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An Algorithm for Computing Digital Cohomology Groups

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Abstract: Several recent papers have discussed the digital version of the cohomology group for digital images, and some researchers calculate digital cohomology groups of some special two or three-dimensional digital images. In this paper, we determine the simplicial cohomology groups of some minimal simple closed curves and the digital surface MSS_6. Also we give a general algorithm for computing digital cohomology groups of finite dimensional digital images.

Keywords: Digital curve, digital surface, digital cohomology group.

1 Introduction

Homology and cohomology are both topological invariants. But there are some differences between them; one of the matter is by the multiplication, called cup product, cohomology groups have also ring structure. This makes cohomology stronger and more useful than homology since cohomology can separate between some certain algebraic objects that homology can not. Thus if there are some spaces that have the same homology and cohomology as groups, there can be differences on their ring structure.

Digital topology [19, 23] has been used in different image processing and computer graphics algorithms for thirty years. It addresses the fundamental properties of binary object connectivity in two dimensional (2D) and three dimensional (3D) digital images. Concepts and results of Digital Topology are used to specify and justify some important low-level image processing algorithms including algorithms for thinning, boundary extraction, object counting and contour filling. The properties of digital images with tools from Topology (including Algebraic Topology) are required to characterize by many researchers [1]- [9] [17, 21, 23, 25]. Simplicial homology groups of digital images have been studied by several researchers [1, 8, 10, 12]. Boxer et al. extend results of [1] about computing simplicial homology groups of digital images.

Gonzalez-Diaz and Real [15] obtain the cohomology ring of a three-dimensional digital binary-valued picture by a simplicial complex topologically representing (up to isomorphisms of pictures) the picture. Gonzalez-Diaz et al. [14] exhibit cohomology in the context of structural pattern recognition and introduce an algorithm to compute representative cocycles in 2D.

Karaca and Ege [12] study on some results about the simplicial homology of 2D digital images. They investigate some fundamental properties of cubical homology groups of digital images. They also calculate cubical homology groups of certain 2-dimensional and 3-dimensional digital images [13].

Burak and Karaca [9] compute a simplicial homology group of some specific digital images, they define ring and algebra structures of digital cohomology with the cup product, and they prove a special case of the Borsuk-Ulam theorem for digital images.

Pilarczyk and Real [22] introduce algorithms to compute homology, cohomology and related operations on cubical cell complexes by using a technique based on a chain contraction from the original chain complex to a reduced one that represents its homology.

Demir and Karaca [10] compute simplicial homology groups of the digital surfaces $MSS_{18}\sharp MSS_{18}$, MSS_6, and $MSS_6\sharp MSS_6$. They also present i-regularity of two ordered pair of digital simplices, give the definition of cup-i product over digital images by using regularity notion, and study some basic properties of the squaring operations [11].

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This paper is organized as follows: First we recall some basic notions on digital images. Then we determine the simplicial cohomology groups of some certain minimal simple closed curves and a surface. Finally, we give a general algorithm for any finite dimensional digital image that shows how we make those calculations.

2 Preliminaries

Let $\mathbb{Z}^n$ be the set of lattice points in the n-dimensional Euclidean space where $\mathbb{Z}$ is the set of integers. We say that (X,κ) is a (binary) digital image where $X \subset \mathbb{Z}^n$ and κ is an adjacency relation for the members of X. We use a variety of adjacency relations in the study of digital images.

For a positive integer l with $1 \leq l \leq n$ and two distinct points $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n) \in \mathbb{Z}^n$, p and q are c_l*-adjacent* [6] if

(1) there are at most l indices i such that $|p_i - q_i| = 1$; and

(2) for all other indices i such that $|p_i - q_i| \neq 1$, $p_i = q_i$.

Another commonly using of the notation c_l reflects the number of neighbors $q \in \mathbb{Z}^n$ that a given point $p \in \mathbb{Z}^n$ may have under the adjacency. For example, if $n = 1$ we have $c_1 = 2$-adjacency; if $n = 2$ we have $c_1 = 4$-adjacency and $c_2 = 8$-adjacency; if $n = 3$ we have $c_1 = 6$-adjacency, $c_2 = 18$-adjacency, and $c_3 = 26$-adjacency [6]. Given a natural number l in conditions (1) and (2) with $1 \leq l \leq n$, l determines each of the κ-adjacency relations of $\mathbb{Z}^n$ in terms of (1) and (2) [16] as follows.

$$\kappa \in \Big\{2n\ (n \geq 1),\ 3^n - 1\ (n \geq 2),\ 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1\ (2 \leq r \leq n-1, n \geq 3)\Big\} \quad (2.1)$$

The pair (X,κ) is considered in a digital picture $(\mathbb{Z}^n, \kappa, \overline{\kappa}, X)$ for $n \geq 1$ in [2, 3, 5, 17], which is called a *digital image* where $(\kappa, \overline{\kappa}) \in \{(\kappa, 2n), (2n, 3^n - 1)\}$. Each of κ and $\overline{\kappa}$ is one of the general κ-adjacency relations. We usually do not permit that κ and $\overline{\kappa}$ both equal $2n$ when $n > 1$, because of the digital connectivity paradox [20]. For instance, $(\kappa, \overline{\kappa}) \in \{(4,8),(8,4)\}$ and $\{(6,18),(6,26),(26,6),(18,6)\}$ are usually considered in $\mathbb{Z}^2$ and $\mathbb{Z}^3$, respectively [5, 17, 23, 24].

A *digital interval* is a set of the form

$$[a,b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$$

where $a, b \in \mathbb{Z}$ with $a < b$.

Let κ be an adjacency relation on $\mathbb{Z}^n$. A κ-neighbor of a lattice point p is κ-adjacent to p. A digital image $X \subset \mathbb{Z}^n$ is κ*-connected* [18] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \ldots, x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$ and x_i and x_{i+1} are κ-neighbors where $i = 0, 1, \ldots, r-1$. A κ*-component* of a digital image X is a maximal κ-connected subset of X.

