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On Bipermutable and S-Bipermutable Subgroups of Finite Groups

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Abstract: Let H be a subgroup of a finite group G. Then we say that H is: *bipermutable* in G provided G has subgroups A and B such that $G = AB$, $H \le A$ and H permutes with all subgroups of A and with all subgroups of B; S-*bipermutable* in G provided G has subgroups A and B such that $G = AB$, $H \le A$ and H permutes with all Sylow subgroups of A and with all Sylow p-subgroups of B such that $(|H|, p) = 1$. In this paper we analyze the influence of bipermutable and S-bipermutable subgroups on the structure of G.

Keywords: finite group, S-bipermutable subgroup, Hall subgroup, Sylow subgroup, p-soluble group, p-supersoluble group, saturated formation.

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover p is always supposed to be a prime dividing $|G|$. We use $\mathscr{M}_\phi(G)$ to denote a set of maximal subgroups of G such that $\Phi(G)$ coincides with the intersection of all subgroups in $\mathscr{M}_\phi(G)$. Let A and B be subgroups of G. If $AB = BA$, then A is said to *permute* with B; if $G = AB$, then B is called a *supplement* of A to G.

A subgroup H is said to be *quasinormal* [1] or *permutable* [2] in G if permutes with all subgroups of G, H is said to be S-*permutable*, S-*quasinormal*, or π-*quasinormal* [3] in G if H permutes with all Sylow subgroups of G. In this paper we study the following generalizations of these concepts.

Definition 1.1. Let H be a subgroup of G. Then we say that H is:

(1) *bipermutable* in G provided G has subgroups A and B such that $G = AB$, $H \le A$ and H permutes with all subgroups of A and with all subgroups of B.

(2) S-*bipermutable* in G provided G has subgroups A and B such that $G = AB$, $H \le A$ and H permutes with all Sylow subgroups of A and with all Sylow p-subgroups of B such that $(|H|, p) = 1$.

In last years, many researches (see, for example [4]–[15]) deal with some interesting subclasses of the class of all bipermutable subgroups and of the class of all S-bipermutable subgroups. Recall, for example, that a subgroup H of G is called semi-normal [16] (SS-*quasinormal* [5]) in G if H permutes with all subgroups (with all Sylow subgroups, respectively) of some supplement of H to G. A subgroup H of G is called S-*semipermutable* [17] in G if H permutes with all Sylow p-subgroups of G for all primes p such that $(|H|, p) = 1$. It is clear that every SS-quasinormal subgroup and every S-semipermutable subgroup are S-bipermutable. Every semi-normal subgroup is bipermutable. The following elementary example shows that, in general, the set of all S-bipermutable subgroups of G is wider than the set of all its SS-quasinormal subgroups and the set of all its S-semipermutable subgroups.

Example 1.2. Let $p > q > r$ be primes such that qr divides $p-1$. Let P be a group of order p and $QR \le Aut(P)$, where Q and R are groups with order q and r, respectively. Let $G = P \rtimes (QR)$. Then R is bipermutable in G. Suppose that R is S-semipermutable in G. Then $Q^xR = RQ^x$ for all $x \in G$. But $Q^xR \simeq G/P$ is cyclic, so $Q^G = PQ \le N_G(R)$. Hence R is normal in G, which implies that $R \le C_G(P) = P$. Therefore R is not S-semipermutable in G. Later, after veiwing of Lemma 2.5, one can easly show that, R is not SS-quasinormal in G too.

Our main goal here is to prove the following results.

Theorem A. Let P be a Sylow p-subgroup of G.

(I) If P is S-bipermutable in G, then G is p-soluble.

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(II) If P is bipermutable in G, then the following statements hold:

(i) G is p-soluble and $P' \leq O_p(G)$. If, in addition, $N_G(P)$ is p-nilpotent, then the focal subgroup $G' \cap P$ of G is contained in $O_p(G)$.

(ii) If p is the largest prime dividing $|G|$, then P is normal in G.

(iii) $l_p(G) \leq 2$.

(iv) If for some prime $q \neq p$ a Hall p'-subgroup of G is q-supersoluble, then G is q-supersoluble.

Corollary 1.3 (See Main result in [6]). Let P be a Sylow p-subgroup of G. If P is semi-normal in G, then the following statements hold:

(i) G is p-soluble and $P' \leq O_p(G)$.

(ii) $l_p(G) \leq 2$.

(iii) If for some prime $q \neq p$ a Hall p'-subgroup of G is q-supersoluble, then G is q-supersoluble.

Corollary 1.4 (See Theorem 3 in [18]). Let P be a Sylow p-subgroup of G, where p is the largest prime dividing $|G|$. If P is semi-normal in G, then P is normal in G.

On the basis of Theorem A we prove also the following results.

Theorem B. Let P be a Sylow p-subgroup of G. Suppose that $|P| > p$. If every member V of some fixed $\mathscr{M}_\phi(P)$ is S-bipermutable in G, then G is p-supersoluble. If, in addition, $(p-1, |G|) = 1$, then G is p-nilpotent.

