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Approximate Analytical Solution by Residual Power Series Method for System of Fredholm Integral Equations

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Abstract: In this paper, we present a new analytical technique for obtaining the analytical approximate solutions for system of Fredholm integral equations based on the use of the residual power series method (RPSM). The proposed method provides the solution in terms of convergent series with easily computable components, as well as it possesses main advantage as compared to other existed methods; it can be applied without any limitation or linearization on the nature of the problem, type of classification, and the number of mesh points. In this sense, some examples are given to demonstrate the simplicity and efficiency of the proposed method. The results obtained by employing the RPSM are compared with exact solutions to reveal that the method is easy to implement, straightforward and convenient to handle a wide range of such system of integral equations.

Keywords: Analytical solutions, Residual power series method, Systems of differential equations, Fredholm integral equations

1 Introduction

Systems of Fredholm integral equations occur frequently in applied mathematics, theoretical physics, engineering, biology, mathematical modeling of real world phenomena in which uncertainty or vagueness pervades and so on. Unfortunately, investigation about system of integral equations is scarce especially discussion on finding solution. Indeed, it is usually difficult to obtain the closed-form solutions to systems of Fredholm integral equations met in practice, so these problems have been attacked using numeric-analytic methods with great interest by several authors. Therefore, a class of system of integral equation takes a central seat in the mathematical modeling literature.

The numerical solvability of such system has been pursued by various approximate numerical methods. To mention a few, the Adomian decomposition method (AMD) [1], Wavelet Galerkin method [2], Taylor-series expansion method [3], Modified homotopy perturbation method [4], homotopy analysis method (HAM) [5], reproducing kernel Hilbert space method (RKHS) [6], hat basis and delta functions [7,8], Chebyshev and Legendre wavelet method [9,10], and others [11,12,13,14,15,16].

In this paper, we apply the residual power-series method for system of Fredholm integral equations in the form:

$$\begin{aligned}
y_1(x) - \int_{x_0}^{b} K(x,t)\, g_1\left(x,t,\overrightarrow{y_i}(t)\right) dt &= f_1(x), \\
y_2(x) - \int_{x_0}^{b} K(x,t)\, g_2\left(x,t,\overrightarrow{y_i}(t)\right) dt &= f_2(x), \\
&\vdots \\
y_n(x) - \int_{x_0}^{b} K(x,t)\, g_n\left(x,t,\overrightarrow{y_i}(t)\right) dt &= f_n(x),
\end{aligned} \tag{1}$$

where $x \in [x_0, b]$, $K(x,t)$ is continuous known kernel such that $K(x,t) = [k_{ij}(x,t)]$, $i,j = 1,2,\ldots,n$, $f_i(x)$, $i = 1,2,\ldots,n$, are analytical functions which satisfy all necessary requirements of the existence of a unique

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solution, g_i are linear or nonlinear function of y_i depend on the problem discussed, $\overrightarrow{y_i}(t) = (y_1(t), y_2(t), ..., y_n(t))$, and $y_i(x)$, $i = 1, 2, \ldots, n$ are unknown analytical functions on the given interval to be determined.

The RPS method is an effective and easy to construct power series solutions for strongly linear and nonlinear differential equations without linearization, perturbation or discretization [17, 18, 19, 20, 21]. This method provides the solution in terms of convergent power series with easily computable components, were computed by chain of linear equations of one or more variables. It is different from the classical Taylor series method that computationally expensive for large orders and suited for the linear problems, which is an alternative procedure for obtaining analytical Taylor series solution for system of Fredholm integral equations. Consequently, the solutions and all of its derivatives are applicable for each arbitrary point in the given interval. On the other aspect as well, the RPSM does not require any conversion while switching from the low-order to the higher-order, so it can be applied directly to given problem by choosing an appropriate initial guess approximation. However, different applications with other versions of linear and nonlinear problems can be found in [22, 23, 24, 25, 26, 27] and references therein.

In this paper, the extension of the RPS scheme and differential of it are used to approximate the solution functions for system of Fredholm integral equation based on Taylor series expansion. The organization of the remainder of this paper is as follows. In Section 2, we present the formulation of the residual power-series method for system (1). The error analysis technique based on the residual function is also developed for the present method. In Section 3, the RPSM is applied and extended to provide symbolic approximate series solutions for system (1) and to illustrate the capability of the proposed method. Results reveal that only few terms are required to deduce the approximate solutions which are found to be accurate and efficient. Finally, a brief discussion and conclusion are presented in Section 4.

2 The residual power-series method

In this section, we review some elementary knowledge and some properties about residual power-series functions which are useful of the remainder of this analysis. Then, we employ the RPSM to find out a series solution for system of Fredholm integral equation (1) by formulate and analyze the proposed method.

For initial point $x = x_0$, we suppose that the expression form solution of system (1) as a power series expansion is given by

$$y_i(x) = \sum_{j=0}^{\infty} y_{i,j}(x), i = 1, 2, ..., n, \tag{2}$$

where $y_{i,j}(x), i = 1, 2, ..., n, j = 0, 1, 2, ...,$ are terms of approximations such that $y_{i,j}(x) = c_{i,j}(x - x_0)^j$.

By truncating the series in Eq. (2), we obtain the kth-truncated series solutions as

$$y_i^k(x) = \sum_{j=0}^{k} c_{i,j}(x - x_0)^j, i = 1, 2, ..., n. \tag{3}$$

To apply the RPS technique, system (1), for a simplification, will be rewritten in the form

$$y_i(x) - \int_{x_0}^{b} K(x,t)\, g_i(x, t, \overrightarrow{y}(t))\, dt - f_i(x) = 0, \quad i = 1, 2, ..., n, \tag{4}$$

where $\overrightarrow{y} = (y_1, y_2, ..., y_n)$.