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0 and κ_1-adjacency respectively. Then the function $f : X \to Y$ is called (κ_0, κ_1)*-continuous* [5,24] if for every κ_0-connected subset U of X, $f(U)$ is a κ_1-connected subset of Y. We say that such a function is digitally continuous.

Let X be a digital image with κ-adjacency. If $f : [0,m]_{\mathbb{Z}} \to X$ is a $(2,\kappa)$-continuous function such that $f(0) = x$ and $f(m) = y$, then f is called a *digital path* from x to y in X. If $f(0) = f(m)$ then the κ-path is said to be *closed*, and the function is called a κ*-loop*. Let $f : [0,m-1]_{\mathbb{Z}} \to X$ be a $(2,\kappa)$-continuous function such that $f(i)$ and $f(j)$ are κ-adjacent if and only if $j = i \pm 1\ mod\ m$. Then the set $f([0,m-1]_{\mathbb{Z}})$ is called a *simple closed* κ*-curve*. A point $x \in X$ is called a κ*-corner*, if x is κ-adjacent to two and only two points $y, z \in X$ such that y and z are κ-adjacent to each other [3]. Moreover, the κ-corner x is called *simple* if y, z are not κ-corners and if x is the only point κ-adjacent to both y, z [2]. X is called a *generalized simple closed* κ*-curve* if what is obtained by removing all simple κ-corners of X is a simple closed κ-curve [3]. If (X,κ) is a κ-connected digital image in $\mathbb{Z}^3$,

$$|X|^x = N_3^*(x) \cap X,$$

where $N_3^*(x) = \{x' \in \mathbb{Z}^3 : x \text{ and } x' \text{ are 26-adjacent}\}$ [2, 3]. Generally, if (X,κ) is a κ-connected digital image in $\mathbb{Z}^n$, $|X|^x = N_n^*(x) \cap X$, where

$$N_n^*(x) = \{x' \in \mathbb{Z}^n : x \text{ and } x' \text{ are } c_n\text{-adjacent}\}\ [17].$$

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0 and κ_1-adjacency respectively. A function $f : X \to Y$ is a (κ_0, κ_1)*-isomorphism* [7] (called (κ_0, κ_1)-homeomorphism in [4]) if f is (κ_0, κ_1)-continuous, bijective and $f^{-1} : Y \to X$ is (κ_1, κ_0)-continuous, in which case we write $X \approx_{(\kappa_0,\kappa_1)} Y$.

Definition 2.1. [17] Let $c^* := \{x_0, x_1, \ldots, x_n\}$ be a closed κ-curve in $\mathbb{Z}^2$ where $\{\kappa, \overline{\kappa}\} = \{4,8\}$. A point x of the complement $\overline{c^*}$ of a closed κ-curve c^* in $\mathbb{Z}^2$ is said to be in the *interior* of c^* if it belongs to the bounded $\overline{\kappa}$-connected component of $\overline{c^*}$. The set of all interior points of c^* is denoted by $Int(c^*)$.

Definition 2.2. [17] Let (X,κ) be a digital image in $\mathbb{Z}^n$, $n \geq 3$ and $\overline{X} = \mathbb{Z}^n - X$. Then X is called a *closed* κ*-surface* if it satisfies the following.

(1) In case that $(\kappa, \overline{\kappa}) \in \{(\kappa, 2n), (2n, 3^n - 1)\}$, where the κ-adjacency is taken from Definition 2.1 with $\kappa \neq 3^n - 2^n - 1$ and $\overline{\kappa}$ is the adjacency on $\overline{X}$, then

(a) for each point $x \in X$, $|X|^x$ has exactly one κ-component κ-adjacent to x;

(b) $|\overline{X}|^x$ has exactly two $\overline{\kappa}$-components $\overline{\kappa}$-adjacent to x; we denote by C^{xx} and D^{xx} these two components; and

(c) for any point $y \in N_\kappa(x) \cap X$, $N_{\overline{\kappa}}(y) \cap C^{xx} \neq \emptyset$ and $N_{\overline{\kappa}}(y) \cap D^{xx} \neq \emptyset$, where $N_\kappa(x)$ means the κ-neighbors of x.

Further, if a closed κ-surface X does not have a simple κ-point, then X is called simple.

(2) In case that $(\kappa,\overline{\kappa}) = (3^n - 2^n - 1, 2n)$, then
(a) X is κ-connected,
(b) for each point $x \in X$, $|X|^x$ is a generalized simple closed κ-curve.
Further, if the image $|X|^x$ is a simple closed κ-curve, then the closed κ-surface X is called simple.

Definition 2.3. [26] Let S be a set of nonempty subsets of a digital image (X,κ). The members of S are called simplexes of (X,κ) if the following holds:

(i) If p and q are distinct points of $s \in S$, then p and q are κ-adjacent.

(ii) If $s \in S$ and $\emptyset \neq t \subset s$, then $t \in S$ (note this implies every point p that belongs to a simplex determines a simplex $\{p\}$).

An m-simplex is a simplex S such that $|S| = m+1$.

Let P be a digital m-simplex. If P' is a nonempty proper subset of P, then P' is called a face of P.

Definition 2.4. [1] Let (X,κ) be a finite collection of digital m-simplices, $0 \le m \le d$ for some nonnegative integer d. If the following statements hold, then (X,κ) is called a finite digital simplicial complex:

(1) If P belongs to X, then every face of P also belongs to X.

(2) If $P,Q \in X$, then $P \cap Q$ is either empty or a common face of P and Q.

The dimension of a digital simplicial complex X is the biggest integer m such that X has an m-simplex.

$C_q^\kappa(X)$ is a free abelian group with basis all digital (κ,q)-simplices in X [1].

Corollary 2.5. [8] Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. Then for all $q > m$, $C_q^\kappa(X)$ is a trivial group.

Definition 2.6. [1] Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. The homomorphism $\partial_q : C_q^\kappa(X) \to C_{q-1}^\kappa(X)$ defined by

$$\partial_q(< p_0, p_1, ..., p_q >) = \begin{cases} \sum_{i=0}^{q} (-1)^i < p_0, p_1, ..., \widehat{p_i}, ..., p_q >, & q \le m; \\ 0, & q > m \end{cases}$$

is called a boundary homomorphism where $\widehat{p_i}$ means deleting the point p_i. Then for all $1 \le q \le m$, we have $\partial_{q-1} \circ \partial_q = 0$.

