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Some Statistical Quantities of a Quantum System in Hypergeometric and Negative Hypergeometric Distributions

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Abstract: In this paper we introduce a quantum system of the interaction between a two-level atom and input field initially prepared in hypergeometric and negative hypergeometric distributions. We study the dynamics of nonlocal correlation measured by von Neumann entropy or the field entropy. The statistical properties of the considered field will be discussed through the evolution of Wehrl space entropy. The relationship between Wehrl space entropy and entanglement will be explored. The effects of the field distribution parameters on the evolution of statistical quantities will be examined. It is shown that when the field is closed to the classical state no quantum correlation can be found and the system return to its separable state.

Keywords: Negative hypergeometric distributions, statistical properties, Wehrl space entropy, nonlocal correlation, atomic motion.

1 Introduction

The probability distribution (PD) plays a central role in quantum optics and quantum information processing. In the class of atom-field interaction PD is acts the distribution of the electromagnetic field elements which is change with time. In this way, the binomial states (BSs) are intermediate number coherent states in the sense that they reduce to the number and coherent states in two different limits. Complementary to the BSs, the negative binomial states (NBSs) are also introduced and investigated [1,2,3,4], they interpolate between the Susskind-Glogower phase states and coherent states [6].

Different generation of the NBSs have been applied. For example the even and odd NBSs are introduced as the interpolation state between the even and odd coherent states and the even and odd quasi-thermal states depending on the values of the parameters involved. In this regard the quasi probability distributions of the of the even and odd NBSs such as Wigner Functions (W-function) and tomograms have been discussed [7]. Furthermore quantum statistical properties of the even and odd NBSs.

Entanglement is one of the most peculiar features of quantum mechanics and the heart of quantum information theory. It is an important kind of quantum correlations between two or more systems [8]. The concept of entropy is used for detecting the squeezing and entanglement [9, 10,11]. The nonlocal nature of entanglement has been used as essential resources to perform different tasks in quantum information processing such as quantum cryptography [12,13], quantum teleportation [14] and quantum estimation [15]. These quantum information tasks depend on finding the quantum states in which entanglement can be created or enhanced. One of these important states is the non-Gaussian quantum states which used to perform certain continuous-variable quantum information tasks, such as quantum error correction [16], quantum entanglement distillation [17], and universal quantum computation [18].

It has been noted that ignoring the effect of time dependence in any quantum system gives an incomplete picture of the phenomena connected with such a system. The Jaynes-Cummings model (JCM) for moving atom or time dependent coupling becomes more realistic model. Therefore it will be important to consider the effect of time dependence when studying physical models. It is not

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an easy task to obtain general solutions for nonconservative quantum system [19], however, some solutions can be obtained for some particular systems or under certain conditions. Therefore, some explicit expressions for the time-dependent dynamical operators may be obtained in such cases [20]. In a previous paper a bimodal time-dependent JCM assuming that the instantaneous position of the particle within the cavity depends on time have considered. Consequently the effects of both the velocity and the acceleration have been taken into account during the interaction process [21]. Also, the problem of the interaction between a three-level atom and a quantized bimodal cavity field when the coupling parameter between the atom and the field is taken to be time dependent by taking the atomic motion into consideration was considered [22]. It was found that both of the velocity and detuning parameters play an essential role in the dynamics of the system entanglement and geometric phase.

Here, we investigate the statistical properties and nonlocal correlation between a two-level atom system and optical field initially prepared the HGSs and NHGSs. The statistical properties of the field will be studied through the evolution of the Wehrl space entropy while the nonlocal correlation or entanglement will be discussed through the evolution of the von Neumann (field) entropy. The influence of the initial state of the input field mode parameters and the two-level atom motion will be examined.

2 Hypergeometric and negative hypergeometric states

Recently three quantum states, Polya states (PSs)[23], the generalized non-classical states related to Hahn polynomials[24], and negative hypergeometric states (NHGSs)[25] are introduced as different intermediate BS-NBS states. The phase properties of the HGSs and NHGSs based on the Hermitial-phase-operator formalism are studied [26]. It is found that the number of peaks of phase probability distribution depicts one peak for the HGSs and M peaks for the NHGSs. The (HGSs) which are complementary to the NHGSs are defined as [26, 27].

