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A Descent Resolvent Method for Mixed Quasi Variational Inequalities

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Abstract: In this paper, we suggest and analyze a new iterative method for solving mixed quasi variational inequalities. The new iteration is obtained by searching the optimal step size along the integrated descent direction from two descent directions. Global convergence of the proposal method is proved under certain assumptions. Our results can be treated as refinement of previously known results. An example is given to illustrate the efficiency of the proposed method.

Keywords: Mixed quasi variational inequalities, self-adaptive rules, resolvent operator, co-coercive operator.
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1 Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let K be a closed convex set in H and $T : H \to H$ be a nonlinear operator. Let $\varphi(.,.) : H\times H \longrightarrow R\cup\{+\infty\}$ be a continuous bifunction. We consider the problem of finding $u \in H$ such that

$$\langle Tu, v-u\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in H. \tag{1}$$

Problem (1) is called the mixed quasi variational inequality. Such type of mixed quasi variational inequalities arise in the study of elasticity with non-local friction laws, fluid flow through porous media and structural analysis. For the finite element analysis, existence results and applications, see [19,21].
If the bifunction $\varphi(.,.)$ is a proper, convex and lower semicontinuous function with respect to the first argument, then problem (1) is equivalent to finding $u \in H$ such that

$$0 \in Tu + \partial\varphi(u,u), \tag{2}$$

which is known as finding the zero of the sum of monotone operators.

For $\varphi(v,u) = \varphi(v), \forall u \in H$, problem (1) reduces to finding $u \in H$ such that

$$\langle Tu, v-u\rangle + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in H, \tag{3}$$

which is called the mixed variational inequality or variational inequality of the second kind, see [1,13,14,15,17,18,19,21].

If $\varphi(.,.) = \varphi(.)$ is an indicator function of a closed convex set K in H, then problem (1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geq 0, \quad \forall v \in K, \tag{4}$$

which is known as the classical variational inequality introduced and studied by Stampacchia [26] in 1964.
Due to the presence of the nonlinear bifunction, the projection method and its variant forms including the Wiener-Hopf equations technique can not be extended to suggest iterative methods for solving mixed quasi variational inequalities (1). To overcome these drawbacks, some iterative methods have been suggested for special cases of the mixed quasi variational inequalities. For example, if the bifunction is proper, convex and lower semicontinuous function with respect to the first argument, then one can show that the mixed quasi variational inequalities are equivalent to the fixed-point problems and the implicit resolvent equations using the resolvent operator technique. This equivalent formulation has been used to suggest and analyze some iterative methods. Several modified resolvent methods have been suggested and developed for solving mixed variational

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inequalities. For recent development of the subject, we refer to [[2]-[11], [19]-[25]].

Inspired by the above cited works, we propose a descent resolvent method for solving mixed quasi variational inequality, the new iterate is obtained along a new descent direction. The new direction is obtained by combining two descent directions. Global convergence of the proposed method is proved under certain assumptions. To illustrate the proposed method and demonstrate its efficiency, some applications and their numerical results are also provided. Our results can be viewed as significant extensions of the previously known results.

2 Preliminaries

In this section, we recall some basic definitions and results, which will be frequently used in our later analysis.

Definition 2.1. The mapping $T : \Omega \subset H \to H$ is said to be (a) monotone over a set Ω if

$$\langle T(x) - T(y), x - y\rangle \geq 0, \qquad \forall x, y \in \Omega; \tag{5}$$

(b) strongly monotone over Ω if there exists an $\alpha > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq \alpha\|x - y\|^2, \qquad \forall x, y \in \Omega; \tag{6}$$

(c) co-coercive over Ω if there exists a $c > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq c\|T(x) - T(y)\|^2, \qquad \forall x, y \in \Omega; \tag{7}$$

(d) Lipschitz continuous over Ω if there exists an $L > 0$ such that

$$\|T(x) - T(y)\| \leq \|x - y\|^2, \qquad \forall x, y \in \Omega. \tag{8}$$

It is clear from Definition 2.1 that co-coercive mappings are monotone but not necessarily strongly monotone. Conversely, strongly monotone and Lipschitz continuous mapping are co-coercive. This shows that co-coercivity is a weaker condition than strongly monotonicity.