Theorem 2.7. [1] Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. Then

$$C_*^\kappa(X) : 0 \xrightarrow{\partial_{m+1}} C_m^\kappa(X) \xrightarrow{\partial_m} \dots \xrightarrow{\partial_1} C_0^\kappa(X) \xrightarrow{\partial_0} 0$$

is a chain complex.

Definition 2.8. [1] Let (X,κ) be a digital simplicial complex. The group of digital simplicial q-cycles is

$$Z_q^\kappa(X) = Ker\ \partial_q = \{\sigma \in C_q^\kappa(X) | \partial_q(\sigma) = 0\}$$

and the group of digital simplicial q-boundaries is

$$\begin{aligned} B_q^\kappa(X) =& Im\ \partial_{q+1} \\ =& \{\tau \in C_q^\kappa(X) | \partial_{q+1}(\sigma) = \tau \text{ for } \sigma \in C_{q+1}^\kappa(X)\}. \end{aligned}$$

So the q^{th} digital simplicial homology group is

$$H_q^\kappa(X) = Z_q^\kappa(X)/B_q^\kappa(X).$$

Theorem 2.9. [1] If $f : X \to Y$ is a digital (κ_0,κ_1)-isomorphism, then for all $q \le m$

$$H_q^{\kappa_0}(X) \cong H_q^{\kappa_1}(Y).$$

Theorem 2.10. [8] Let (X,κ) be a directed digital simplicial complex of dimension m.

(1) $H_q^\kappa(X)$ is a finitely generated abelian group for every $q \ge 0$.

(2) $H_q^\kappa(X)$ is a trivial group for all $q > m$.

(3) $H_q^\kappa(X)$ is a free abelian group, possibly zero.

Definition 2.11. [21] Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simlicial complex and C_q^κ be an abelian group whose bases are all (κ,q)-simplexes in X. $C^{*,\kappa}(X) = \{C^{q,\kappa}(X), \delta_q\}_{q \ge 0}$ is the digital cochain complex of X where

$$\begin{aligned} C^{q,\kappa}(X) =& Hom(C_q^\kappa(X), G) \\ =& \{c : C_q^\kappa(X) \to G | c \text{ is a homomorphism}\}. \end{aligned}$$

Here $\delta_q : C^{q,\kappa}(X) \to C^{q+1,\kappa}(X)$ is the *digital cochain homomorphism* and defined as $\delta_q(c)(a) = c(\partial_{q+1}(a))$ for $c \in C^{q,\kappa}(X)$, $a \in C_{q+1}^\kappa(X)$. $Z^{q,\kappa}(X;G)$ is the kernel of δ_q and called group of *digital cocycles* of (X,κ) with coefficients in G, $B^{q,\kappa}(X;G)$ is the image of δ_{q-1} and called group of *digital coboundaries* of (X,κ) with coefficients in G, and (noting that since $\partial^2 = 0$, $\delta^2 = 0$)

$$H^{q,\kappa}(X;G) = Z^{q,\kappa}(X;G)/B^{q,\kappa}(X;G)$$

is called the *digital q^{th} cohomology group* of (X,κ) with coefficients in G.

We use the $\langle c^q, c_q \rangle$ representation to denote the value of c^q on c_q where c^q is the q-dimensional digital cochain and c_q is the q-dimensional digital chain. Using this notation, we can state the cohomology operator

$$\langle \delta c^q, d_{q+1} \rangle = \langle c^q, \partial d_{q+1} \rangle$$

such that $d_{q+1} \in C_{q+1}^\kappa(X)$. Recall that the group $C_q^\kappa(X)$ of digital q-chains is free abelian; it has a standard basis obtained by orienting the digital q-simplices of X arbitrarily and using the corresponding elementary chains as a basis. Let $\{\sigma_\alpha\}_{\alpha \in I}$ be the collection of oriented digital (κ,q)-simplices. Under this circumstance the elements of $C_q^\kappa(X)$ are represented as finite linear combinations $\sum n_\alpha \sigma_\alpha$ of the elementary digital chains σ_α.

Let σ be the elementary digital cochain with $\mathbb{Z}$ coefficients such that

$$\langle \sigma_\alpha^*, \sigma_\alpha \rangle = 1 \text{ and } \langle \sigma_\alpha^*, \sigma_\beta \rangle = 0 \text{ for all } \beta \neq \alpha.$$

Then if $g \in G$, we let $g\sigma_\alpha^*$ denote the digital cochain such that

$$\langle g\sigma_\alpha^*, \sigma_\alpha \rangle = g \text{ and } \langle g\sigma_\alpha^*, \sigma_\beta \rangle = 0 \text{ for all } \beta \neq \alpha.$$

By using this notation, we write

$$c^q = \sum g_\alpha \sigma_\alpha^*.$$

Then

$$\delta c^q = \sum g_\alpha (\delta \sigma_\alpha^*)$$

where $\delta\sigma_\alpha^* = \sum \varepsilon_j \tau_j^*$. In this representation, the summation is taken over all digital $q+1$-simplices τ_j having σ as a face and $\varepsilon_j = \pm 1$ is the sign with which σ appears in the expression for $\partial \tau_j$ where

$$\partial \tau_j = \sum_{i=0}^{q+1} \varepsilon_i \sigma_{\alpha_i}.$$

Theorem 2.13. [21] If (X, κ) is a singleton digital image, then

$$H^{q,\kappa}(X;G) = \begin{cases} G, & q = 0; \\ 0, & q > 0 \end{cases}$$

where G is an abelian group.

3 Simplicial Cohomology Groups of Some Digital Images

By using the analogue argument in [21], simplicial cohomology groups of several digital images have been computed in following theorems.