Theorem B has many corollaries. In particular, this theorem covers Theorems 1.1–1.4 in [5] (see Section 4).

The following our theorem covers main result in [18].

Theorem C. If every Sylow subgroup of G is bipermutable in G, then G is supersoluble.

All unexplained notation and terminology are standard. The reader is referred to [19], [20], [21] and [22] if necessary.

2 Preliminaries

Lemma 2.1 (see Theorem 4.6 in [30, Chapter VI]). Let A and B be subgroups of G such that $G = AB$.

(1) If G is p-soluble, then there are Hall p'-subgroups $A_{p'}$, $B_{p'}$ and $G_{p'}$ of A, B and G, respectively, such that $G_{p'} = A_{p'}B_{p'}$

(2) For any prime p dividing $|G|$, there are Sylow p-subgroups A_p, B_p and G_p of A, B and G, respectively, such that $G_p = A_pB_p$.

Lemma 2.2 (see [20, Chapter A, Lemma 1.6]). Let H, K and N be subgroups of G. If $HK = KH$ and $HN = NH$, then $H\langle K, N\rangle = \langle K, N\rangle H$.

Lemma 2.3. Let H be an S-bipermutable subgroup of G and N a normal subgroup of G such that for every prime p dividing $|H|$ and for every Sylow p-subgroup H_p we have $H_p \nleq N$. Then

(1) HN/N is S-bipermutable in G/N.

(2) If H is bipermutable in G, then HN/N is bipermutable in G/N.

(3) H permutes with some Sylow p-subgroup of G for all primes p such that $(|H|, p) = 1$.

(4) If G is p-soluble and H is a p-group, then H permutes with some Hall p'-subgroup of G.

Proof. (1) By hypothesis there are subgroups A_1 and A_2 of G such that $G = A_1A_2$, $H \leq A_1$ and H permutes with all Sylow subgroups of A_1 and with all Sylow p-subgroups of A_2 for all primes p satisfying $(|H|, p) = 1$.

Then $G/N = (A_1N/N)(A_2N/N)$ and $HN/N \leq A_1N/N$. Let K/N be any Sylow p-subgroup of A_2N/N such that $(|HN/N|, p) = 1$. Since for every prime q dividing $|H|$ and for any Sylow q-subgroup H_q of H we have $H_q \nleq N$, $(|H|, p) = 1$. Moreover, $K = (K \cap A_2)N$, so by Lemma 2.1, there are Sylow p-subgroups K_p, P and N_p of K, $K \cap A_2$ and N, respectively, such that $K_p = PN_p$. Let $P \leq A_p$, where A_p is a Sylow p-subgroup of A_2. Then $K/N \leq A_pN/N$, which implies that $K/N = A_pN/N$. But H permutes with A_p, so that HN/N permutes with K/N. Similarly, it may be proved that HN/N permutes with all Sylow subgroups of A_1N/N. Therefore HN/N is S-bipermutable in G/N.

(2) See the proof of (1).

(3) By Lemma 2.1 there are Sylow p-subgroups P_1, P_2 and P of A_1, A_2 and G, respectively, such that $P = P_1P_2$. Then

$$HP = H(P_1P_2) = (HP_1)P_2 = (P_1H)P_2 =$$

$$P_1(HP_2) = P_1(P_2H) = (P_1P_2)H = PH.$$

(4) See the proof of (3) and use Lemma 2.2.

A group G is said to be p-closed provided G has a normal Sylow p-subgroup.

Lemma 2.4. Let P be a Sylow p-subgroup of G and A a subgroup of G. If P permutes with all Sylow p-subgroups of A, then A is p-closed.

Proof. Let A_p be a Sylow p-subgroup of A. By hypothesis, $PA_p = A_pP$. Hence $A_p \leq P$. Thus $(A_p)^A \leq P$. But then $(A_p)^A$ is a p-group and so $A_p = (A_p)^A$ is normal in A.

Lemma 2.5. Let H and B be subgroups of G. If $G = AB$, where $A \leq N_G(H)$, and $HV^b = V^bH$ for some subgroup V of B and for all $b \in B$, then $HV^x = V^xH$ for all $x \in G$.

Proof. Since $G = AB = N_G(H)B$ we have $x = bn$ for some $b \in B$ and $n \in N_G(H)$. Hence $HV^x = HV^{bn} = Hn(V^b)n^{-1} = n(V^b)n^{-1}H = V^xH$.

Lemma 2.6. Let A and B be subgroups of G. If $A^xB = BA^x$ for all $x \in G$, then $AB^x = B^xA$ for all $x \in G$.

Proof. Indeed, from $A^{x^{-1}}B = BA^{x^{-1}}$ we get $AB^x = (A^{x^{-1}}B)^x = (BA^{x^{-1}})^x = B^xA$.

Lemma 2.7 (O. Kegel [24]). Let A and B be subgroups of G such that $G \neq AB$ and $AB^x = B^xA$, for all $x \in G$. Then G has a proper normal subgroup N such that either $A \leq N$ or $B \leq N$.