Now, by substituting the kth-truncated series $y_i^k(x)$ into Eq. (4), we obtain the mth-residual functions system as

$$Res_i^m(x) = y_i^k(x) - \int_{x_0}^{b} K(x,t)\, g_i(x, t, \overrightarrow{y_k}(t))\, dt - f_i(x), \quad i = 1, 2, ..., n, \tag{5}$$

where $\overrightarrow{y_k} = \left(y_1^k, y_2^k, ..., y_n^k\right)$, and the ∞th residual function is given by $Res_i^m(x) = \lim_{m\to\infty} Res_i^m(x)$, $i = 1, 2, ..., n$.

Here, it is worth mentioning that $Res_i^m(x) = 0, i = 1, 2, ...,$ for each $x \in [x_0, x_0 + b]$. This show that $Res_i^m(x)$ are infinitely differentiable functions at $x = x_0$ such that $\frac{d^m}{dx^m} Res_i^\infty(x_0) = \frac{d^m}{dx^m} Res_i^m(x_0) = 0$, $m = 0, 1, 2, ..., k$. This relation is a fundamental rule in the RPSM and its applications.

As a consequence, we have the following

$$\begin{aligned} Res_i^m(x) = {} & \textstyle\sum_{j=0}^{k} c_{i,j}(x - x_0)^j - \displaystyle\int_{x_0}^{b} K(x,t) \\ & g_i(x, t, \textstyle\sum_{j=0}^{k} c_{1,j}(t - t_0)^j, \\ & \textstyle\sum_{j=0}^{k} c_{2,j}(t - t_0)^j, ..., \\ & \textstyle\sum_{j=0}^{k} c_{n,j}(t - t_0)^j) dt - f_i(x), \\ & i = 1, 2, ..., n, \end{aligned} \tag{6}$$

which contained n equations in j variables.

The unknown coefficients $c_{i,j}, i = 1, 2, \ldots, n$, $j = 0, 1, \ldots, k$, in Eq. (6) can be obtained by straightforward steps using the following procedure, which leads to algebraic systems of $n \times (k+1)$ equations that solved directly using Mathematica software package.

Firstly, putting $m = 0$ in Eq. (6) and using the fact that $Res_i^0(x_0) = 0$ for $i = 1, 2, \ldots, n$, leads to system of algebraic equations in the form

$$c_{i,0} - G_i(x_0, \overrightarrow{y}_{i,j}) = f_i(x_0), \; i = 1, 2, ..., n, \tag{7}$$

where

$$G_i(x_0,\overrightarrow{y}_{i,j})=\int_{x_0}^{b}K(x_0,t)\,g_i\left(x_0,t,y_{1,j},y_{2,j},...,y_{n,j}\right)dt,$$

$y_{i,j}=\sum_{j=0}^{k}c_{i,j}\,(t-t_0)^j$, $i=1,2,\dots,n$, $j=0,1,\dots,k$.

Now, differentiate both sides of Eq. (6) with respect to x, and then set $m=1$, we get that

$$\begin{aligned}\frac{d}{dx}Res_i^1(x)=&\sum_{j=1}^{k}jc_{i,j}\,(x-x_0)^{j-1}\\&-\frac{d}{dx}\Big[\int_{x_0}^{b}K(x,t)\,g_i(x,t,\sum_{j=0}^{k}c_{1,j}\,(t-t_0)^j,\\&\sum_{j=0}^{k}c_{2,j}\,(t-t_0)^j,...,\\&\sum_{j=0}^{k}c_{n,j}\,(t-t_0)^j)dt\Big]-f_i'(x),\\&i=1,2,...,n.\end{aligned}\tag{8}$$

Following relation (8) and using the fact that $\left(\frac{d}{dx}Res_i^1(x_0)\right)=0$ for $i=1,2,\dots,n$, we obtain other system of algebraic equations in the form

$$c_{i,1}-G_i'(x_0,\overrightarrow{y}_{i,j})=f_i'(x_0),\ i=1,2,...,n,\tag{9}$$

where

$$G_i'(x_0,\overrightarrow{y}_{i,j})=\frac{d}{dx}\Big[\int_{x_0}^{b}K(x_0,t)\,g_i\left(x_0,t,y_{1,j},y_{2,j},...,y_{n,j}\right)dt\Big],$$

$i=1,2,\dots,n$, $j=0,1,\dots,k$.

Similarly, differentiate both sides of Eq. (6) twice with respect to x, and set $m=2$, we obtain that

$$\begin{aligned}\frac{d^2}{dx^2}Res_i^2(x)=&\sum_{j=2}^{k}j(j-i)c_{i,j}\,(x-x_0)^{j-2}\\&-\frac{d^2}{dx^2}\Big[\int_{x_0}^{b}K(x,t)\,g_i(x,t,\sum_{j=0}^{k}c_{1,j}\,(t-t_0)^j,\\&\sum_{j=0}^{k}c_{2,j}\,(t-t_0)^j,...,\sum_{j=0}^{k}c_{n,j}\,(t-t_0)^j)dt\Big]\\&-f_i''(x),\ i=1,2,...,n.\end{aligned}$$

Consequently, by using the fact that $\left(\frac{d^2}{dx^2}Res_i^2(x_0)\right)=0$ for $i=1,2,\dots,n$, we also obtain other system of algebraic equations in the form

$$2c_{i,2}-G_i''(x_0,\overrightarrow{y}_{i,j})=f_i''(x_0),\ i=1,2,...,n,\tag{10}$$

where $G_i''(x_0,\overrightarrow{y}_{i,j})=$
$\frac{d^2}{dx^2}\Big[\int_{x_0}^{b}K(x_0,t)\,g_i\left(x_0,t,y_{1,j},y_{2,j},...,y_{n,j}\right)dt\Big]$, $i=1,2,\dots,n$,
$j=0,1,\dots,k$.