Theorem 3.1. Let X be a digital image in $\mathbb{Z}^2$ with the points $\{c_0 = (0,0), c_1 = (1,0), c_2 = (1,1)\}$ and adjacency relation $\kappa = 8$ (see Figure 1). The digital simplicial cohomology groups of X are

$$H^{q,8}(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0; \\ 0, & q \neq 0. \end{cases}$$

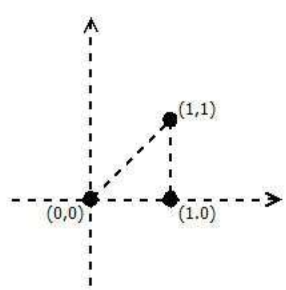

Fig. 1: $X = \{c_0 = (0,0), c_1 = (1,0), c_2 = (1,1)\}$

Proof. If we use the dictionary ordering, we can direct X as $c_0 < c_1 < c_2$. Then we have the following simplicial chain complexes: $C_0^8(X)$ has for a basis $\{\langle c_0 \rangle, \langle c_1 \rangle, \langle c_2 \rangle\}$, $C_1^8(X)$ has for a basis $\{e_0 = \langle c_0c_1 \rangle, e_1 = \langle c_0c_2 \rangle, e_2 = \langle c_1c_2 \rangle\}$, and $C_2^8(X)$ has for a basis $\{\sigma = \langle c_0c_1c_2 \rangle\}$. Hence we get the following short sequence

$$0 \xrightarrow{\partial_3} C_2^8(X) \xrightarrow{\partial_2} C_1^8(X) \xrightarrow{\partial_1} C_0^8(X) \xrightarrow{\partial_0} 0,$$

by using the sequence above and Definition 2.11 we get the following short sequence

$$0 \xrightarrow{\delta^{-1}} C^{0,8}(X) \xrightarrow{\delta^0} C^{1,8}(X) \xrightarrow{\delta^1} C^{2,8}(X) \xrightarrow{\delta^2} 0$$

where $C^{q,8}(X) = Hom(C_q^8(X), \mathbb{Z})$ and $q \in \{0,1,2\}$. Since $Ker\ \delta^q \cong \{0\}$ for all $q \geq 3$, $H^{q,8}(X)$ is a trivial group.

We first determine the kernel of δ^0. Let's take any general 0-cochain $p^0 = \sum_{i=0}^{2} n_i c_i^*$. p^0 is a cocycle if and only if $\delta^0(p^0) = 0$ if and only if $n_0 = n_1 = n_2 = n$. So we can write 0-cochain as $p^0 = n\sum_{i=0}^{2} c_i^*$ and this gives us $Z^{0,8}(X) \cong \mathbb{Z}$. And since $Im\ \ \delta^{-1} \cong \{0\}$, we get $H^{0,8}(X) = \mathbb{Z}$.

Since

$$\langle \delta^1 r^1, \sigma \rangle = \langle r^1, \partial_2 \sigma \rangle = r^1(e_0 + e_2 - e_1) = 0$$

and

$$\langle \delta^1 s^1, \sigma \rangle = \langle s^1, \partial_2 \sigma \rangle = s^1(e_0 + e_2 - e_1) = 0$$

such that $r^1 = e_0^* + e_1^*$ and $s^1 = e_1^* + e_2^*$, r^1 and s^1 are 1-cocycles. So

$$Z^{1,8}(X) = Span\{r^1, s^1\} \cong \mathbb{Z}^2.$$

We need to find the image of δ^0. Let $p^0 = \sum_{i=0}^{2} n_i c_i^*$ be any general 0-cochain. Since

$$\begin{aligned} \langle \delta^0 c_0^*, e_0 \rangle &= -1 & \langle \delta^0 c_0^*, e_1 \rangle &= -1 \\ \langle \delta^0 c_1^*, e_0 \rangle &= 1 & \langle \delta^0 c_1^*, e_2 \rangle &= -1 \\ \langle \delta^0 c_2^*, e_1 \rangle &= 1 & \langle \delta^0 c_2^*, e_2 \rangle &= 1 \end{aligned}$$

we can write $\delta^0 c_0^* = -e_0^* - e_1^*$, $\delta^0 c_1^* = e_0^* - e_2^*$ and $\delta^0 c_2^* = e_1^* + e_2^*$. Accordingly, from the equation below

$$\begin{aligned} \delta^0(p^0) &= \sum_{i=0}^{2} n_i \delta^0(c_i^*) \\ &= (-n_0 + n_1)e_0^* + (-n_0 + n_2)e_1^* + (-n_1 + n_2)e_2^*, \end{aligned}$$

we find

$$\begin{aligned} B^{1,8}(X) &= Im\ \delta^0 \\ &= \{n_0 e_0^* + n_1 e_1^* + (-n_0 + n_1)e_2^* : n_0, n_1 \in \mathbb{Z}\} \cong \mathbb{Z}^2. \end{aligned}$$

Thus $H^{1,8}(X) = \{0\}$.

Since

$$\langle \delta^1 p^1, \sigma \rangle = \langle p^1, \partial_2 \sigma \rangle = p^1(e_0 - e_1 + e_2) = 1,$$

$\delta^1(p^1) = \{\sigma^*\}$ for any general 1-cochain $p^1 = \sum_{i=0}^{2} n_i e_i^*$. So $B^{2,8}(X) = Im\ \delta^1 \cong \mathbb{Z}$ and since $Ker\ \delta^2 \cong \mathbb{Z}$, we can write $H^{2,8}(X) = \{0\}$. □

Theorem 3.2. If

$$\begin{aligned} MSC_4 = \{&c_0 = (-1,-1), c_1 = (-1,0), c_2 = (-1,1), \\ &c_3 = (0,1), c_4 = (1,1), c_5 = (1,0), \\ &c_6 = (1,-1), c_7 = (0,-1)\} \end{aligned}$$

(see Figure 2), then the digital simplicial cohomology groups of MSC_4 are

$$H^{q,4}(MSC_4;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0,1; \\ 0, & q \neq 0,1. \end{cases}$$