In our proofs we shall need the following well-known properties of supersoluble and p-supersoluble groups.

Lemma 2.8. Let N and R be normal subgroups of G.

(1) If $N \leq \Phi(G) \cap R$ and R/N is p-supersoluble, then R is p-supersoluble.

(2) If G is p-supersoluble and $O_{p'}(G) = 1$, then p is the largest prime dividing $|G|$, G is supersoluble and $F(G) = O_p(G)$ is a normal Sylow p-subgroup of G.

(3) If G is supersoluble, then $G' \leq F(G)$ and G is a Sylow tower group of supersoluble type.

Lemma 2.9 (O. Kegel [25]). If G has three nilpotent subgroups A_1, A_2 and A_3 whose indices $|G:A_1|$, $|G:A_2|$, $|G:A_3|$ are pairwise coprime, then G is itself nilpotent.

Lemma 2.10 (V. N. Knyagina and V. S. Monakhov [12]). Let H, K and N be subgroups of G. If N is normal in G, H permutes with K and H is a Hall subgroup of G, then

$$N \cap HK = (N \cap H)(N \cap K).$$

Lemma 2.11 (See Lemma 1.2.16 in [27]). If H is S-permutable in G and H is a p-group for some prime p, then $O^p(G) \leq N_G(H)$.

Lemma 2.12 (See Lemma 2.15 in [28]). Let E be a normal non-identity quasinilpotent subgroup of G. If $\Phi(G) \cap E = 1$, then E is the direct product of some minimal normal subgroups of G.

Lemma 2.13. Suppose that G is p-soluble and $O_{p'}(G) = 1$. Then $F^*(G) = O_p(G)$.

Proof. It is clear that $F(G) = O_p(G) \leq F^*(G)$. Suppose that $O_p(G) \neq F^*(G)$ and let $H/O_p(G)$ be a chief factor of G below $F^*(G)$. Then, since G is p-soluble, $H/O_p(G)$ is a non-abelian p'-group and $O_p(G) \leq Z_\infty(H)$ by [26, Chapter X, Theorems 13.6 and 13.7]. Hence $H/C_H(O_p(G))$ is a p-group by [31, Chapter 5, Theorem 1.4]. On the other hand, by the Schur-Zassenhaus theorem, $O_p(G)$ has a complement E in H. Then $E \leq C_H(O_p(G))$, which implies that E is normal in H. Thus E is a characteristic subgroup of E, so $E \leq O_{p'}(G) = 1$, a contradiction. The lemma is proved.

Let $\mathscr{F}$ be a class of groups. A chief factor H/K of G is called $\mathscr{F}$*-central* in G provided $(H/K) \rtimes (G/C_G(H/K)) \in \mathscr{F}$.

Lemma 2.14 (See [29, Theorem B]) Let $\mathscr{F}$ be any formation and E a normal subgroup of G. If each chief factor of G below $F^*(E)$ is $\mathscr{F}$-central in G, then each chief factor of G below E is $\mathscr{F}$-central in G as well.

Lemma 2.15. Let P be a Sylow p-subgroup of E such that $(p-1, |G|) = 1$. If either P is cyclic or G is p-supersoluble, then G is p-nilpotent.

Proof. Suppose that this lemma is false let G be a counterexample of minimal order. Then G is a minimal non-p-nilpotent group. Hence, by [30, Chaper IV, Satz 5.4], P is normal in G, G/P is nilpotent and $P/\Phi(P)$ is a chief factor of G. Since either P is cyclic or G is p-supersoluble, $|P/\Phi(P)| = p$. But since $(p-1, |G|) = 1$, we have $C_G(P/\Phi(P)) = G$, which implies the nilpotency of G. This contradiction completes the proof of the lemma.

3 Proofs of Theorems A, B and C

Proof of Theorem A. Suppose that this theorem is false and let G be a counterexample of minimal order.

(I) By hypothesis there are subgroups A and B of G such that $G = AB$, $P \leq A$ and P permutes with all Sylow subgroups of A and with all Sylow q-subgroups of B for all primes $q \neq p$.

(1) $P^G = P(P^G \cap B)$.

Since P permutes with all Sylow q-subgroups of B for all primes $q \neq p$, P permutes with $O^p(B)$ by Lemma 2.11. By Lemma 2.1, there are Sylow p-subgroups A_p, B_p and G_p of A, B and G, respectively, such that $G_p = A_pB_p$. By Lemma 2.4, P is normal in A. Hence $B_p \leq N_G(P)$. Therefore $PB = P(B_pO^p(B)) = (B_pO^p(B))P = BP$ is a subgroup of G. Thus $P^G = P^{AB} = P^B \leq \langle P, B\rangle = PB$ since $PB = BP$ is a subgroup of G. Hence $P^G = P^G \cap PB = P(P^G \cap B)$.

(2) If N is a non-identity normal subgroup of G, then N is not p-soluble.