Correspondingly, by continuing with this technique till to $m=1$, we get that

$$\begin{aligned}\frac{d^k}{dx^k}Res_i^k(x)=&\sum_{j=k}^{k}j!c_{i,j}\,(x-x_0)^{j-k}\\&-\frac{d^k}{dx^k}\Big[\int_{x_0}^{b}K(x,t)\,g_i(x,t,\sum_{j=0}^{k}c_{1,j}\,(t-t_0)^j,\\&\sum_{j=0}^{k}c_{2,j}\,(t-t_0)^j,...,\sum_{j=0}^{k}c_{n,j}\,(t-t_0)^j)dt\Big]\\&-f_i^{(k)}(x),\ i=1,2,...,n,\end{aligned}$$

and by using the fact that $\left(\frac{d^k}{dx^k}Res_i^k(x_0)\right)=0$ for $i=1,2,\dots,n$, the kth algebraic system can be generated as follows

$$k!c_{i,k}-G_i^{(k)}(x_0,\overrightarrow{y}_{i,j})=f_i^{(k)}(x_0),\ i=1,2,...,n,\tag{11}$$

where $G_i^{(k)}(x_0,\overrightarrow{y}_{i,j})=$
$\frac{d^k}{dx^k}\Big[\int_{x_0}^{b}K(x_0,t)\,g_i\left(x_0,t,y_{1,j},y_{2,j},...,y_{n,j}\right)dt\Big]$, $i=1,2,\dots,n$,
$j=0,1,\dots,k$.

Hence, by solving these package of algebraic systems (7), (9), (10) up to (11), the kth approximate solutions, $y_i^k(x)$, $i=1,2,...,n$, of Eq. (7) can be obtained.

However, higher accuracy can be achieved by evaluating more components of the solution. It will be convenient to have a notation for the error in the approximation $y_i(x)\approx y_i^k(x)$. Accordingly, let $Rem_i^k(x)$, $i=1,2,...,n$, be the kth remainder for the RPS approximation, which is the difference between $y_i(x)$ and its kth Taylor polynomial obtained by RPSM; that is,

$$\begin{aligned}Rem_i^k(x)&=y_i(x)-y_i^k(x)\\&=\sum_{j=k+1}^{\infty}\frac{1}{j!}y_i^{(j)}(x_0)\,(x-x_0)^j.\end{aligned}$$

In fact, it often happens that the remainders $Rem_i^k(x)$ become smaller and smaller, approaching zero, as k gets large. The concept of accuracy refers to how closely a computed or measured value agrees with the truth value. Taylor's theorem allows us to represent fairly general functions exactly in terms of polynomials with a known, specified, and bounded error. To show the accuracy of the RPSM for some tested problems, we report four types of error; The residual error $Resd_i^k(x)$, the absolute error $Abs_i^k(x)$, the relative error $Rel_i^k(x)$, and the consecutive error $Con_i^k(x)$, which are defined respectively by

$$Resd_i^k(x)=\left|y_i^k(x)-\int_{x_0}^{b}K(x,t)\,g_i\left(x,t,\overrightarrow{y_{i,k}}(t)\right)dt-f_i(x)\right|,$$

$$\begin{aligned}Abs_i^k(x)&=\left|y_i(x)-y_i^k(x)\right|=\left|Rem_i^k(x)\right|,\\Rel_i^k(x)&=\frac{\left|y_i(x)-y_i^k(x)\right|}{|y_i(x)|},\\Con_i^k(x)&=\left|y_i^{k+1}(x)-y_i^k(x)\right|,\ i=1,2,...,n,\end{aligned}$$

where $y_i(x)$, $i=1,2,...,n$, are the exact solutions, and $y_i^k(x)$, are the kth-order approximation obtained by the RPSM.

Next, we present a convergence theorem of the RPS technique to capture the behavior of solutions.

Theorem 1. [28] *Suppose that $y_i(x)$, $i=1,2,...,n$, are the exact solutions for Eq. (1). Then, the approximate* *solutions obtained by the RPS technique are in fact the Taylor expansion of $y_i(x)$ for $i = 1,2,...,n$.*

Theorem 2. [28] *Let $y_i(x)$, $i = 1,2,\ldots,n$, be a polynomial for some i, then the RPS technique will obtain the exact solution.*

The reader is referred to [28,29,30,31] and the references therein in order to know more details and principles about the RPS technique, including their applications in various kinds of differential equations.

3 Illustrative problems

In order to assess the accuracy and the performance of the new adaption of the RPSM, we apply this approach to some examples. Results obtained by the method are compared with the analytical solution of each example and are found to be in good agreement with each other. We highlight the significant features of the developed adaption in reducing the size of required computational work. Through this paper, all of the symbolic and numerical computations are performed by using Mathematica software package.

Example 1. [3] Consider the system of Fredholm integral equations in the form

$$\begin{aligned} &y_1(x) - \int_0^1 \left((x-t)^3 y_1(t) + (x-t)^2 y_2(t)\right) dt \\ &= f_1(x), \\ &y_2(x) - \int_0^1 \left((x-t)^4 y_1(t) + (x-t)^3 y_2(t)\right) dt \\ &= f_2(x), \end{aligned} \tag{12}$$

where $f_1(x) = \frac{1}{20} - \frac{11}{30}x + \frac{5}{3}x^2 - \frac{1}{3}x^3$ and $f_2(x) = \frac{-1}{30} - \frac{41}{60}x + \frac{3}{20}x^2 + \frac{23}{12}x^3 - \frac{1}{3}x^4$, $x \in [0,1]$.