The hypergeometric states (HGSs) which are complementary to the NHGSs are defined as

$$|L,M,\eta\rangle = \sum_{n=0}^{M} \binom{\eta L}{n} \binom{L-\eta L}{M-n}^{-1/2} \binom{\eta L}{n}^{-1/2} |n\rangle \qquad (1)$$

where L is a real number satisfying $L \geq \max\{M/\eta, M/(1-\eta)\}$, and

$$\binom{x}{n} = \frac{x(x-1) \quad (x-n+1)}{n!} \text{ and } \binom{x}{0} = 1 \qquad (2)$$

The HGS can be reduced to the BS in certain limit and the BS to the number and coherent state. The NHGSs is defined as [27]

$$\begin{aligned} |\beta,M,s\rangle &= \sum_{n=0}^{M} \Omega_n^M(\beta,s)\,|n\rangle \\ &= \sum_{n=0}^{M} \sqrt{\binom{n+s}{n}\binom{\frac{M}{1-\beta}-n-s-1}{M-n}\binom{\frac{M}{1-\beta}}{M}^{-1}}\,|n\rangle\,. \end{aligned} \qquad (3)$$

where β is real number and s a non-negative integer satisfying $s < \frac{M\beta}{1-\beta} < \frac{M}{1-\beta}$. The NHGSs is also claimed to be a intermediate BS-NBS state. One can see that it is equivalent to the PS and the generalized non-classical state. Using the following identities

$$\binom{n+s}{n} = \frac{(n+s)_n}{n!}$$

$$\binom{\frac{M}{1-\beta}-n-s-1}{M-n} = \frac{\left(\frac{M\beta}{1-\beta}-s\right)_{M-n}}{(M-n)!}$$

It is found that the NHGS and PS are equivalent. Thus the three intermediate BS-NBS states, the PS, the generalized non-classical state related to Hahn polynomials and the NHGS are equivalent. In Section 3 of this article, we will discuss the dynamical properties of the Wehrl space entropy and entanglement of the single two-level atom system and optical field initially prepared in the HGS and NHGS.

Model and its dynamics

The field-atom interaction is a main application in quantum information and quantum statistic. In this regard the important and simplest model is known as JCM [28], which describes interaction between a two-level atom and optical radiation field. JCM have important significance because JCM is experimentally realized and it have many theoretical investigations [29]. Stimulated by the JCM success, more researchers have paid special attention to the generalizations by considering new quantum effects [30].

Here, the model under consideration is an intensity-dependent JCM of a two-level atom interacting resonantly with a single mode of the radiation field in a cavity via multi-photon process where the coupling is intensity dependent; this coupling preserves the energy of the system. Under the rotating-wave approximation, the interaction Hamiltonian of the system reservoir is given by

$$\hat{H}_{in} = G(t)\left(\sqrt{\hat{a}^{\dagger}\hat{a}}\hat{a}|0\rangle\langle 1| + \hat{a}^{\dagger}\sqrt{\hat{a}^{\dagger}\hat{a}}|1\rangle\langle 0|\right). \qquad (4)$$

Here, $|0\rangle$ ($|1\rangle$) is the upper (lower) state of a two-level atom, $\hat{a}^{\dagger}(\hat{a})$ is the creation (annihilation) operator of the field mode, $G(t)$ is the time dependent coupling between the two-level atom and field. In the case of neglecting the atomic motion effect $G(t) = g = const$. When the time dependent coupling $G(t)$ is taken into account, the transient regime where the coupling varies rapidly with time t. The generalization from the constant coupling g to arbitrary time dependent coupling $G(t)$ enables us to model several new physical situations not discussed before. A realization of particular interest when $G(t)$ may be the time-dependent alignment or orientation of the atomic/molecular dipole moment using laser pulse [31] and motion of the atom through the cavity. So, we assume that the coupling is modeled approximately to be sinusoidal $G(t) = g\sin^2(t)$.