Definition 2.2. The bifunction $\varphi(.,.)$ is said to be *skew-symmetric*, if,

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) \geq 0, \quad \forall u, v \in H. \tag{9}$$

Clearly, if the bifunction $\varphi(.,.)$ is linear in both arguments, then,

$$\begin{aligned}\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) &= \varphi(u-v, u-v)\\ &\geq 0, \quad \forall u, v \in H,\end{aligned}$$

which shows that the bifunction $\varphi(.,.)$ is nonnegative.

Definition 2.3.[12] Let A be a maximal monotone operator, then the resolvent operator associated with A is defined as

$$J_A(u) = (I + \rho A)^{-1}(u), \quad \forall u \in H,$$

where $\rho > 0$ is a constant and I is the identity operator.

Remark 2.1. It is well known that the subdifferential $\partial\varphi(.,.)$ of a convex, proper and lower-semicontinuous function $\varphi(.,.) : H \times H \longrightarrow R \cup \{+\infty\}$ is a maximal monotone with respect to the first argument, we can define its resolvent by

$$J_{\varphi(u)} = (I + \rho\partial\varphi(.,u))^{-1}) = (I + \rho\partial\varphi(u))^{-1}, \tag{10}$$

where $\partial\varphi(u) = \partial\varphi(.,u)$, unless otherwise specified.

The resolvent operator $J_{\varphi(u)}$defined by (10) has the following characterization,

Lemma 2.1.[23] For a given $u \in H$, $z \in H$ satisfies the inequality

$$\langle u - z, v - u\rangle + \rho\varphi(v,u) - \rho\varphi(u,u) \geq 0, \quad \forall v \in H, \tag{11}$$

if and only if

$$u = J_{\varphi(u)}[z],$$

where $J_{\varphi(u)}$ is resolvent operator defined by (10).
It follows from Lemma 2.1 that

$$\langle J_{\varphi(u)}[z] - z, v - J_{\varphi(u)}[z]\rangle + \rho\varphi(v, J_{\varphi(u)}[z]) - \rho\varphi(J_{\varphi(u)}[z], J_{\varphi(u)}[z]) \geq 0, \quad \forall u, v, z \in H \tag{12}$$

]

The following result can be proved by using Lemma 2.1.

Lemma 2.2. u^* is solution of problem (1) if and only if $u^* \in H$ satisfies the relation:

$$u^* = J_{\varphi(u^*)}[u^* - \rho T(u^*)], \tag{13}$$

] where $\rho > 0$.

From Lemma 2.2, it is clear that u is solution of (1) if and only if u is a zero point of the function

$$r(u,\rho) := u - J_{\varphi(u)}[u - \rho T(u)].$$

The following lemma shows that $\|r(u,\rho)\|$ is a non-decreasing function, while $\dfrac{\|r(u,\rho)\|}{\rho}$ is a non-increasing one with respect to ρ.

Lemma 2.3.[5] For all $u \in H$ and $\rho' \geq \rho > 0$, it holds that

$$\|r(u,\rho')\| \geq \|r(u,\rho)\| \tag{14}$$

] and

$$\frac{\|r(u,\rho')\|}{\rho'} \leq \frac{\|r(u,\rho)\|}{\rho}. \tag{15}$$

]

Throughout this paper, we make following assumptions.

Assumptions:

– H is a finite dimension space.
– T is continuous and co-coercive with modulus $c > 0$ on H.
– The bifunction $\varphi(.,.)$ is skew-symmetric.
– The solution set of problem (1) denoted by S^* is nonempty.

3 The proposed method and some properties

In this section, we suggest and analyze the new descent resolvent method for solving mixed quasi variational inequality (1). To describe our method, we need ρ_k satisfies

$$0 < \rho_l \leq \inf_{k=0}^{\infty} \rho_k \leq \sup_{k=0}^{\infty} \rho_k \leq \rho_u < 4c.$$

Algorithm 3.1

Step 0. Given $\varepsilon > 0, \gamma \in [1,2), \mu \in (0,1), \sigma \in (0,1), \delta \in (0,1), \delta_0 \in (0,1)$ and $u^0 \in H$, set $k = 0$.