Indeed, if $P \leq N$, then G/N is a p'-group and so the p-solubility of N implies the p-solubility of G. On the other hand, if $P \nleq N$, then the hypothesis holds for G/N by Lemma 2.3 (1). Hence G/N is p-soluble by the choice of G since $|G/N| < |G|$. Therefore in the case, when N is p-soluble, G is also p-soluble, which contradicts the choice of G.

(3) $P^G = G$.

From (1) we know that $P^G = P(P^G \cap B)$. Let Q be any Sylow q-subgroup of $P^G \cap B$, where $q \neq p$. Then for some Sylow q-subgroup B_q of B we have $Q = B_q \cap (P^G \cap B) = B_q \cap P^G$. Hence $PB_q \cap P^G = P(B_q \cap P^G) = PQ = QP$ is a subgroup of P^G. Therefore P is S-bipermutable in P^G, so the hypothesis holds for P^G. If $P^G \neq G$, then P^G is p-soluble by the choice of G. But this contradicts (2). Hence we have (3).

(4) If Q is a Sylow q-subgroup of $P^G \cap B$, where $q \neq p$ is a prime divisor of $|P^G \cap B|$, then the hypothesis is true for Q^G.

Let R be a Sylow r-subgroup of $Q^G \cap B$, where $r \neq p$. Then for some Sylow r-subgroup B_r of B we have $R = B_r \cap (Q^G \cap B) = B_r \cap Q^G$. By Lemma 2.10 we know also that $PB_r \cap Q^G = (P \cap Q^G)(B_r \cap Q^G) = (P \cap Q^G)R = R(P \cap Q^G)$, where $P \cap Q^G$ is a Sylow p-subgroup of Q^G. Therefore the hypothesis holds for Q^G.

Final contradiction for (I). From (2) and (4) it follows that $Q^G = G$. The choice of G implies by Burnside's p^aq^b-theorem that $PQ \neq G$. On the other hand, by Lemma 2.5, $PQ^x = Q^xP$ for all $x \in G$. Hence by Lemma 2.7, $P^G \neq G$. This contradiction completes the proof of Assertion (I).

(II) By (I), G is p-soluble. By hypothesis there are subgroups A and B of G such that $G = AB$, $P \leq A$ and P permutes with all subgroups of A and with all subgroups of B. The subgroup P is normal in A by Lemma 2.4, and P permutes with B. Therefore $= P^G = P^{AB} = P^B \leq PB$, which implies that $P^G = P(P^G \cap B)$.

(i) Suppose that this assertion is false. Then:

(1) $O_p(N) = 1$ for any normal subgroup N of G.

Indeed, suppose that $O_p(G) \neq 1$. By Lemma 2.3 (2), the hypothesis holds for $G/O_p(G)$. Hence Assertion (i) is true for $G/O_p(G)$ by the choice of G. Thus

$$P'O_p(G)/O_p(G) \leq (PO_p(G)/O_p(G))' \leq O_p(G/O_p(G)) = 1,$$

and if $N_G(P)$ is p-nilpotent, then

$$(G/O_p(G))' \cap (P/O_p(G)) = (G'O_p(G)/O_p(G)) \cap (P/O_p(G)) =$$
$$= O_p(G)(G' \cap P)/O_p(G) \leq O_p(G/O_p(G)) = 1.$$

Hence we have $P' \leq O_p(G)$ in the former case, and $G' \cap P \leq O_p(G)$ in the case, when $N_G(P)$ is p-nilpotent. Thus Assertion (i) is true for G, a contradiction. Therefore $O_p(G) = 1$. Finally, if N is a normal subgroup of G, then $O_p(N)$ is characteristic in N and so $O_p(N) \leq O_p(G) = 1$. Hence we have (1).

(2) P is not abelian.

Suppose that P is abelian. Then in the case, when $N_G(P)$ is p-nilpoten, $P \leq Z(N_G(P))$, so G is p-nilpotent by Burnside's theorem [30, IV, 2.6]. Hence a Hall p'-subgroup E of G is normal in G. Since P is abelian, it follows that $G' \leq E$. Therefore $G' \cap P = 1 \leq O_p(G)$, contrary to our assumption on G. Hence we have (2).

(3) $C_G(O_{p'}(G)) \leq O_{p'}(G) \neq 1$.

By (1), $O_p(G) = 1$. Therefore, since G is p-soluble, $O_{p',p}(G) = O_{p'}(G) \neq 1$ and so $C_G(O_{p'}(G)) \leq O_{p'}(G) \neq 1$ by [30, VI, 6.9].

(4) $P^G = G$ and $G = PB$.

Since $P^G = P(P^G \cap B)$, P is bipermutable in P^G. Therefore in the case, when $P^G \neq G$, $P' \leq O_p(P^G)$ by the choice of G. But by (1), $O_p(P^G) = 1$. Therefore $P' = 1$, so P is abelian, which contradicts (2). Thus $P^G = G$ and $G = PB$.

(5) G is not supersoluble. Suppose that G is supersoluble. Then $G' \leq F(G)$, so $G' \cap P \leq O_p(G) = 1$. Hence $P' = 1$, contrary to (2). Hence we have (5).