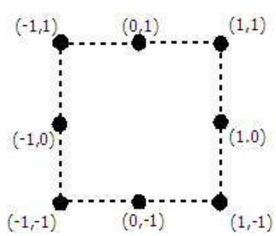

Fig. 2: MSC_4

Proof. By using the dictionary ordering, we can direct the points of MSC_4 as $c_0 < c_1 < c_2 < c_7 < c_3 < c_6 < c_5 < c_4$. Then we have the following simplicial chain complexes: $C_0^4(MSC_4)$ has for a basis

$$\{\langle c_0\rangle, \langle c_1\rangle, \langle c_2\rangle, \langle c_3\rangle, \langle c_4\rangle, \langle c_5\rangle, \langle c_6\rangle, \langle c_7\rangle\},$$

$C_1^4(MSC_4)$ has for a basis

$$\begin{aligned} \{&e_0 = \langle c_0 c_1\rangle, e_1 = \langle c_1 c_2\rangle, e_2 = \langle c_2 c_3\rangle, e_3 = \langle c_3 c_4\rangle, \\ &e_4 = \langle c_5 c_4\rangle, e_5 = \langle c_6 c_5\rangle, e_6 = \langle c_7 c_6\rangle, e_7 = \langle c_0 c_7\rangle\}, \end{aligned}$$

and $C_q^4(MSC_4) = \{0\}$ for all $q \geq 2$.

Thus, we obtain the following short sequence

$$0 \xrightarrow{\partial_2} C_1^4(MSC_4) \xrightarrow{\partial_1} C_0^4(MSC_4) \xrightarrow{\partial_0} 0,$$

by using the sequence above and Definition 2.11 we obtain the following short sequence

$$0 \xrightarrow{\delta^{-1}} C^{0,4}(MSC_4) \xrightarrow{\delta^0} C^{1,4}(MSC_4) \xrightarrow{\delta^1} 0$$

where $C^{q,4}(MSC_4) = Hom(C_q^4(MSC_4), \mathbb{Z})$ and $q \in \{0,1\}$. Since $Ker\ \delta^q \cong \{0\}$ for all $q \geq 2$, $H^{q,4}(MSC_4)$ is a trivial group.

Let's $p^0 = \sum_{i=0}^{7} n_i c_i^*$ be any general 0-cochain. Since

$$\begin{array}{lll} \langle \delta^0 c_0^*, e_0\rangle = -1 & \langle \delta^0 c_3^*, e_2\rangle = 1 & \langle \delta^0 c_5^*, e_5\rangle = 1 \\ \langle \delta^0 c_0^*, e_7\rangle = -1 & \langle \delta^0 c_3^*, e_3\rangle = -1 & \langle \delta^0 c_6^*, e_5\rangle = -1 \\ \langle \delta^0 c_1^*, e_0\rangle = 1 & \langle \delta^0 c_4^*, e_3\rangle = 1 & \langle \delta^0 c_6^*, e_6\rangle = 1 \\ \langle \delta^0 c_1^*, e_1\rangle = -1 & \langle \delta^0 c_4^*, e_4\rangle = 1 & \langle \delta^0 c_7^*, e_6\rangle = -1 \\ \langle \delta^0 c_2^*, e_1\rangle = 1 & \langle \delta^0 c_5^*, e_4\rangle = -1 & \langle \delta^0 c_7^*, e_7\rangle = 1 \\ \langle \delta^0 c_2^*, e_2\rangle = -1 & & \end{array}$$

we can write

$$\begin{array}{ll} \delta^0 c_0^* = -e_0^* - e_7^* & \delta^0 c_4^* = e_3^* + e_4^* \\ \delta^0 c_1^* = e_0^* - e_1^* & \delta^0 c_5^* = -e_4^* + e_5^* \\ \delta^0 c_2^* = e_1^* - e_2^* & \delta^0 c_6^* = -e_5^* + e_6^* \\ \delta^0 c_3^* = e_2^* - e_3^* & \delta^0 c_7^* = -e_6^* + e_7^* \end{array}$$

p^0 is a cocycle if and only if

$$\begin{aligned} \delta^0(p^0) = &\sum_{i=0}^{7} n_i \delta^0(c^i) \\ = &(-n_0 + n_1)e_0^* + (-n_1 + n_2)e_1^* + (-n_2 + n_3)e_2^* \\ &+ (-n_3 + n_4)e_3^* + (n_4 - n_5)e_4^* + (n_5 - n_6)e_5^* \\ &+ (n_6 - n_7)e_6^* + (n_7 - n_0)e_7^* \\ = &0 \end{aligned} \tag{3.1}$$

if and only if $n_0 = n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = n_7 = n$. Thus we can state 0-cochain as $p^0 = n \sum_{i=0}^{7} c_i^*$ and this means $Z^{0,4}(MSC_4) \cong \mathbb{Z}$. Since $Im\ \delta^{-1} \cong \{0\}$, we find $H^{0,4}(MSC_4) = \mathbb{Z}$.

We need to find the image of δ^0. By the equation (3.1), we get

$$\begin{aligned} B^{1,4}(MSC_4) = &Im\ \delta^0 \\ = &\{\sum_{i=0}^{3} n_i e_i^* + \sum_{i=4}^{6} (-n_i) e_i^* + \sum_{i=1}^{7} n_i e_7^* : n_i \in \mathbb{Z}\} \\ \cong &\mathbb{Z}^7. \end{aligned}$$

Since $Ker\ \delta^1 \cong \mathbb{Z}^8$, we have $H^{1,4}(MSC_4) = \mathbb{Z}$. □

Theorem 3.3. Let

$$\begin{aligned} MSC_8 = \{&c_0 = (-1,-1), c_1 = (-1,0), c_2 = (0,1), \\ &c_3 = (1,0), c_4 = (1,-1), c_5 = (0,-2)\} \end{aligned}$$

(see Figure 3), then we have

$$H^{q,8}(MSC_8;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0,1; \\ 0, & q \neq 0,1. \end{cases}$$

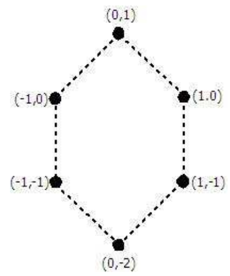

Fig. 3: MSC_8

Proof. By using the dictionary ordering, we can direct the points of MSC_8 as $c_0 < c_1 < c_5 < c_2 < c_4 < c_3$. Then we have the following simplicial chain complexes:
$C_0^8(MSC_8)$ has for a basis

$$\{\langle c_0\rangle, \langle c_1\rangle, \langle c_2\rangle, \langle c_3\rangle, \langle c_4\rangle, \langle c_5\rangle\},$$