Step 1. Set $\rho_k = \rho$. If $\|r(u^k,\rho)\| < \varepsilon$, then stop; otherwise, find the smallest no-negative integer m_k, such that $\rho_k = \rho\mu^{m_k}$ satisfying

$$\|\rho_k(T(u^k) - T(w^k))\| \leq \delta\|r(u^k,\rho_k)\|, \tag{16}$$

] where

$$w^k = J_{\varphi(u^k)}[u^k - \rho_k T(u^k)].$$

Step 2. For each $u^* \in S^*$, choose a direction d_k satisfying the following inequality

$$\langle d_k, u^k - u^* \rangle \geq ||r(u^k,\rho_k)||^2 - \rho_k\langle r(u^k,\rho_k), T(u^k) - T(w^k)\rangle. \tag{17}$$

] Compute

$$D_k = (1-\sigma)r(u^k,\rho_k) + \sigma d_k. \tag{18}$$

] Step 3. Get the next iterate

$$u^{k+1} = u^k - \gamma\alpha_k D_k$$

where

$$\alpha_k = \beta\frac{||r(u^k,\rho_k)||^2}{||D_k||^2}$$

and

$$\beta = (1-\sigma)(1-\frac{\rho_k}{4c}) + \sigma(1-\delta).$$

Step 4. If

$$\|\rho_k(T(u^k) - T(w^k))\| \leq \delta_0\|r(u^k,\rho_k)\|,$$

then set $\rho = \frac{\rho_k}{\mu}$, else set $\rho = \rho_k$. Set $k := k+1$, and go to Step 1.

If $\varphi(v,u) = \varphi(v), \forall u \in H$, and φ is an indicator function of a closed convex set K in H, then $J_\varphi \equiv P_K$ [20], the projection of H onto K. Consequently Algorithm 3.1 reduces to Algorithm 3.2 for solving variational inequalities (4).

Algorithm 3.2

Step 0. Given $\varepsilon > 0, \gamma \in [1,2), \mu \in (0,1), \sigma \in (0,1), \delta \in (0,1), \delta_0 \in (0,1)$ and $u^0 \in K$, set $k = 0$.

Step 1. Set $\rho_k = \rho$. If $\|r(u^k,\rho)\| < \varepsilon$, then stop; otherwise, find the smallest no-negative integer m_k, such that $\rho_k = \rho\mu^{m_k}$ satisfying

$$\|\rho_k(T(u^k) - T(w^k))\| \leq \delta\|r(u^k,\rho_k)\|,$$

where

$$w^k = P_K[u^k - \rho_k T(u^k)].$$

Step 2. For each $u^* \in S^*$, choose a direction d_k satisfying the following inequality

$$\langle d_k, u^k - u^* \rangle \geq ||r(u^k,\rho_k)||^2 - \rho_k\langle r(u^k,\rho_k), T(u^k) - T(w^k)\rangle.$$

Compute

$$D_k = (1-\sigma)r(u^k,\rho_k) + \sigma d_k.$$

Step 3. Get the next iterate

$$u^{k+1} = u^k - \gamma\alpha_k D_k.$$

where

$$\alpha_k = \beta\frac{||r(u^k,\rho_k)||^2}{||D_k||^2}$$

and

$$\beta = (1-\sigma)(1-\frac{\rho_k}{4c}) + \sigma(1-\delta).$$

Step 4. If

$$\|\rho_k(T(u^k) - T(w^k))\| \leq \delta_0\|r(u^k,\rho_k)\|,$$

then set $\rho = \frac{\rho_k}{\mu}$, else set $\rho = \rho_k$. Set $k := k+1$, and go to Step 1.