According to the proposed method, the kth-truncated series solution $y_i^k(x)$, $i = 1,2$, about $x_0 = 0$ for system (12) is given by

$$\begin{aligned} y_1^k(x) &= \sum_{j=0}^{k} c_{1,j} x^j = c_{1,1}x + c_{1,2}x^2 + \ldots + c_{1,k}x^k, \\ y_2^k(x) &= \sum_{j=0}^{k} c_{2,j} x^j = c_{2,1}x + c_{2,2}x^2 + \ldots + c_{2,k}x^k. \end{aligned} \tag{13}$$

Using the RPS procedure, we first construc the following mth-residual functions $Res_i^m(x)$, $i = 1,2$, in order to find out the values of the coefficients $c_{1,j}, c_{2,j}$, $j = 1,2,3,\ldots,k$, in Eq. (13):

$$\begin{aligned} Res_1^m(x) &= \sum_{j=0}^{k} c_{1,j} x^j \\ &- \int_0^1 \left((x-t)^3 \sum_{j=0}^{k} c_{1,j} t^j + (x-t)^2 \sum_{j=0}^{k} c_{2,j} t^j\right) dt \\ &- \frac{1}{20} + \frac{11}{30}x - \frac{5}{3}x^2 + \frac{1}{3}x^3, \\ Res_2^m(x) &= \sum_{j=0}^{k} c_{2,j} x^j \\ &- \int_0^1 \left((x-t)^4 \sum_{j=0}^{k} c_{1,j} t^j + (x-t)^3 \sum_{j=0}^{k} c_{2,j} t^j\right) dt \\ &+ \frac{1}{30} + \frac{41}{60}x - \frac{3}{20}x^2 - \frac{23}{12}x^3 + \frac{1}{3}x^4. \end{aligned} \tag{14}$$

Consequently, the expression forms of algebraic systems with respect to $c_{1,j}, c_{2,j}$, $j = 1,2,3,\ldots,k$, can be found through the following steps: Firstly, by setting $m = 0$ in Eq. (14) and using the facts $Res_1^0(0) = 0$ and $Res_2^0(0) = 0$, we get that

$$c_{1,0} + \sum_{j=0}^{k} \int_0^1 \left(c_{1,j} t^{j+3} - c_{2,j} t^{j+2}\right) dt = \frac{1}{20},$$

$$c_{2,0} - \sum_{j=0}^{k} \int_0^1 \left(c_{1,j} t^{j+4} - c_{2,j} t^{j+3}\right) dt = \frac{-1}{30},$$

which implies

$$\begin{aligned} c_{1,0} + \sum_{j=0}^{k} \left(\frac{1}{j+4} c_{1,j} - \frac{1}{j+3} c_{2,j}\right) &= \frac{1}{20}, \\ c_{2,0} - \sum_{j=0}^{k} \left(\frac{1}{j+5} c_{1,j} + \frac{1}{j+4} c_{2,j}\right) &= -\frac{1}{30}. \end{aligned} \tag{15}$$

Secondly, differentiate both sides of Eq. (14) with respect to x and set $m = 1$ in order to obtain

$$\begin{aligned} \left(\frac{d}{dx} Res_1^1(x)\right) &= \sum_{j=1}^{k} j c_{1,j} x^{j-1} \\ &- \int_0^1 \left(3(x-t)^2 \sum_{j=0}^{k} c_{1,j} t^j + 2(x-t) \sum_{j=0}^{k} c_{2,j} t^j\right) dt \\ &+ \frac{11}{30} - \frac{10}{3}x + x^2, \\ \left(\frac{d}{dx} Res_2^1(x)\right) &= \sum_{j=1}^{k} j c_{2,j} x^{j-1} \\ &- \int_0^1 \left(4(x-t)^3 \sum_{j=0}^{k} c_{1,j} t^j + 3(x-t)^2 \sum_{j=0}^{k} c_{2,j} t^j\right) dt \\ &+ \frac{41}{60} - \frac{3}{10}x - \frac{23}{4}x^2 + \frac{4}{3}x^3, \end{aligned}$$

as well as use the facts $\left(\frac{d}{dx}Res_1^1(0)\right) = 0$ and $\left(\frac{d}{dx}Res_2^1(0)\right) = 0$ leads to

$$\begin{aligned} &c_{1,1} - \sum_{j=0}^{k}\left(\tfrac{3}{j+3}c_{1,j} - \tfrac{2}{j+2}c_{2,j}\right) = -\tfrac{11}{30}, \\ &c_{2,1} + \sum_{j=0}^{k}\left(\tfrac{4}{j+4}c_{1,j} - \tfrac{3}{j+3}c_{2,j}\right) = -\tfrac{41}{60}. \end{aligned} \tag{16}$$

Thirdly, differentiate both sides of Eq. (14) twice with respect to x and set $m = 2$ in order to obtain

$$\begin{aligned} \left(\tfrac{d^2}{dx^2}Res_1^2(x)\right) = &\sum_{j=2}^{k} j(j-1)c_{1,j}x^{j-2} \\ &-\int_0^1\left(6(x-t)\sum_{j=0}^{k}c_{1,j}t^j + \sum_{j=0}^{k}2c_{2,j}t^j\right)dt \\ &-\tfrac{10}{3} + 2x, \\ \left(\tfrac{d^2}{dx^2}Res_2^2(x)\right) = &\sum_{j=2}^{k} j(j-1)c_{2,j}x^{j-2} \\ &-\int_0^1\left(12(x-t)^2\sum_{j=0}^{k}c_{1,j}t^j + 6(x-t)\sum_{j=0}^{k}c_{2,j}t^j\right)dt \\ &-\tfrac{3}{10} - \tfrac{23}{2}x + 4x^2, \end{aligned}$$

and thus use the facts $\left(\frac{d^2}{dx^2}Res_1^2(0)\right) = 0$ and $\left(\frac{d^2}{dx^2}Res_2^2(0)\right) = 0$ leads to

$$\begin{aligned} &c_{1,2} + \sum_{j=0}^{k}\left(\tfrac{3}{j+2}c_{1,j} - \tfrac{1}{j+1}c_{2,j}\right) = \tfrac{5}{3}, \\ &c_{2,2} - \sum_{j=0}^{k}\left(\tfrac{6}{j+3}c_{1,j} - \tfrac{3}{j+2}c_{2,j}\right) = \tfrac{3}{20}. \end{aligned} \tag{17}$$