The initial state is given by $|\psi(0)\rangle = |\psi_A(0)\rangle \otimes |\psi_F(0)\rangle$, where $|\psi_A(0)\rangle$ is the initial state of the two-level atom and $|\psi_F(0)\rangle$ is the initial state of the input field. The combined two-level atom-field system can be written as

$$|\psi(0)\rangle = |0\rangle \otimes |\psi_F(0)\rangle = \begin{cases} |L,M,\eta,0\rangle & \text{for the HGSs given by Eq. (1)} \\ \sum\limits_{n=0}^{M} \Omega_n^M(\beta,s)\,|n,0\rangle & \text{for the NHGSs given by Eq. (3)} \end{cases}. \tag{5}$$

The wave function can be obtained as

$$|\psi(t)\rangle = \exp\left[-i\int_0^t H_I(\tau)\,d\tau\right]|\psi(0)\rangle. \tag{6}$$

All information about the system is carried by either the wave function (6) or the total (atom-field) density matrix $\hat{\rho}(t) = \hat{\rho}_{AF}(t) = |\psi(t)\rangle\langle\psi(t)|$. Therefore, we evaluate the field reduced density matrix $\hat{\rho}_F(t)$ via the relation

$$\hat{\rho}_F(t) = \mathrm{Tr}_A\{\hat{\rho}(t)\}, \tag{7}$$

where the subscript Q means that the trace is taken over the two-level atom basis. We close this section by evaluating the Husimi Q function Q_F of the field mode in terms of the diagonal elements of the density operator in the coherent state basis. We get

$$Q_F(\beta,\beta^*) = \frac{1}{\pi}\langle\beta|\rho_F(t)|\beta\rangle \tag{8}$$

where ρ_F is the field's reduced density operator.

3 Statistical properties,Wehrl space entropy and nonlocal correlation

In this section we turn our attention to the concept of the classical-like (semiclassical) Wehrl entropy [32], as a very useful measure for describing the time evolution of a quantum system in phase-space. The atomic Wehrl entropy is used for detecing the entanglement in quantum systems [33,34,35]. The Wehrl entropy, introduced as a classical entropy of a quantum state yields additional insights into the dynamics of the system, as compared to other entropies [32]. This semiclassical information entropy is defined as the coherent-state representation of the density matrix [32,36] via

$$S_W(t) = -\int Q_F(\beta,t)\ln Q_F(\beta,t)\,d^2\beta, \tag{9}$$

where $Q_F(\beta,t)$ is given by (8) and $d^2\beta = |\beta|\,d|\beta|\,d\Theta$. We can specialize things by recourse to the Wehrl phase distribution (Wehrl PD), defined to be the phase density of the Wehrl entropy [37,38], i.e.,

$$S_\Theta(t) = -\int Q_F(\beta,t)\ln Q_F(\beta,t)\,|\beta|\,d|\beta| \tag{10}$$

where $\Theta = \arg(\beta)$.

It is well known that the nonlocal correlation or entanglement between the two-level atom and field state can be quantified by the von Neumann entropy [39,40], which is generally defined in terms of the reduced field (atom) density matrix as

$$S_F = -Tr(\rho_F \ln\rho_F) = -\sum_{j=0}^{\infty}\lambda_j\ln\lambda_j \tag{11}$$

where $\rho_F = Tr_A(\rho_{AF})$ is the reduced density operator of field F, and λ_j are its eigenvalues.

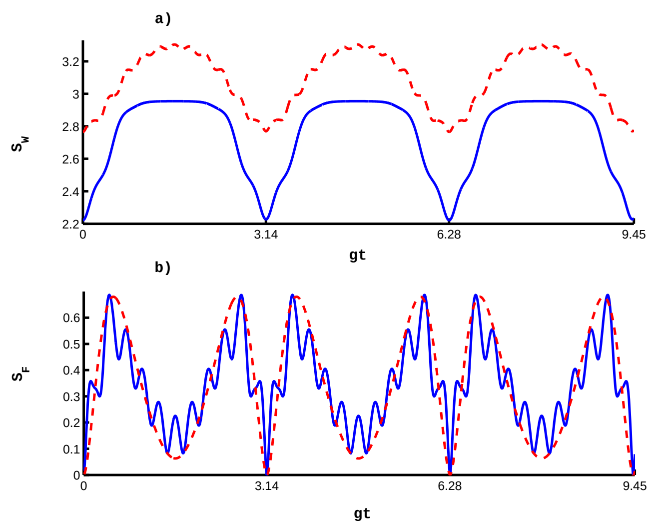

Fig. 1: The time evolution of the: a) Wehrl space entropy S_W and b) von Neumann (field) entropy S_F of a stationary two-level atom interacting with field initially prepared in HGD for $M = 10$, and with $\eta = 0.5$ (solid line) and $\eta = 0.9$ (dashed line).