(6) $PO_{p'}(G) = G$. Suppose that $E = PO_{p'}(G) \neq G$. By (4) we have $O_{p'}(G) \leq B$. Hence P is bipermutable in E. Thus $P' \leq O_p(E)$ by the choice of G. Therefore $P' \leq C_G(O_{p'}(G)) \leq O_{p'}(G)$ by (3). Hence P is abelian, which contradicts (2). Thus $PO_{p'}(G) = G$.

(7) Final contradiction for (i). Let V be any subgroup of $O_{p'}(G)$. Then by (4) for any $x \in G$ we have $PV^x = V^xP$. Hence $VP^x = P^xV$ for all $x \in G$ by Lemma 2.6. Now note that $V = P^xV \cap O_{p'}(G)$ is normal in P^xV, so $P^x \leq N_G(V)$. But then, by (4), $G = P^G \leq N_G(V)$. Therefore every subgroup of $O_{p'}(G)$ is normal in G. But by (6), $PO_{p'}(G) = G$. Hence G is supersoluble, contrary to (5). Therefore Assertion (i) is true for G.

(ii) Suppose that this assertion is false. Then:

(a) $O_p(G) = 1$.

Suppose that $O_p(G) \neq 1$. Then Assertion (ii) is true for $G/O_p(G)$ by the choice of G, so $P/O_p(G)$ is normal in $G/O_p(G)$, which implies that P is normal in G. Hence $O_p(G) = 1$.

(b) $P^G = G$ and $G = PB$. Suppose that $P^G \neq G$. Since $P^G = P(P^G \cap B)$, the hypothesis holds for P^G. Hence P is normal in P^G by the choice of G. Hence P is characteristic in P^G, which implies that P is normal in G. This contradiction shows that $P^G = G$, so $G = PB$.

(c) G is not supersoluble (Since p is the largest prime dividing $|G|$, this assertion directly follows from (a) and Lemma 2.8 (2)).

(d) $C_G(O_{p'}(G)) \leq O_{p'}(G) \neq 1$ (See (3) in the proof of (i)).

Final contradiction for (ii). In view of (b), the hypothesis holds for $PO_{p'}(G)$. Therefore in the case, when $PO_{p'}(G) \neq G$, P is normal in $PO_{p'}(G)$, which implies that $P \leq C_G(O_{p'}(G)) \leq O_{p'}(G)$ by (4). This contradiction shows that $PO_{p'}(G) = G$. But then, in view of (b), G is supersoluble (see the final contradiction in the proof of (i)), which contradicts (c). Therefore Assertion (ii) is true for G.

(iii) Since by (i), $P' \leq O_p(G)$, every Sylow p-subgroup of $G/O_p(G)$ is abelian. Hence, by [30, Chapter VI, Satz 6.6], we have $l_p(G/O_p(G)) \leq 1$. But then $l_p(G) \leq 2$.

(iv) Suppose that this assertion is false. Let N be a minimal normal subgroup of G. Assume that $N \leq O_{q'}(G)$. Then the hypothesis holds for G/N, so G/N is q-supersoluble by the choice of G. But then G is q-supersoluble, contrary to our assumption on G. Therefore $O_{q'}(G) = 1$. In particular, $N \nleq P$, which implies that N is p'-group since G is p-soluble by (i). Let E be a Hall p'-subgroup of G. Then $N \leq E$, so N is a q-group since $O_{q'}(G) = 1$ and E is q-supersoluble. Thus the hypothesis holds for G/N. Therefore G/N is q-supersoluble. Hence N is the only minimal normal subgroup of G and $N \nleq \Phi(G)$ by Lemma 2.8 (1). Hence $N \leq P^G$, and $N = C_G(N)$ by [20, Chapter A , Theorem 15.2]. Since $P^G = P(P^G \cap B)$, it follows that $N \leq B$. Thus P permutes with all subgroups of N. Since E is q-supersoluble, N has a maximal subgroup V such that V is normal in E. On the other hand, $PV \cap N = V$ is normal in PV. Hence $G = PE \leq N_G(V)$, which in view of the minimality of N implies that $V = 1$. Hence $|N| = q$, so $G/N = G/C_G(N)$ is a cyclic group of exponent dividing $q-1$. But then G is supersoluble. This contradiction completes the proof of Assertion (iv). The theorem is proved.

Proof of Theorem B. Suppose that this theorem is false and let G be a counterexample of minimal order.

First we shall show that G is p-supersoluble. Assume that this is false. Let $V \in \mathscr{M}_\phi(P)$. By hypothesis there are subgroups A and B of G such that $G = AB$, $V \leq A$ and V permutes with all Sylow subgroups of A and with all Sylow q-subgroups of B for all primes $q \neq p$.

(1) V is normal in B, $V^G = V(V^G \cap B)$ and V permutes with every Sylow q-subgroup of $V^G \cap B$ for all primes $q \neq p$.

