1}{\beta+1}}\mathrm{d}h\right\}^{\frac{\beta+1}{k}},\end{aligned}$$

where we have used the classical Hölder inequality for integrals with conjugate exponents $\frac{k}{k-\beta-1}$ and $\frac{k}{\beta+1}>1$. Therefore, using the time scales chain rule three times, we get

$$\begin{aligned}&\left|(f^{\alpha})^{\Delta}(f^{\Delta})^{\beta}\right|=\alpha\left|f^{\Delta}\right|^{\beta+1}\left|\int_0^1(hf^{\sigma}+(1-h)f)^{\alpha-1}\mathrm{d}h\right|\\&=\frac{\alpha(g^{\Delta})^{\frac{\beta+1}{k}}}{r^{\frac{\beta+1}{k}}}\left|\int_0^1(hf^{\sigma}+(1-h)f)^{\alpha-1}\mathrm{d}h\right|\\&\le\frac{\alpha(g^{\Delta})^{\frac{\beta+1}{k}}}{r^{\frac{\beta+1}{k}}}\left\{\int_0^1(hR^{\sigma}+(1-h)R)^{\frac{(k-1)(\alpha-1)}{k-\beta-1}}\mathrm{d}h\right\}^{\frac{k-\beta-1}{k}}\\&\quad\times\left\{\int_0^1(hg^{\sigma}+(1-h)g)^{\frac{\alpha-1}{\beta+1}}\mathrm{d}h\right\}^{\frac{\beta+1}{k}}\\&=\frac{c}{r^{\frac{\beta}{k-1}}}\left\{\frac{k\alpha-\alpha-\beta}{k-\beta-1}R^{\Delta}\right.\\&\qquad\left.\times\int_0^1(hR^{\sigma}+(1-h)R)^{\frac{k\alpha-\alpha-\beta}{k-\beta-1}-1}\mathrm{d}h\right\}^{\frac{k-\beta-1}{k}}\\&\quad\times\left\{\frac{\alpha+\beta}{\beta+1}g^{\Delta}\int_0^1(hg^{\sigma}+(1-h)g)^{\frac{\alpha+\beta}{\beta+1}-1}\mathrm{d}h\right\}^{\frac{\beta+1}{k}}\\&=\frac{c}{r^{\frac{\beta}{k-1}}}\left\{\left(R^{\frac{k\alpha-\alpha-\beta}{k-\beta-1}}\right)^{\Delta}\right\}^{\frac{k-\beta-1}{k}}\left\{\left(g^{\frac{\alpha+\beta}{\beta+1}}\right)^{\Delta}\right\}^{\frac{\beta+1}{k}},\end{aligned}$$

and thus finally

$$\begin{aligned}&\int_a^b s(t)\left|(f^{\alpha})^{\Delta}(t)\left(f^{\Delta}(t)\right)^{\beta}\right|\Delta t\\&\le c\int_a^b s(t)\frac{\left\{\left(R^{\frac{k\alpha-\alpha-\beta}{k-\beta-1}}\right)^{\Delta}(t)\right\}^{\frac{k-\beta-1}{k}}}{(r(t))^{\frac{\beta}{k-1}}}\\&\qquad\times\left\{\left(g^{\frac{\alpha+\beta}{\beta+1}}\right)^{\Delta}(t)\right\}^{\frac{\beta+1}{k}}\Delta t\\&\le c\left\{\int_a^b(s(t))^{\frac{k}{k-\beta-1}}\frac{\left(R^{\frac{k\alpha-\alpha-\beta}{k-\beta-1}}\right)^{\Delta}(t)}{(r(t))^{\frac{k\beta}{(k-\beta-1)(k-1)}}}\Delta t\right\}^{\frac{k-\beta-1}{k}}\\&\quad\times\left\{\int_a^b\left(g^{\frac{\alpha+\beta}{\beta+1}}\right)^{\Delta}(t)\Delta t\right\}^{\frac{\beta+1}{k}}\\&=K\left\{g^{\frac{\alpha+\beta}{\beta+1}}(b)\right\}^{\frac{\beta+1}{k}}\\&=K(g(b))^{\frac{\alpha+\beta}{k}},\end{aligned}$$

where we have used one last time the time scales Hölder inequality with conjugate exponents $\frac{k}{k-\beta-1}$ and $\frac{k}{\beta+1}>1$. The proof is complete.

The next result follows from Theorem 3 by choosing $\beta=0$.

Corollary 1. *Assume that $a\in\mathbb{T}$, $b\in(a,\infty)_{\mathbb{T}}$,*

$$r,s\in\mathrm{C_{rd}}([a,b]_{\mathbb{T}},(0,\infty)),\quad\textit{and}\quad f\in\mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R}).$$

Let $\alpha\ge1$ and $k>1$. If $f(a)=0$, then

$$\int_a^b s(t)\left|(f^{\alpha})^{\Delta}(t)\right|\Delta t\le K\left\{\int_a^b r(t)\left|f^{\Delta}(t)\right|^k\Delta t\right\}^{\frac{\alpha}{k}},$$

where

$$K=\left\{\int_a^b(s(t))^{\frac{k}{k-1}}(R^{\alpha})^{\Delta}(t)\Delta t\right\}^{\frac{k-1}{k}}$$

with

$$R(t)=\int_a^t \frac{\Delta\tau}{(r(\tau))^{\frac{1}{k-1}}}.$$

The next result follows from Corollary 1 by choosing $k=\alpha$ (see also [5, Corollary 3.2]).

Corollary 2. *Assume that* $a\in\mathbb{T}$, $b\in(a,\infty)_{\mathbb{T}}$,

$$r,s\in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}},(0,\infty)),\quad \textit{and}\quad f\in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R}).$$

Let $\alpha>1$. *If* $f(a)=0$, *then*

$$\int_a^b s(t)\left|(f^{\alpha})^{\Delta}(t)\right|\Delta t\le K\int_a^b r(t)\left|f^{\Delta}(t)\right|^{\alpha}\Delta t,$$

where

$$K=\left\{\int_a^b (s(t))^{\frac{\alpha}{\alpha-1}}\left(R^{\alpha}\right)^{\Delta}(t)\Delta t\right\}^{\frac{\alpha-1}{\alpha}}$$

with

$$R(t)=\int_a^t \frac{\Delta\tau}{(r(\tau))^{\frac{1}{\alpha-1}}}.$$

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