4 Numerical results and discursion

In this section, we discuss a atom-field system whose dynamics is described by the JC-model with and without

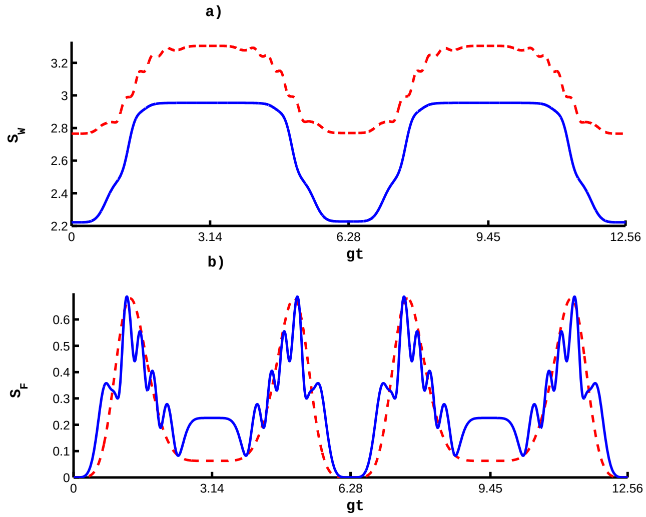

Fig. 2: The same as Fig.1 but for moving two-level atom case where the atomic motion is considered through $G(t) = g\sin^2(t)$.

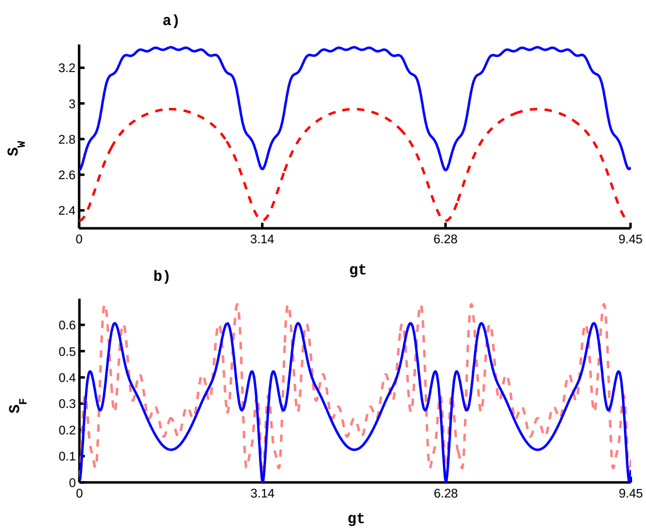

Fig. 3: The time evolution of the: a) Wehrl space entropy S_W and b) von Neumann (field) entropy S_F of a stationary two-level atom interacting with field initially prepared in NHGD for $M = 10$ and with $\beta = 0.5$ (solid line) and $\beta = 0.9$ (dashed line).

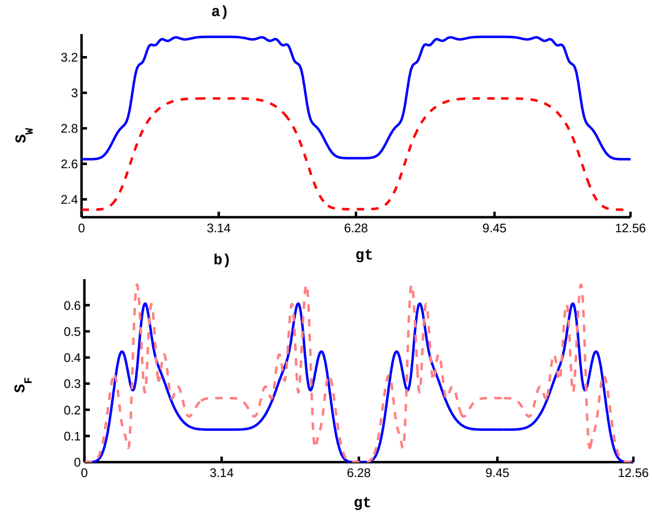

Fig. 4: The same as Fig.3 but for moving two-level atom case where the atomic motion is considered through $G(t) = g\sin^2(t)$.