By Lemma 2.11, $O^p(A) \leq N_A(V)$. Hence, since V is maximal in P, $A = A_pO^p(A) \leq N_A(V)$ for any Sylow p-subgroup A_p of A. Therefore V is normal in A. Now arguing similarly as in the proof of Theorem A (I) one can show that V permutes with B. Hence

$$V^G = V^{AB} = V^{N_G(V)B} = V^B \leq \langle V, B\rangle = VB$$

since $VB = BV$ is a subgroup of G. Therefore $V^G = V^G \cap VB = V(V^G \cap B)$. Now let Q any Sylow q-subgroup of $V^G \cap B$, where $q \neq p$. Then for some Sylow q-subgroup B_q of B we have $Q = (V^G \cap B) \cap B_q$. On the other hand, $VB_q = B_qV$ is a subgroup of G, so

$$VB_q \cap V^G = V(B_q \cap V^G) = V(B_q \cap (Q \cap V^G)) = VQ = QV$$

is a subgroup of G. Hence we have (1).

(2) $O_{p'}(N) = 1$ for every normal subgroup N of G.

Indeed, suppose that for some normal subgroup N of G we have $O_{p'}(N) \neq 1$. Since $O_{p'}(N)$ is a characteristic subgroup of N, it is normal in G. On the other hand, by Lemma 2.3 (1), the hypothesis holds for $G/O_{p'}(N)$. Hence $G/O_{p'}(N)$ is p-supersoluble by the choice of G. Thus G is p-supersoluble, a contradiction.

(3) If L is a minimal normal subgroup of G, then $L \nleq \Phi(P)$. Indeed, in the case, where $L \leq \Phi(P)$, we have $L \leq \Phi(G)$ and the hypothesis holds for G/L by Lemma 2.3 (1). Hence G/L is p-supersoluble by the choice of L. Therefore G is p-supersoluble by Lemma 2.8 (1), contrary to the choice of G.

(4) Every normal p-soluble subgroup D of G is supersoluble and p-closed. By (2), $O_{p'}(D) = 1$. Therefore $O_p = O_p(D) \neq 1$ since D is p-soluble. Let N be a minimal normal subgroup of G contained in O_p. In view of (3) we have $N \nleq \Phi(P)$. Hence for some subgroup $W \in \mathscr{M}_\phi(P)$ we have $P = NW$. Let $S = N \cap W$. Then S is normal in P. On the other hand, by Lemma 2.3 (3), for any prime $q \neq p$, there is a Sylow q-subgroup Q of G such that $WQ = QW$. Hence $S = QW \cap N$ is a normal subgroup of QW and so $Q \leq N_G(S)$. Thus S is normal in G. Hence $|S| = 1$ and $|N| = p$. But then W is a complement of N in P, which implies by Gaschütz's theorem [30, Chapter I, Satz 17.4], that L has a complement M in G. Thus $N \nleq \Phi(G)$. It is clear that $\Phi(G) \cap O_p$ is normal in G. Therefore $\Phi(G) \cap O_p = 1$. Hence $O_p = L_1 \times \ldots \times L_t$, where $L_1, \ldots, L_t$ are minimal normal subgroups of G by Lemma 2.12. By (3) we have $L_i \nleq \Phi(P)$. Thus, as above, one can show that $|L_i| = p$. Therefore every chief factor of G below O_p is cyclic. On the other hand, by Lemma 2.13, $F^*(D) = O_p$. Hence D is supersoluble by Lemma 2.14. But $O_{p'}(D) = 1$, so O_p is a Sylow p-subgroup of D by Lemma 2.8 (2).

(5) G is p-soluble. Assume that G is not p-soluble. Then:

(a) If $O_p(G) \neq 1$, then P is not cyclic. Suppose that P is cyclic. Let L be a minimal normal subgroup of G contained in $O_p(G) \leq P$. Suppose that $C_G(L) = G$, so $L \leq Z(G)$. Let $N = N_G(P)$. If $P \leq Z(N)$, then G is p-nilpotent by Burnside's theorem [30, IV, 2.6], which contradicts the choice of G. Hence $N \neq C_G(P)$. Let $x \in N \backslash C_G(P)$ with $(|x|, |P|) = 1$ and $E = P \rtimes \langle x\rangle$. By [30, III, 13.4], $P = [E, P] \times (P \cap Z(E))$. Since $L \leq P \cap Z(E)$ and P is cyclic, it follows that $P = P \cap Z(E)$ and so $x \in C_G(P)$. This contradiction shows that $C_G(L) \neq G$.

Since P is cyclic, $|L| = p$. Hence $G/C_G(L)$ is a cyclic group of order dividing $p-1$. If $|P/L| > p$, then the hypothesis holds for G/L by Lemma 2.3 (1). Hence G/L is p-supersoluble by the choice of G and then G is p-soluble, a contradiction. Thus $|P/L| = p$, so $V = L$ is normal in G. Therefore the hypothesis holds for $(C_G(L), P)$. Hence $C_G(L)$ is p-soluble by the choice of G since $C_G(L) \neq G$. But then G is p-soluble. This contradiction shows that we have (a).