$C_1^8(MSC_8)$ has for a basis

$$\{e_0 = \langle c_0c_1\rangle, e_1 = \langle c_1c_2\rangle, e_2 = \langle c_2c_3\rangle, e_3 = \langle c_4c_3\rangle,$$
$$e_4 = \langle c_5c_4\rangle, e_5 = \langle c_0c_5\rangle\},$$

and $C_q^8(MSC_8) = \{0\}$ for all $q \geq 2$. Thus, we obtain the following short sequence

$$0 \xrightarrow{\partial_2} C_1^8(MSC_8) \xrightarrow{\partial_1} C_0^8(MSC_8) \xrightarrow{\partial_0} 0,$$

by using the sequence above and Definition 2.11 we have the following short sequence

$$0 \xrightarrow{\delta^{-1}} C^{0,8}(MSC_8) \xrightarrow{\delta^0} C^{1,8}(MSC_8) \xrightarrow{\delta^1} 0.$$

where $C^{q,8}(MSC_8) = Hom(C_q^8(MSC_8), \mathbb{Z})$ and $q \in \{0,1\}$. Since $Ker\ \delta^q \cong \{0\}$ for all $q \geq 2$, $H^{q,8}(MSC_8)$ is a trivial group.

Let's $p^0 = \sum_{i=0}^{5} n_i c_i^*$ be any general 0-cochain. Since

$$\begin{array}{lll}
\langle\delta^0 c_0^*, e_0\rangle = -1 & \langle\delta^0 c_2^*, e_1\rangle = 1 & \langle\delta^0 c_4^*, e_3\rangle = -1 \\
\langle\delta^0 c_0^*, e_5\rangle = -1 & \langle\delta^0 c_2^*, e_2\rangle = -1 & \langle\delta^0 c_4^*, e_4\rangle = 1 \\
\langle\delta^0 c_1^*, e_0\rangle = 1 & \langle\delta^0 c_3^*, e_2\rangle = 1 & \langle\delta^0 c_5^*, e_4\rangle = -1 \\
\langle\delta^0 c_1^*, e_1\rangle = -1 & \langle\delta^0 c_3^*, e_3\rangle = 1 & \langle\delta^0 c_5^*, e_5\rangle = 1
\end{array}$$

we can write

$$\begin{array}{ll}
\delta^0 c_0^* = -e_0^* - e_5^* & \delta^0 c_3^* = e_2^* + e_3^* \\
\delta^0 c_1^* = e_0^* - e_1^* & \delta^0 c_4^* = -e_3^* + e_4^* \\
\delta^0 c_2^* = e_1^* - e_2^* & \delta^0 c_5^* = -e_4^* + e_5^*
\end{array}$$

p^0 is a cocycle if and only if

$$\begin{aligned}
\delta^0(p^0) =& \sum_{i=0}^{5} n_i \delta^0(c^i) \\
=& (-n_0 + n_1)e_0^* + (-n_1 + n_2)e_1^* + (-n_2 + n_3)e_2^* \\
& + (n_3 - n_4)e_3^* + (n_4 - n_5)e_4^* + (-n_0 + n_5)e_5^* \\
=& 0 \qquad (3.2)
\end{aligned}$$

if and only if $n_0 = n_1 = n_2 = n_3 = n_4 = n_5 = n$. Thus we can state 0-cochain as $p^0 = n\sum_{i=0}^{5} c_i^*$ and this means $Z^{0,8}(MSC_8) \cong \mathbb{Z}$. Since $Im\ \delta^{-1} \cong \{0\}$, we find $H^{0,8}(MSC_8) = \mathbb{Z}$.

By the equation (3.2), we have

$$\begin{aligned}
B^{1,8}(MSC_8) =& Im\ \delta^0 \\
=& \{n_1e_0^* + n_2e_1^* + n_3e_2^* - n_4e_3^* - n_5e_4^* \\
& + \sum_{i=1}^{5} n_i e_5^* : n_i \in \mathbb{Z}\} \\
\cong& \mathbb{Z}^5.
\end{aligned}$$

Since $Ker\ \delta^1 \cong \mathbb{Z}^6$, we get $H^{1,8}(MSC_8) = \mathbb{Z}$. □

Theorem 3.4. The digital simplicial cohomology groups of MSS_6 (see Figure 4) are

$$H^{q,6}(MSS_6;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0; \\ \mathbb{Z}^{23}, & q = 1; \\ 0, & q \neq 0,1. \end{cases}$$

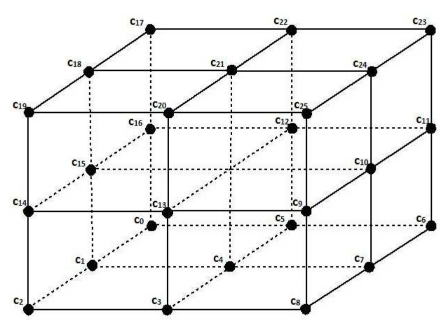

Fig. 4: MSS_6

Proof. Here we direct MSS_6 again with using the dictionary ordering. We have the following simplicial chain complexes:
$C_0^6(MSS_6)$ has for a basis $\{\langle c_0\rangle, \langle c_1\rangle, \langle c_2\rangle, \dots, \langle c_{25}\rangle\}$,