Lemma 3.1. Let $u^* \in S^*$ and $\forall u^k \in H$, we have

$$\langle r(u^k,\rho_k), u^k - u^* \rangle \geq (1-\frac{\rho_k}{4c})||r(u^k,\rho_k)||^2. \tag{19}$$

Proof: Substituting $z = u^k - \rho_k T(u^k)$ and $v = u^*$ into (12), and using the definition of $r(u^k,\rho_k)$, we get

$$\langle r(u^k,\rho_k) - \rho_k T(u^k), w^k - u^* \rangle + \rho_k\varphi(u^*,w^k) - \rho_k\varphi(w^k,w^k) \geq 0. \tag{20}$$

] From (1) we have

$$\langle \rho_k T(u^*), w^k - u^* \rangle + \rho_k\varphi(w^k,u^*) - \rho_k\varphi(u^*,u^*) \geq 0. \tag{21}$$

] Adding (20) and (21), and using the skew-symmetry of the bifunction $\varphi(.,.)$, we have

$$\langle r(u^k,\rho_k) - \rho_k[T(u^k) - T(u^*)], w^k - u^* \rangle \geq 0$$

which can be rewritten as

$$\langle r(u^k,\rho_k) - \rho_k[T(u^k) - T(u^*)], u^k - u^* - r(u^k,\rho_k) \rangle \geq 0.$$

Using the co-coercivity of T, we get

$$\begin{aligned}\langle u^k-u^*, r(u^k,\rho_k)\rangle &\geq \|r(u^k,\rho_k)\|^2-\rho_k\langle T(u^k)-T(u^*),\\ &\quad r(u^k,\rho_k)\rangle+\rho_k\langle T(u^k)-T(u^*),u^k-u^*\rangle\\ &\geq \|r(u^k,\rho_k)\|^2-\rho_k\langle T(u^k)-T(u^*),\\ &\quad r(u^k,\rho_k)\rangle+\rho_k c\|T(u^k)-T(u^*)\|^2\\ &= \|r(u^k,\rho_k)\|^2+\|\sqrt{\rho_k c}(T(u^k)-\\ &\quad T(u^*))-\frac{1}{2}\sqrt{\frac{\rho_k}{c}}r(u^k,\rho_k)\|^2\\ &\quad -\frac{\rho_k}{4c}\|r(u^k,\rho_k)\|^2\\ &\geq (1-\frac{\rho_k}{4c})\|r(u^k,\rho_k)\|^2.\end{aligned}$$

Hence, (19) holds and the proof is completed. □

Lemma 3.2. Let $u^* \in S^*$ and $\forall u^k \in H$, then we have

$$\langle D_k, u^k-u^*\rangle \geq \beta\|r(u^k,\rho_k)\|^2. \tag{22}$$

Proof: Using the definition of D_k, Lemma 3.1, (16) and (17). For any solution $u^* \in S^*$, we have

$$\begin{aligned}\langle D_k,u^k-u^*\rangle &= \langle(1-\sigma)r(u^k,\rho_k)+\sigma d_k, u^k-u^*\rangle\\ &= (1-\sigma)\langle r(u^k,\rho_k),u^k-u^*\rangle+\sigma\langle d_k,u^k-u^*\rangle\\ &\geq (1-\sigma)(1-\frac{\rho_k}{4c})\|r(u^k,\rho_k)\|^2+\sigma\|r(u^k,\rho_k)\|^2\\ &\quad -\sigma\langle r(u^k,\rho_k),T(u^k)-T(w^k)\rangle\\ &\geq (1-\sigma)(1-\frac{\rho_k}{4c})\|r(u^k,\rho_k)\|^2+\sigma\|r(u^k,\rho_k)\|^2\\ &\quad -\delta\sigma\|r(u^k,\rho_k)\|^2\\ &= [(1-\sigma)(1-\frac{\rho_k}{4c})+\sigma(1-\delta)]\|r(u^k,\rho_k)\|^2.\end{aligned}$$

Using the definition of β, we get the assertion of this lemma. □

Remark 3.1.
-Lemma 3.2 shows that $-D_k$ is a descent direction at x^k for the merit function $\frac{1}{2}\|x-x^*\|^2$.
-At iteration k, the two directions

$$d_k^1 = r(u^k,\rho_k)+\rho_k T(w^k)$$

and

$$d_k^2 = r(u^k,\rho_k)+\rho_k[T(w^k)-T(u^k)]$$

satisfied (17). For the proof, see Lemma 3.3 in [3] for d_k^1 and Lemma 3.2 in [22] for d_k^2.