time-dependent coupling effect. To explore the influence of the different parameters on the dynamical behavior of the quantum entanglement, nonclassical properties and Wehrl entropy of the system under consideration, we have plotted in Figs 1 and 2 the time evolution of the field entropy, S_F and Wehrl entropy S_W as a function of the scaled time gt when the time-dependent coupling is neglected (i.e. $G(t) = g = const.$) and considered ($G(t) = g\sin^2(t)$) for various values of the HGSs. Figs. 3,4 are the same as Figs. 1,2 but the field starts from NHGSs.

Fig. 1 depicts the dynamical behavior of the Wehrl entropy $S_W(t)$ of the field initially prepared in HGS for $M = 10$. The dashed red line line is for $\eta = 0.9$ and the solid blue line is for $\eta = 0.5$. Generally, the Wehrl entropy increases with increasing time and stabilizes at the maximal values after long time, indicating that the field becomes more quantum mechanical in this limit. From another side, $S_W(m\pi) = 1 + \ln(\pi)$ detect the field is more classical for $\eta = 0.5$, but in the case of high values of the parameter η (e.g. $\eta = 0.9$) the initial value of S_W at $t = m\pi$ increase to be 2.8. As the parameter η increases the field being more quantum. A saturation behavior of S_W is obtained through the time interval $\pi/4 \leq t \leq 3\pi/4$. A growth of S_W is observed $0 \leq gt \leq \pi/4$ and the decreasing $\pi/4 \leq gt \leq 3\pi/4$.

To describe the dynamical behavior of the entanglement in this model, it is useful to investigate the variation of the field entropy as shown in Fig. 1(b). It is observed that the field entropy has a different order as a function of the scaled time in the stationary two-level atom case. Interestingly, after an initial change with rapid oscillatory, in a periodic manner through every periodic interval $m\pi \leq gt \leq (m+1)\pi$. Also, the system returns to its separable state (zero value of field entropy) which corresponding to minimum value of Wehrl entropy and the classicality of the field. These results resported that the strong correlation between the field entropy and Wehrl entropy. On the other hand through the comparison between the solid curve and dashed curve it is clear that Wehrl entropy is very sensitive to the initial field distribution parameter so it is gives a good description for the statistical properties of the field.

According to Fig.2 we can see that the time-dependent coupling does not a strong effect the time evolution of the Wehrl entropy in the presence of the atomic motion. In this case the only effect appear though the change the periodic time from $gt = m\pi$ in the stationary atom case to $gt = 2m\pi$ the moving atom case. The atom-field entanglement or nonlocal correlation is affected by the atomic motion (see Fig.2 (b)). It is observed that a short time around the periodic time these no quantum correlation observed where the field is more classical in this interval.

Fig. 3 illustrates the influence of the changing the initial field state or the distribution of the input field elements from the HGSs to the NHGSs on the time evolution of the atom-field entanglement. From another side, the dependance on the time-dependent coupling is shown in Fig. 4 (a,b), where the field entropy and Wehrl entropy are plotted as a function of the index gt. As can be seen, the field entropy and Wehrl entropy have a periodic behavior during the time evolution in the case case of high and low value of the NHGSs parameters. Also, the field entropy exhibiting the phenomena of sudden death and sudden birth of entanglement.

5 Conclusion:

In conclusion, we have analyzed the dynamics of a two-level atom interacting with field initially in HGSs and NHGSs with time-dependent coupling effect. We have investigated the quantum features sudden birth and sudden death of entanglement, and statistical properties. We have established the analytical results for certain parametric conditions and we analyze the influence of initial field distribution parameters on the entanglement and Wehrl entropy. We have determined the different situations of the atom-field system for which the time-dependent coupling effect and atomic motion are very significant for this model. Finally, we have explored an interesting relation between the atom-field entanglement and Wehrl entropy behavior during the time evolution where it is shown that the amount of atom-field entanglement can be enhanced as the field tends to be more quantum. Also, when the field is closed to the classical state no quantum correlations can be obtained and the system return to its separable state.

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