$C_1^6(MSS_6)$ has for a basis

$$\{e_0=\langle c_0c_1\rangle,e_1=\langle c_0c_5\rangle,e_2=\langle c_0c_{16}\rangle,e_3=\langle c_1c_2\rangle,$$
$$e_4=\langle c_1c_4\rangle,e_5=\langle c_1c_{15}\rangle,e_6=\langle c_2c_{14}\rangle,e_7=\langle c_2c_3\rangle,$$
$$e_8=\langle c_4c_3\rangle,e_9=\langle c_3c_8\rangle,e_{10}=\langle c_3c_{13}\rangle,e_{11}=\langle c_5c_4\rangle,$$
$$e_{12}=\langle c_4c_7\rangle,e_{13}=\langle c_5c_6\rangle,e_{14}=\langle c_5c_{12}\rangle,e_{15}=\langle c_6c_7\rangle,$$
$$e_{16}=\langle c_6c_{11}\rangle,e_{17}=\langle c_7c_8\rangle,e_{18}=\langle c_7c_{10}\rangle,e_{19}=\langle c_8c_9\rangle,$$
$$e_{20}=\langle c_{10}c_9\rangle,e_{21}=\langle c_{13}c_9\rangle,e_{22}=\langle c_9c_{25}\rangle,e_{23}=\langle c_{11}c_{10}\rangle,$$
$$e_{24}=\langle c_{10}c_{24}\rangle,e_{25}=\langle c_{12}c_{11}\rangle,e_{26}=\langle c_{11}c_{23}\rangle,$$
$$e_{27}=\langle c_{16}c_{12}\rangle,e_{28}=\langle c_{12}c_{22}\rangle,e_{29}=\langle c_{14}c_{13}\rangle,$$
$$e_{30}=\langle c_{13}c_{20}\rangle,e_{31}=\langle c_{15}c_{14}\rangle,e_{32}=\langle c_{14}c_{19}\rangle,$$
$$e_{33}=\langle c_{16}c_{15}\rangle,e_{34}=\langle c_{15}c_{18}\rangle,e_{35}=\langle c_{16}c_{17}\rangle,$$
$$e_{36}=\langle c_{17}c_{18}\rangle,e_{37}=\langle c_{17}c_{22}\rangle,e_{38}=\langle c_{18}c_{19}\rangle,$$
$$e_{39}=\langle c_{18}c_{21}\rangle,e_{40}=\langle c_{19}c_{20}\rangle,e_{41}=\langle c_{21}c_{20}\rangle,$$
$$e_{42}=\langle c_{20}c_{25}\rangle,e_{43}=\langle c_{22}c_{21}\rangle,e_{44}=\langle c_{21}c_{24}\rangle,$$
$$e_{45}=\langle c_{22}c_{23}\rangle,e_{46}=\langle c_{23}c_{24}\rangle,e_{47}=\langle c_{24}c_{25}\rangle\},$$

and $C_q^6(MSS_6)=0$ for all $q\geq 2$. Hence, we get the following short sequence

$$0\xrightarrow{\partial_2}C_1^6(MSS_6)\xrightarrow{\partial_1}C_0^6(MSS_6)\xrightarrow{\partial_0}0,$$

and by using above we have the following short sequence

$$0\xrightarrow{\delta^{-1}}C^{0,6}(MSS_6)\xrightarrow{\delta^0}C^{1,6}(MSS_6)\xrightarrow{\delta^1}0.$$

where $C^{q,6}(MSS_6)=Hom(C_1^6(MSS_6);\mathbb{Z})$ and $q\in\{0,1\}$. Since $Ker\ \delta^q\cong\{0\}$ for all $q\geq 2$, $H^{q,6}(MSS_6)$ is a trivial group.

From the definition

$$\begin{array}{lll}
\partial_1e_0=c_1-c_0 & \partial_1e_{16}=c_{11}-c_6 & \partial_1e_{32}=c_{19}-c_{14}\\
\partial_1e_1=c_5-c_0 & \partial_1e_{17}=c_8-c_7 & \partial_1e_{33}=c_{15}-c_{16}\\
\partial_1e_2=c_{16}-c_0 & \partial_1e_{18}=c_{10}-c_7 & \partial_1e_{34}=c_{18}-c_{15}\\
\partial_1e_3=c_2-c_1 & \partial_1e_{19}=c_9-c_8 & \partial_1e_{35}=c_{17}-c_{16}\\
\partial_1e_4=c_4-c_1 & \partial_1e_{20}=c_9-c_{10} & \partial_1e_{36}=c_{18}-c_{17}\\
\partial_1e_5=c_{15}-c_1 & \partial_1e_{21}=c_9-c_{13} & \partial_1e_{37}=c_{22}-c_{17}\\
\partial_1e_6=c_{14}-c_2 & \partial_1e_{22}=c_{25}-c_9 & \partial_1e_{38}=c_{19}-c_{18}\\
\partial_1e_7=c_3-c_2 & \partial_1e_{23}=c_{10}-c_{11} & \partial_1e_{39}=c_{21}-c_{18}\\
\partial_1e_8=c_3-c_4 & \partial_1e_{24}=c_{24}-c_{10} & \partial_1e_{40}=c_{20}-c_{19}\\
\partial_1e_9=c_8-c_3 & \partial_1e_{25}=c_{11}-c_{12} & \partial_1e_{41}=c_{20}-c_{21}\\
\partial_1e_{10}=c_{13}-c_3 & \partial_1e_{26}=c_{23}-c_{11} & \partial_1e_{42}=c_{25}-c_{20}\\
\partial_1e_{11}=c_4-c_5 & \partial_1e_{27}=c_{12}-c_{16} & \partial_1e_{43}=c_{21}-c_{22}\\
\partial_1e_{12}=c_7-c_4 & \partial_1e_{28}=c_{22}-c_{12} & \partial_1e_{44}=c_{24}-c_{21}\\
\partial_1e_{13}=c_6-c_5 & \partial_1e_{29}=c_{13}-c_{14} & \partial_1e_{45}=c_{23}-c_{22}\\
\partial_1e_{14}=c_{12}-c_5 & \partial_1e_{30}=c_{20}-c_{13} & \partial_1e_{46}=c_{24}-c_{23}\\
\partial_1e_{15}=c_7-c_6 & \partial_1e_{31}=c_{14}-c_{15} & \partial_1e_{47}=c_{25}-c_{24}
\end{array}$$