Fourthly and similarly, differentiate both sides of Eq. (14) again and set $m = 3$ to obtain that

$$\begin{aligned} \left(\tfrac{d^3}{dx^3}Res_1^3(x)\right) = &\sum_{j=3}^{k} j(j-1)(j-2)c_{1,j}x^{j-3} \\ &-\left(\sum_{j=0}^{k}\int_0^1\left(6c_{1,j}t^j\right)dt\right) + 2, \\ \left(\tfrac{d^3}{dx^3}Res_2^3(x)\right) = &\sum_{j=3}^{k} j(j-1)(j-2)c_{2,j}x^{j-3} \\ &-\int_0^1\left(24(x-t)\sum_{j=0}^{k}c_{1,j}t^j + \sum_{j=0}^{k}6c_{2,j}t^j\right)dt \\ &-\tfrac{23}{2} + 8x, \end{aligned}$$

and thus use the fact $\left(\frac{d^3}{dx^3}Res_1^3(0)\right) = 0$ and $\left(\frac{d^3}{dx^3}Res_2^3(0)\right) = 0$ leads also to

$$\begin{aligned} &c_{1,3} - \sum_{j=0}^{k}\tfrac{1}{j+1}c_{1,j} = -\tfrac{1}{3}, \\ &c_{2,3} + \sum_{j=0}^{k}\left(\tfrac{4}{j+2}c_{1,j} - \tfrac{1}{j+1}c_{2,j}\right) = \tfrac{23}{12}. \end{aligned} \tag{18}$$

Finally, differentiate both sides of Eq. (14) again and set $m = 4$ to obtain that

$$\begin{aligned} \left(\tfrac{d^4}{dx^4}Res_1^4(x)\right) = &\sum_{j=4}^{k} j(j-1)(j-2)(j-3)c_{1,j}x^{j-4}, \\ \left(\tfrac{d^4}{dx^4}Res_2^4(x)\right) = &\sum_{j=4}^{k} j(j-1)(j-2)(j-3)c_{2,j}x^{j-4} \\ &-\left(\sum_{j=0}^{k}\int_0^1 24c_{1,j}t^j dt\right) + 8, \end{aligned} \tag{19}$$

and thus use the fact $\left(\frac{d^4}{dx^4}Res_1^4(0)\right) = 0$ and $\left(\frac{d^4}{dx^4}Res_2^4(0)\right) = 0$ leads also to

$$\begin{aligned} &c_{1,4} = 0, \\ &c_{2,4} - \sum_{j=0}^{k}\tfrac{1}{j+1}c_{1,j} = -\tfrac{1}{3}; \end{aligned} \tag{20}$$

As well as by differentiating both sides of Eq. (19), setting $m = 5$ and using $\left(\frac{d^5}{dx^5}Res_1^5(0)\right) = \left(\frac{d^5}{dx^5}Res_2^5(0)\right) = 0$, we get that

$$\begin{aligned} \left(\tfrac{d^5}{dx^5}Res_1^5(0)\right) = &\\ &\sum_{j=5}^{k} j(j-1)(j-2)(j-3)(j-4)c_{1,j}x^{j-5} = 0, \\ \left(\tfrac{d^5}{dx^5}Res_2^5(0)\right) = &\\ &\sum_{j=5}^{k} j(j-1)(j-2)(j-3)(j-4)c_{2,j}x^{j-5} = 0, \end{aligned}$$

which implies that $c_{1,5} = 0$ and $c_{2,5} = 0$. Hence, the coefficients $c_{1,j}$ and $c_{2,j}$ of expansion (13) vanish for $5 \le j \le k$.

Therefore, the kth series solution of system (12) will be given by

$$y_1^k(x) = \sum_{j=0}^{3}c_{1,j}x^j,\; y_2^k(x) = \sum_{j=0}^{4}c_{2,j}x^j,$$

whereas the coefficients $c_{1,j}$ for $0 \leq j \leq 3$ and $c_{2,j}$ for $0 \leq j \leq 4$ can be found by solving the following collections

$$\begin{aligned}
&c_{1,0} + \sum_{j=0}^{3} \tfrac{1}{j+4} c_{1,j} - \sum_{j=0}^{4} \tfrac{1}{j+3} c_{2,j} = \tfrac{1}{20},\\
&c_{2,0} - \sum_{j=0}^{3} \tfrac{1}{j+5} c_{1,j} + \sum_{j=0}^{4} \tfrac{1}{j+4} c_{2,j} = -\tfrac{1}{30},\\
&c_{1,1} - \sum_{j=0}^{3} \tfrac{3}{j+3} c_{1,j} + \sum_{j=0}^{4} \tfrac{2}{j+2} c_{2,j} = -\tfrac{11}{30},\\
&c_{2,1} + \sum_{j=0}^{3} \tfrac{4}{j+4} c_{1,j} - \sum_{j=0}^{4} \tfrac{3}{j+3} c_{2,j} = -\tfrac{41}{60},\\
&c_{1,2} + \sum_{j=0}^{3} \tfrac{3}{j+2} c_{1,j} - \sum_{j=0}^{4} \tfrac{1}{j+1} c_{2,j} = \tfrac{5}{3}, \qquad (21)\\
&c_{2,2} - \sum_{j=0}^{3} \tfrac{6}{j+3} c_{1,j} + \sum_{j=0}^{4} \tfrac{3}{j+2} c_{2,j} = \tfrac{3}{20},\\
&c_{1,3} - \sum_{j=0}^{3} \tfrac{1}{j+1} c_{1,j} = -\tfrac{1}{3},\\
&c_{2,3} + \sum_{j=0}^{3} \tfrac{4}{j+2} c_{1,j} - \sum_{j=0}^{4} \tfrac{1}{j+1} c_{2,j} = \tfrac{23}{12},\\
&c_{2,4} - \sum_{j=0}^{3} \tfrac{1}{j+1} c_{1,j} = -\tfrac{1}{3}.
\end{aligned}$$