(b) If $P \nleq V^G$, then V is normal in G. Since $P \nleq V^G$, V is a Sylow p-subgroup of V^G. On the other hand, by (1) we have $V^G = V(V^G \cap B)$ and V is S-bipermutable in V^G. Therefore V^G is p-soluble by Theorem A. Thus V is normal in V^G by (4). Since V is a Sylow p-subgroup of V^G, V is characteristic in V^G. Hence $V = V^G$ is normal in G.

(c) P is not cyclic. Suppose that P is cyclic. Then $\mathscr{M}_\phi(P) = \{V\}$, and by (a) and (b) we have $P \leq V^G = V(V^G \cap B)$ and V permutes with every Sylow q-subgroup of $V^G \cap B$ for all primes $q \neq p$. Hence the hypothesis holds for V^G. Assume that $V^G \neq G$. Then V^G is p-supersoluble by the choice of G. Hence by (4), P is normal in G, which contradicts (a). Therefore $V^G = G$, which implies that $G = VB$ by (1). Hence $P = P \cap VB = V(P \cap B)$, so $P \leq B$ since P is cyclic. Therefore $B = G$. Let q be any prime dividing $|G|$ with $q \neq p$ and Q a Sylow q-subgroup of B. Then $VQ^x = Q^xV$ for all $x \in G$. Since $V^G = G$, it follows that $D = Q^G \neq G$ by Lemma 2.7. Let R be a Sylow r-subgroup of D, where $r \neq p$. Then for some sylow r-subgroup G_r of G we have $R = G_r \cap D$ and $VG_r = G_rV$. Assume that $P \leq D$. Then $VG_r \cap D = V(G_r \cap D) = VR = RV$. Therefore V is S-bipermutable in D. But then, since $D \neq G$, D is p-supersoluble by the choice of G. Thus P is normal in G, contrary to (a). Therefore $P \nleq D$. Hence $D_p = D \cap V$ is a Sylow p-subgroup of D. By Lemma 2.10 we have

$$VG_r \cap D = (V \cap D)(G_r \cap D) = D_pR = RD_p.$$

Therefore the subgroup D_p is S-bipermutable in D. Hence D is p-soluble by Theorem A, which contradicts (a). Hence P is not cyclic.

(d) P is S-bipermutable in P^D. Let $D = P^G$. In view (c), there is a subgroup $W \in \mathscr{M}_\phi(P)$ such that $V \neq W$. Then $P = VW$. Hence in view of Lemma 2.2 we have only to show that V and W permute with all Sylow q-subgroups of P^G for all primes $q \neq p$. In view of (b) we may suppose that $P \leq V^G$ and $P \leq W^G$. Then $P^G \leq V^G$, so by (1), $P^G = V(V^G \cap B)$ and V permutes with every Sylow q-subgroup Q of $V^G \cap B$. It is clear that Q is a Sylow q-subgroup of P^G. Hence V permutes with every Sylow q-subgroup of P^G by Lemma 2.5. Similarly, it may be proved that W permutes with every Sylow q-subgroup of P^G.

Final contradiction for (5). By (d) and Theorem A, P^D is p-soluble. Hence by (4), P is normal in G. Hence G is p-soluble. This contradiction completes the proof of (5).

By (5), G is p-soluble. Hence G is supersoluble by (4).

Finally, suppose that $(p-1,|G|)=1$. Then for every chief factor H/K of H with $|H/K|=p$ we have $C_G(H/K)=G$. Hence G is p-nilpotent. This contradiction completes the proof.

Proof of Theorem C. Suppose that this theorem is false and let G be a counterexample of minimal order. By Theorem A, G is soluble.

Let N be a minimal normal subgroup of G. Then the hypothesis holds for G/N by Lemma 2.3 (1). Hence G/N is supersoluble by the choice of G. Since the class of all supersoluble groups is a saturated formation, N is the only minimal normal subgroup of G, $|N|>p$ and $N \nleq \Phi(G)$. Hence $N=C_G(N)=F(G)$ is a p-group for some prime p by [20, Chapter A, Theorem 15.2]. On the other hand, by Theorem A, a Sylow q-subgroup of G, where q is the largest prime dividing $|G|$, is normal in G. Hence $q=p$ and $N=P$.

Since $N \nleq \Phi(G)$, for some maximal subgroup E of G we have $G=N \rtimes E$. Hence $E \simeq G/N$ is supersoluble Let $p_1 > \ldots > p_t$ be the set of all primes divisors of $|E|$. Let P_i be a Sylow p_i-subgroup of E. First assume that $t=2$. Then P_1 is normal in E, so $N_G(P_1)\cap P=1$. Therefore P_1 permutes with all subgroups of P. If $P\leq N_G(P_2)$, then $PP_2=P\times P_2$. Hence in this case $P_2\leq C_G(P)=P$. This contradiction shows that $N_G(P_2)\cap P\neq P$, so there is a non-identity subgroup $B<P$ such that $P_2B=BP_2$. Hence $BE=B(P_1P_2)=(P_1P_2)B=BE$ is a subgroup of G, which contradicts the maximality of $E=P_1P_2$.