Thus we can write digital zero cochains as follows:

$$\begin{array}{ll}
\delta^0c_0^*=-e_0^*-e_1^*-e_2^* & \delta^0c_{13}^*=e_{10}^*-e_{21}^*+e_{29}^*-e_{30}^*\\
\delta^0c_1^*=e_0^*-e_3^*-e_4^*-e_5^* & \delta^0c_{14}^*=e_6^*-e_{29}^*+e_{31}^*-e_{32}^*\\
\delta^0c_2^*=e_3^*-e_6^*-e_7^* & \delta^0c_{15}^*=e_5^*-e_{31}^*+e_{33}^*-e_{34}^*\\
\delta^0c_3^*=e_7^*+e_8^*-e_9^*-e_{10}^* & \delta^0c_{16}^*=e_2^*-e_{27}^*-e_{33}^*-e_{35}^*\\
\delta^0c_4^*=e_4^*-e_8^*+e_{11}^*-e_{12}^* & \delta^0c_{17}^*=e_{35}^*-e_{36}^*-e_{37}^*\\
\delta^0c_5^*=e_1^*-e_{11}^*-e_{13}^*-e_{15}^* & \delta^0c_{18}^*=e_{34}^*+e_{36}^*-e_{38}^*-e_{39}^*\\
\delta^0c_6^*=e_{13}^*-e_{15}^*-e_{16}^* & \delta^0c_{19}^*=e_{32}^*+e_{38}^*-e_{40}^*\\
\delta^0c_7^*=-e_{12}^*+e_{15}^*-e_{17}^*-e_{18}^* & \delta^0c_{20}^*=e_{30}^*+e_{40}^*+e_{41}^*-e_{42}^*\\
\delta^0c_8^*=e_9^*+e_{17}^*-e_{19}^* & \delta^0c_{21}^*=e_{39}^*-e_{41}^*-e_{43}^*-e_{44}^*\\
\delta^0c_9^*=e_{19}^*+e_{20}^*+e_{21}^*-e_{22}^* & \delta^0c_{22}^*=e_{28}^*+e_{37}^*-e_{43}^*-e_{45}^*\\
\delta^0c_{10}^*=e_{18}^*-e_{20}^*+e_{23}^*-e_{24}^* & \delta^0c_{23}^*=e_{26}^*+e_{45}^*-e_{46}^*\\
\delta^0c_{11}^*=e_{16}^*-e_{23}^*+e_{25}^*-e_{26}^* & \delta^0c_{24}^*=e_{24}^*+e_{44}^*+e_{46}^*-e_{47}^*\\
\delta^0c_{12}^*=e_{14}^*-e_{25}^*+e_{27}^*-e_{28}^* & \delta^0c_{25}^*=e_{22}^*+e_{42}^*+e_{47}^*
\end{array}$$

Let's consider any general 0-cochain $p^0=\sum_{i=0}^{25}n_ic_i^*$. p^0 is a cocycle if and only if $\delta^0p^0=0$ if and only if

$$n_0=n_1=\cdots=n_{25}=n.$$

By virtue of this, we can write $p^0=n\sum_{i=0}^{25}c_i^*$ and we say $Z^{0,6}(MSS_6)=Ker\ \delta^0\cong\mathbb{Z}$. Beside $Im\ \delta^{-1}\cong\{0\}$, we have $H^{0,6}(MSS_6)\cong\mathbb{Z}$.

When we solve the equation system above, we get $B^{1,6}(MSS_6)\cong\mathbb{Z}^{25}$ and since we have $Ker\ \delta^1\cong\mathbb{Z}^{48}$, we get $H^{1,6}(MSS_6)\cong\mathbb{Z}^{23}$. □

4 Conclusion

The purpose of this paper is to determine digital cohomology groups of some special digital images such as digital circle MSC_4 and digital sphere MSS_6, and to give an algorithm for computing cohomology groups of digital images. In this work, we first compute digital cohomology groups of some certain digital closed curves and a surface. Since these are minimal structures for digital images, we hope that these computations and especially the algorithm will be useful in the study of digital cohomology groups.

An Algorithm for Calculating Cohomology Group of a Digital Image

```
 Input: A digital simplicial complex
        of dimension m, (X,κ) ⊂ ℤ^n.
Output: Cohomology group of given
        digital simplicial complex with
        coefficients in ℤ.

BEGIN

  Take the coordinates of p+1 points of
digital simplicial complex into an
integer array A[p+1][n].
```

$(c_0 = (c_{01}, c_{02}, \ldots, c_{0n}), c_1 = (c_{11}, c_{12}, \ldots, c_{1n}),$
$c_2 = (c_{21}, c_{22}, \ldots, c_{2n}), \cdots, c_p = (c_{(p)1}, c_{(p)2}, \ldots, c_{(p)n}))$

```
  Order the points with respect to
dictionary order.

  FOR i ← 0 TO n DO
      if (i <= m){
            detect C_i^κ(X)
                   C^{i,κ}(X;ℤ) := Hom(C_i^κ(X),ℤ)
                }
      else {C_i^κ(X) = 0
           C^{i,κ}(X;ℤ) = 0
           }
  REPEAT
```

//While constructing ∂_i, use Definition 2.6.

```
  FOR i ← m TO 1 DO
      ∂_i : C_i^κ(X) → C_{i-1}^κ(X)
  REPEAT
```

//Define ∂_{m+1} as zero homomorphism and ∂_0 as trivial homomorphism.

//While constructing δ_i, use Definition 2.11.

```
  FOR i ← 0 TO m−1 DO
      δ_i : C^{i,κ}(X) → C^{i+1,κ}(X)
  REPEAT
```

//Define δ_{-1} as zero homomorphism and δ_m as trivial homomorphism.

//While constructing $Z^{i,\kappa}(X,\mathbb{Z})$, $B^{i,\kappa}(X,\mathbb{Z})$ and $H^{i,\kappa}(X,\mathbb{Z})$, use Definition 2.11.

```
  FOR i ← 0 TO m DO
      detect Z^{i,κ}(X,ℤ)
             B^{i,κ}(X,ℤ)
             H^{i,κ}(X,ℤ) = Z^{i,κ}(X,ℤ)/B^{i,κ}(X,ℤ)
  REPEAT
END
```


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