Consequently, by using Mathematica software package, the coefficients $c_{1,j}$ for $0 \leq j \leq 3$ and $c_{2,j}$ for $0 \leq j \leq 4$ are given by

$$\begin{aligned}
&c_{1,0} = 0, c_{1,1} = 0, = c_{1,2} = 1, c_{1,3} = 0,\\
&c_{2,0} = 0, c_{2,1} = -1, = c_{2,2} = 1, c_{2,3} = 1, c_{2,4} = 0.
\end{aligned}$$

Thus, the approximate solution is

$$\begin{aligned}
y_1(x) &= \sum_{j=0}^{\infty} c_{1,j} x^j = x^2,\\
y_2(x) &= \sum_{j=0}^{\infty} c_{2,j} x^j = -x + x^2 + x^3,
\end{aligned}$$

which is the closed-form solution. The same solution was obtained using Taylor-series expansion method in [3].

The RPSM provides analytical approximate solutions in terms of an infinite power series. In addition, there are practical needs to evaluate these solutions and to obtain numerical values from the infinite power series. The consequent series truncation and the corresponding practical procedure are realized to accomplish this task. The truncation transforms the otherwise analytical results into exact solutions, which is evaluated to a finite degree of accuracy.

Example 2. Consider the system of Fredholm integral equations in the form

$$\begin{aligned}
y_1(x) + \int_0^1 \pi^3 \left(x^2 y_1(t) + \pi t y_2(t)\right) dt &= f_1(x),\\
y_2(x) - \int_0^1 \pi^3 (x(t+1) y_1(t) - \pi x t y_2(t)) dt &= f_2(x),
\end{aligned} \qquad (22)$$

where $f_1(x) =$
$\pi^2 x^2 + x\sin(\pi x) - 2(48 + 24\pi + 6\pi^2 + \pi^3)e^{-\frac{1}{2}\pi} + 96$ and $f_2(x) =$
$x^2 e^{-\frac{1}{2}\pi x} + 2x(50 - \pi^2 - (48 + (24\pi + \pi^2(6+\pi)e^{-\frac{1}{2}\pi}), x \in [0,1]$.

The exact solution of the system of integral Eq. (22) is

$$y_1(x) = x\sin(\pi x),\ y_2(x) = x^2 e^{-\frac{1}{2}\pi x}. \qquad (23)$$

Now, according to the proposed adaption of RPSM, the mth-residual functions $Res_i^m(x)$ for $i = 1,2$ about $x_0 = 0$ is

$$\begin{aligned}
Res_1^m(x) &= \sum_{j=0}^{k} c_{1,j} x^j\\
&+ \int_0^1 \pi^3 \left(x^2 \sum_{j=0}^{k} c_{1,j} t^j + \pi \sum_{j=0}^{k} c_{2,j} t^{j+1} \right) dt\\
&- \left(\begin{array}{l} \pi^2 x^2 + x\sin(\pi x) \\ -2(48 + 24\pi + 6\pi^2 + \pi^3)e^{-\frac{1}{2}\pi} + 96 \end{array} \right),\\
Res_2^m(x) &= \sum_{j=0}^{k} c_{2,j} x^j\\
&- \int_0^1 \pi^3 \left(x(t+1) \sum_{j=0}^{k} c_{1,j} t^j - \pi x \sum_{j=0}^{k} c_{2,j} t^{j+1} \right) dt\\
&- \left(\begin{array}{l} x^2 e^{-\frac{1}{2}\pi x} + 2x(50 - \pi^2 \\ -(48 + (24\pi + \pi^2(6+\pi))e^{-\frac{1}{2}\pi}) \end{array} \right).
\end{aligned} \qquad (24)$$

Following the residual functions (24) and the fact $\frac{d^m}{dx^m} Res_1^m(0) = \frac{d^m}{dx^m} Res_2^m(0) = 0$ for $m = 0,1,2,\ldots,k$, the coefficients $c_{i,j}$, $i = 1,2$, $j = 0,1,\ldots,k$, can be obtained. Hence, the series solution of Eq. (22) is derived, which converge to the closed-form solution given in Eq. (23).

Consequently, to illustrate the efficiency of the present method, some numerical comparisons between exact and series solutions for Eq. (22) at some selected grid points in $[0,1]$ with step size of 0.16 using the 20th-order approximation are listed in Tables 1 and 2. Here, we can observe that the present method provides us with an accurate approximate solutions that found to be in good

Table 1: Numerical comparison for Example 2 using the 20th-order approximation of $y_1(x)$.

x	Exact solution	Approximate solution	$Abs_1^{20}(x)$	$Rel_1^{20}(x)$
0.16	0.07708058785627446	0.07708058773318506	1.23089×10^{-10}	1.59689×10^{-9}
0.32	0.27018493616064480	0.27018493607056815	9.00767×10^{-11}	3.33389×10^{-10}
0.48	0.47905282964557033	0.47905282961051476	3.50556×10^{-11}	7.31768×10^{-11}
0.64	0.57908931357825240	0.57908931362019770	4.19452×10^{-11}	7.24331×10^{-11}
0.80	0.47022820183397860	0.47022820197106080	1.37082×10^{-10}	2.91523×10^{-10}
0.96	0.12031990422173235	0.12031990426801054	4.62782×10^{-11}	3.84626×10^{-10}