Therefore $t>2$. Let E_i be a Hall p_i'-subgroup of E. Then the hypothesis holds for PE_i, so PE_i is p-supersoluble by the choice of G. Moreover, since $P=C_G(P)$ we have $O_{p'}(PE_i)=1$. Therefore PE_i is supersoluble by Lemma 2.8 (2), and $F(PE_i)=P$. Thus $PE_i/P\simeq E_i$ is an abelian group of exponent dividing $p-1$. Therefore E has at least three abelian subgroups E_i, E_j and E_k of exponent dividing $p-1$ whose indices $|E:E_i|$, $|E:E_j|$, $|E:E_k|$ are pairwise coprime. But then by Lemma 2.9 , E is nilpotent, and every Sylow subgroup of E is an abelian group of exponent dividing $p-1$. Hence E is an abelian group of exponent dividing $p-1$, which implies that $|P|=p$. But then $G/P=G/C_G(P)$ is a cyclic group of exponent dividing $p-1$, so G is supersoluble.

The theorem is proved.

4 Some applications of Theorem B

In view of Lemma 2.15 and Theorem B we have

Corollary 4.1. Let P be a Sylow subgroup of G, where p is the smallest prime dividing $|G|$. If every member V of some fixed $\mathscr{M}_\phi(P)$ is S-bipermutable in G, then G is p-nilpotent.

Corollary 4.2 (See Theorem 1.1 in [5]). Let P be a Sylow subgroup of G, where p is the smallest prime dividing $|G|$. If every member V of some fixed $\mathscr{M}_\phi(P)$ is SS-quasinormal in G, then G is p-nilpotent.

Corollary 4.3. Let P be a Sylow subgroup of G, where p is the smallest prime dividing $|G|$. If every member V of some fixed $\mathscr{M}_\phi(P)$ is S-semipermutable in G, then G is p-nilpotent.

Corollary 4.4. Let P be a Sylow subgroup of G. If $N_G(P)$ is p-nilpotent and every member V of some fixed $\mathscr{M}_\phi(P)$ is S-bipermutable in G, then G is p-nilpotent.

Proof of Corollary 4.4. If $|P|=p$, then G is p-nilpotent by Burnside's theorem [30, IV, 2.6]. Otherwise, G is p-supersoluble by Theorem B. The hypothesis holds for $G/O_{p'}(G)$ by Lemma 2.3(1), so in the case, where $O_{p'}(G)\neq 1$, $G/O_{p'}(G)$ is p-nilpotent by induction. Hence G is p-nilpotent. Therefore we may assume that $O_{p'}(G)=1$. But then, by Lemma 2.8 (2), P is normal in G. Hence G is p-nilpotent by hypothesis.

From Corollary 4.4 we get

Corollary 4.5. Let P be a Sylow subgroup of G. If $N_G(P)$ is p-nilpotent and every member V of some fixed $\mathscr{M}_\phi(P)$ is S-semipermutable in G, then G is p-nilpotent.

Corollary 4.6 (See Theorem 1.2 in [5]). Let P be a Sylow subgroup of G. If $N_G(P)$ is p-nilpotent and every member V of some fixed $\mathscr{M}_\phi(P)$ is SS-quasinormal in G, then G is p-nilpotent.

Corollary 4.7. Let P be a Sylow subgroup of G. If G is p-soluble and every number V of some fixed $\mathscr{M}_d(P)$ is S-bipermutable in G, then G is p-supersoluble.

Proof. In the case, when $|P|=p$, this directly follows from the p-solubility of G. If $|P|>p$, this corollary follows from Theorem B.

Corollary 4.8. Let P be a Sylow subgroup of G. If G is p-soluble and every number V of some fixed $\mathscr{M}_d(P)$ is S-semipermutable in G, then G is p-supersoluble.

Corollary 4.9 (See Theorem 1.3 in [5]). Let P be a Sylow subgroup of G. If G is p-soluble and every member V of some fixed $\mathscr{M}_\phi(P)$ is SS-quasinormal in G, then G is p-supersoluble.

Corollary 4.10. If, for every prime p dividing $|G|$ and $P\in Syl_p(G)$, every member V of some fixed $\mathscr{M}_\phi(P)$ is S-bipermutable in G, then G is supersoluble.

Proof. Let p be the smallest prime dividing $|G|$. Then G is p-nilpotent by Corollary 4.1, so G is soluble by Fait-Thompson's theorem. Hence G is supersoluble by Corollary 4.7.

Corollary 4.12. If, for every prime p dividing $|G|$ and $P\in Syl_p(G)$, every member V of some fixed $\mathscr{M}_\phi(P)$ is S-semipermutable in G, then G is supersoluble.

Corollary 4.12 (See Theorem 1.4 in [5]). If, for every prime p dividing $|G|$ and $P\in Syl_p(G)$, every member V of some fixed $\mathscr{M}_\phi(P)$ is SS-quasinormal in G, then G is supersoluble.

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