4 Convergence analysis

In this section, we prove the global convergence of the proposed method. The following theorem plays a crucial role in the convergence of the proposed method.

Theorem 4.1 Let $u^* \in S^*$ and u^{k+1} be the sequence obtained from algorithm 3.1. Then u^k is bounded and

$$\|u^{k+1}-u^*\|^2 \leq \|u^k-u^*\|^2-\gamma(2-\gamma)\beta^2\frac{\|r(u^k,\rho_k)\|^4}{\|D_k\|^2}. \tag{23}$$

Proof: Let $u^* \in H$ be a solution of problem (1), then

$$\begin{aligned}\|u^{k+1}-u^*\|^2 &= \|u^k-u^*-\alpha_k\gamma D_k\|^2\\ &= \|u^k-u^*\|^2+\alpha_k^2\gamma^2\|D_k\|^2-2\alpha_k\gamma\langle u^k-u^*,D_k\rangle\\ &\leq \|u^k-u^*\|^2+\alpha_k^2\gamma^2\|D_k\|^2-2\alpha_k\gamma\beta\|r(u^k,\rho_k)\|^2\\ &= \|u^k-u^*\|^2-\gamma(2-\gamma)\beta\alpha_k\|r(u^k,\rho_k)\|^2.\end{aligned}$$

Then

$$\|u^{k+1}-u^*\|^2 \leq \|u^k-u^*\|^2-\gamma(2-\gamma)\beta^2\frac{\|r(u^k,\rho_k)\|^4}{\|D_k\|^2}.$$

Since $\gamma\in[1,2)$, we have

$$\|u^{k+1}-u^*\| \leq \|u^k-u^*\| \leq \ldots \leq \|u^0-u^*\|.$$

Then, the sequence u^k is bounded. □

Now, the convergence of the proposed method could be proved as follows

Theorem 4.2. The sequence u^k generated by the proposed method converges to a solution point of problem (1).

Proof: It follows from (23) that

$$\sum_{k=0}^{\infty}\frac{\|r(u^k,\rho_k)\|^4}{\|D_k\|^2} < \infty$$

which means that

$$\lim_{k\to\infty}\|r(u^k,\rho_k)\| = 0, \tag{24}$$

] and it follows from Lemma 2.3 that

$$\min\{1,\rho_k\}\|r(u^k,1)\| \leq \|r(u^k,\rho_k)\|. \tag{25}$$

] Combining (24) and (25), we get

$$\lim_{k\to\infty}\rho_k\|r(u^k,1)\| = 0. \tag{26}$$

] We have two possible cases. Firstly, suppose that

$$\lim_{k\to\infty}\sup\rho_k > 0.$$

It follows from (26) that

$$\lim_{k\to\infty}\inf\|r(u^k,1)\| = 0.$$

Since $\{u^k\}$ is bounded, it has a cluster point $\bar{u}$ such that $\|r(\bar{u},1)\| = 0$, which implies $\bar{u}$ is a solution of problem (1).

Now, we consider the second possible case

$$\lim_{k\to\infty}\rho_k = 0.$$

By the choice of ρ_k we know that (16) was not satisfied for m_k-1. Then for k large enough such that $\rho_k<\mu$, we obtain

$$\begin{aligned}&\|T(u^k)-T(J_{\varphi(u^k)}[u^k-(\rho_k/\mu)T(u^k)])\|\\ &\geq \delta\mu\|r(u^k,\rho_k/\mu)\|/\rho_k\\ &\geq \delta\|r(u^k,1)\|\end{aligned}$$

where the second inequality follows from Lemma 2.3.
Let $\bar{u}$ be a cluster point of $\{u^k\}$ and the subsequence $\{u^{k_j}\}$ converges to $\bar{u}$. Then, we have