Table 2: Numerical comparison for Example 2 using the 20th-order approximation of $y_2(x)$.

x	Exact solution	Approximate solution	$Abs_2^{20}(x)$	$Rel_2^{20}(x)$
0.16	0.01991085258679780	0.01991085258745783	6.60031×10^{-13}	3.31493×10^{-11}
0.32	0.06194407042706135	0.06194407042838140	1.32005×10^{-12}	2.13103×10^{-11}
0.48	0.10840071576264605	0.10840071576462612	1.98007×10^{-12}	1.82662×10^{-11}
0.64	0.14988546332315372	0.14988546332579380	2.64008×10^{-12}	1.76140×10^{-11}
0.80	0.18215010773505877	0.18215010773835938	3.30061×10^{-12}	1.81203×10^{-11}
0.96	0.20400547183774270	0.20400547184172010	3.97740×10^{-12}	1.94965×10^{-11}

Table 3: Absolute error of 10th, 15th, 20th and 25th-order approximations of $y_1(x)$ for Example 2.

Node	$Abs_1^{10}(x)$	$Abs_1^{15}(x)$	$Abs_1^{20}(x)$	$Abs_1^{25}(x)$
0.0	2.87980×10^{-3}	6.97763×10^{-6}	1.34094×10^{-10}	3.12639×10^{-13}
0.2	2.50594×10^{-3}	6.07759×10^{-6}	1.16899×10^{-10}	2.76140×10^{-13}
0.4	1.38449×10^{-3}	3.37749×10^{-6}	6.53171×10^{-11}	1.66700×10^{-13}
0.6	4.69235×10^{-4}	1.11659×10^{-6}	2.06464×10^{-11}	1.56541×10^{-14}
0.8	2.61533×10^{-3}	6.82014×10^{-6}	1.37082×10^{-10}	2.70339×10^{-13}
1.0	4.58633×10^{-4}	5.61931×10^{-6}	2.33159×10^{-10}	4.29150×10^{-13}

Table 4: Absolute error of 10th, 15th, 20th and 25th-order approximations of $y_2(x)$ for Example 2.

Node	$Abs_2^{10}(x)$	$Abs_2^{15}(x)$	$Abs_2^{20}(x)$	$Abs_2^{25}(x)$
0.0	0.00000	0.00000	0.00000	0.00000
0.2	2.41914×10^{-5}	4.27857×10^{-8}	8.25034×10^{-13}	2.03657×10^{-15}
0.4	4.83764×10^{-5}	8.55715×10^{-8}	1.65007×10^{-12}	4.06619×10^{-15}
0.6	7.20425×10^{-5}	1.28359×10^{-7}	2.47519×10^{-12}	6.16174×10^{-15}
0.8	8.45366×10^{-5}	1.71309×10^{-7}	3.30061×10^{-12}	8.21565×10^{-15}
1.0	1.74612×10^{-5}	2.19707×10^{-7}	4.16575×10^{-12}	1.02973×10^{-14}

Table 5: Numerical comparison of 10-truncated series approximation $y_1^{10}(x)$ for Example 3.

x	$y_1(x)$	$y_1^{10}(x)$	$Abs_1^{10}(x)$	$Rel_1^{10}(x)$
0.16	1.3335108709918102	1.3335108709918100	2.22045×10^{-16}	1.66511×10^{-16}
0.32	1.6971277643359572	1.6971277643358644	9.28146×10^{-14}	5.46893×10^{-14}
0.48	2.0960744021928934	2.0960744021847620	8.13127×10^{-12}	3.87929×10^{-12}
0.64	2.5364808793049516	2.5364808791097320	1.95219×10^{-10}	7.69647×10^{-11}
0.80	3.0255409284924680	3.0255409261876824	2.30479×10^{-9}	7.61776×10^{-10}
0.96	3.5716964734231180	3.5716964560533597	1.73698×10^{-8}	4.86317×10^{-9}

Table 6: Numerical comparison of 10-truncated series approximation $y_2^{10}(x)$ for Example 3.

x	$y_2(x)$	$y_2^{10}(x)$	$Abs_2^{10}(x)$	$Rel_2^{10}(x)$
0.16	1.173510870991810	1.1735108709918098	4.44089×10^{-16}	3.78428×10^{-16}
0.32	1.377127764335957	1.3771277643358644	9.28146×10^{-14}	6.73973×10^{-14}
0.48	1.616074402192893	1.6160744021847617	8.13172×10^{-12}	5.03177×10^{-12}
0.64	1.896480879304952	1.8964808791097323	1.95219×10^{-10}	1.02938×10^{-10}
0.80	2.225540928492468	2.2255409261876826	2.30479×10^{-9}	1.03561×10^{-9}
0.96	2.611696473423118	2.6116964560533598	1.73698×10^{-8}	6.65076×10^{-9}

Table 7: Numerical comparison of 10-truncated series approximation $y_3^{10}(x)$ for Example 3.

x	$y_3(x)$	$y_3^{10}(x)$	$Abs_3^{10}(x)$	$Rel_3^{10}(x)$
0.16	1.9872272833756268	1.9872272833756270	2.22045×10^{-16}	1.11736×10^{-16}
0.32	1.9492354180824410	1.9492354180824387	2.22045×10^{-15}	1.13914×10^{-15}
0.48	1.8869949227792842	1.8869949227789724	3.11751×10^{-13}	1.65210×10^{-13}
0.64	1.8020957578842927	1.8020957578744559	9.83680×10^{-12}	5.45853×10^{-12}
0.80	1.6967067093471653	1.6967067092042047	1.42961×10^{-10}	8.42577×10^{-11}
0.96	1.5735199860724567	1.5735199847997698	1.27269×10^{-9}	8.08815×10^{-10}

Table 8: Consecutive error functions $Con_i^{10}(x)$, $i=1,2,3$, for Example 3.

Node	$Con_1^{10}(x)$	$Con_2^{10}(x)$	$Con_3^{10}(x)$
0.16	2.22045×10^{-16}	4.44089×10^{-16}	0.00000
0.32	9.01501×10^{-14}	9.25926×10^{-14}	2.22045×10^{-15}
0.48	7.80753×10^{-12}	8.11973×10^{-12}	3.12195×10^{-13}
0.64	1.84852×10^{-10}	1.94710×10^{-10}	9.85878×10^{-12}
0.80	2.15196×10^{-9}	2.29542×10^{-9}	1.43464×10^{-10}
0.96	1.59892×10^{-8}	1.72684×10^{-8}	1.27914×10^{-9}

agreement with exact solutions for all values of x in $[0,1]$, as well as the results reported in the tables confirm the effectiveness of RPSM.

Regarding the error analysis of the RPSM for Eq. (22), the absolute errors $Abs_i^k(x)$, $i=1,2$, $x\in[0,1]$ for $k=10$, 15, 20 and $k=25$ with step size of 0.2 are shown in Tables 3 and 4, respectively. As a result, it is clear from these tables that we can control the error by evaluating more components of the solution.

Example 3. Consider the system of Fredholm integral equations in the form

$$\begin{aligned}
&y_1(x)+\int_0^1 3te^x y_2(t)dt = f_1(x),\\
&y_2(x)-\int_0^1 \left(6xe^t y_1(t)-x^2 y_3(t)\right)dt = f_2(x), \qquad (25)\\
&y_3(x)-\int_0^1 \left(4(y_3(t)-1)+y_2(t)\right)dt = f_3(x),
\end{aligned}$$

where $f_1(x)=x+4e^x$, $f_2(x)=e^x-3x\left(e^2+1\right)+x^2\left(\sin(1)+1\right)$, and $f_3(x)=\cos(x)-\left(e+4\sin(1)\right)+2$, $x\in[0,1]$.

The exact solution of Eq. (25) is

$$y_1(x)=x+e^x,\ y_2(x)=e^x,\ y_3(x)=1+\cos(x). \qquad (26)$$

Without loss of generality, to show the accuracy that refers to how closely a computed and measured value agrees with exact value of Example 3 using RPS technique, we obtain the 10-truncated series approximation of the RPS, $y_i^{10}(x)$, $i=1,2,3$, at some grid nodes in $[0,1]$ with step size of 0.16. Anyhow, Tables $5-7$ show, respectively, its absolute and relative errors for various x. It is clear from the tables that the errors can be measure the extent of agreement between the 10-truncated series approximation of the RPS solutions and unknowns closed form solutions. Anyhow, in Table 8, the values of the consecutive error functions $Con_i^k(x)$, $i=1,2,3$, have been calculated for various values of x in $[0,1]$ and step size of 0.16. The goal here was to measure the difference between the consecutive solutions obtained by the 10th-order RPS solutions. In contrast, the residual error functions $Resd_i^k(x)$, $i=1,2,3$, for $k=5,10,15$, have

Table 9: Residual error functions $Resd_i^k(x), i = 1,2,3,\ k = 5,10,15$, for Example 3.

$Resd_i^k$		$x = 0.25$	$x = 0.50$	$x = 0.75$	$x = 1.00$
$i = 1$	$k = 5$	1.07811×10^{-11}	5.66417×10^{-9}	2.23560×10^{-7}	3.05862×10^{-7}
	$k = 10$	6.01470×10^{-15}	1.27625×10^{-11}	1.12824×10^{-9}	2.73127×10^{-8}
	$k = 15$	8.73782×10^{-17}	3.91002×10^{-17}	6.16045×10^{-16}	5.06664×10^{-14}
$i = 2$	$k = 5$	1.53607×10^{-11}	8.07871×10^{-9}	3.19215×10^{-7}	4.37238×10^{-6}
	$k = 10$	9.40043×10^{-15}	1.96813×10^{-11}	1.74147×10^{-9}	4.21977×10^{-8}
	$k = 15$	1.10199×10^{-18}	3.56191×10^{-17}	8.29278×10^{-16}	9.5463×10^{-14}
$i = 3$	$k = 5$	2.62679×10^{-13}	2.68605×10^{-10}	1.54526×10^{-8}	2.73497×10^{-7}
	$k = 10$	1.11022×10^{-16}	5.09037×10^{-13}	6.59260×10^{-11}	2.07625×10^{-9}
	$k = 15$	0.00000	1.11022×10^{-16}	4.44089×10^{-16}	4.77396×10^{-14}

been listed in Table 9 for $x_i = i/4,\ i = 1,2,3,4$, in order to demonstrate the rapid convergence of the present method by increasing the order of RPS approximation. However, the computational results provide a numerical estimate for convergence of the RPSM, as well as it is clear that the accuracy that is obtained using the method is advanced by using an approximation with only a few additional terms. Further,we can conclude that higher accuracy can be achieved by evaluating more components of the solution.

4 Conclusions and discussion

In this paper, the RPSM is implemented successfully to find out the analytical solution of system of Fredholm integral equations in terms of a rapidly convergent series with easily computable components using symbolic computation software. The steps of the RPS method are summarized, and the relevant applications are developed. The proposed solutions by RPSM are obtained without any transformation, perturbation, discretization or any other restrictive conditions, as well as are found in the closed form of a convergent series, which is coincides with exact solution. The results reveal that the RPSM is a powerful tool, very effective, straightforward, and convenient for solving different forms of system of integral equations.

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