$$\begin{aligned}\|r(\bar{u},1)\| &= \lim_{j\to\infty}\|r(u^{k_j},1)\| \\ &\le \lim_{j\to\infty}\frac{\|T(u^{k_j})-T(J_{\varphi(u^{k_j})}[u^{k_j}-(\rho_{k_j}/\mu)T(u^{k_j})])\|}{\delta} \\ &= 0,\end{aligned}$$

which means that $\bar{u}$ is a solution of problem (1).
In the following, we prove that the sequence $\{u^k\}$ has exactly one cluster point. Assume that $\tilde{u}$ is another cluster point and satisfies

$$\tau := \|\tilde{u}-\bar{u}\| > 0.$$

Since $\bar{u}$ is a cluster point of the sequence $\{u^k\}$, there is a $k_0 > 0$ such that

$$\|u^{k_0}-\bar{u}\| \le \frac{\tau}{2}.$$

On the other hand, since $\bar{u}\in S^*$ and from (23), we have

$$\|u^k-\bar{u}\| \le \|u^{k_0}-\bar{u}\| \quad \text{for all} \quad k\ge k_0,$$

it follows that

$$\|u^k-\tilde{u}\| \ge \|\tilde{u}-\bar{u}\| - \|u^k-\bar{u}\| \ge \frac{\tau}{2} \quad \forall k\ge k_0.$$

This contradicts the assumption that $\tilde{u}$ is cluster point of $\{u^k\}$, thus the sequence $\{u^k\}$ converges to $\bar{u}\in S^*$. □

5 Preliminary Computational Results

In the section, we give some numerical results for the proposed method. We consider the nonlinear complementarity problems
Find $u\in R^n$ such that

$$u\ge 0, \quad T(u)\ge 0, \quad \langle u,T(u)\rangle = 0, \tag{27}$$

] where $T(u) = D(u)+Mu+q$, $D(u)$ and $Mu+q$ are the nonlinear part and linear parts of $T(u)$ respectively. Problem (27) is a special case of problem (1), by taking

$$\varphi(v,u) = \begin{cases} 0, & \text{if} \quad v\in R^n_+; \\ +\infty, & \text{otherwise}. \end{cases}$$

In this case Algorithm 3.1 collapses to Algorithm 3.2.

We form the test problems similarly as in Harker and Pang [16]. The matrix $M = A^TA+B$, where A is an $n\times n$ matrix whose entries are randomly generated in the interval $(-5,+5)$ and a skew-symmetric matrix B is generated in the same way. The vector q is generated from a uniform distribution in the interval $(-500,500)$ (easy problems) and $(-500,0)$ (hard problems), respectively. In $D(u)$, the nonlinear part of $T(u)$, the components are $D_j(u) = d_j * \arctan(u_j)$ and d_j is a random variable in $(0,1)$.

In all tests we took $\mu = 2/3$, $\delta = 0.95, c = 0.9$, $\delta_0 = 0.2$, $\gamma = 1.95$ and $d_k = r(u^k,\rho_k)+\rho_k[T(w^k)-T(u^k)]$, the starting point $u^0 = (0,\dots,0)^T$. All codes are written in Matlab. The computation begins with $\rho_0 = 1$ and stops as soon as $\|r(u^k,\rho_k)\|_\infty \le 10^{-7}$. The test results for easy problems ($q\in(-500,500)$) and hard problems ($q\in(-500,0)$) are reported in tables 1-2.

Table 1 Numerical results for easy problems

	Method in [22]		Algorithm 3.2	
n	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)
200	53	0.03	20	0.04
300	46	0.04	21	0.06
500	24	0.11	23	0.31
700	41	0.15	23	0.44

Table 2 Numerical results for hard problems

	Method in [22]		Algorithm 3.2	
n	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)
200	85	0.04	23	0.09
300	85	0.08	26	0.13
500	31	0.37	23	0.56
700	64	0.81	70	1.05

From Tables 1-2, we can see that our Algorithm 3.2 is more efficient than the method in [22], the number of iterations is much less than that of [22].

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