

On a Generalized Fractional Integral Operator in a Complex Domain

Adem Kılıçman^{1,*}, Rabha W. Ibrahim² and Zainab Esa Abdalnaby¹

¹ Department of Mathematics and Institute for Mathematical Research, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia

² Faculty of Computer Science and Information Technology, University Malaya, 50603, Malaysia

Received: 21 Oct. 2015, Revised: 21 Jan. 2016, Accepted: 22 Jan. 2016

Published online: 1 May 2016

Abstract: In this paper we define a new fractional integral operator in complex z -plane \mathbb{C} . We are also interested a fractional integration operator to be compact and bounded, provide some examples in Bergmann spaces. By considering the properties of Gauss hypergeometric function we study the univalence and convexity for the new operator. Finally, we follow modification formal to ensure existence for this operator to be in class of univalent function in \mathbb{U} .

Keywords: Univalent function; Fractional calculus; analytic functions; Gauss hypergeometric function.

1 Introduction

Fractional calculus is considered one of the important branches of mathematical analysis. Actually, in the recent years a lot of attention has been given to fractional integral and differential operators in geometric function theory. There are several kinds of fractional integral and differential operators, such as the familiar operators of Biernacki, Carlson, Ruscheweyh, Srivastava and Owa, (see [1], [8], [9], [10], [11], [17]). One of the important problem in the geometric function theory is univalent functions and how to construct a linear operators that preserves the class of the univalent functions and some of its subclasses. Biernacki [8] claimed that a certain integral operator maps class of univalent into itself, but later Krzyz and Lewandowski provided a counterexample in [6] that the claimed was not true. While, Libera considered another linear integral operator in [7], which maps each of the subclasses of the starlike, convex and close-to-convex functions into itself, for more information can be found in [16] and [15]. Among these operators in geometric function theory, two operators were investigated by Srivastava and Owa (see [9], [10]), which are defined as follows:

Definition 1. The fractional integral of order α is defined, for a function $f(z)$ by:

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \quad (1)$$

where $0 \leq \alpha < 1$, and the function $f(z)$ is analytic in simply-connected region of the complex z -plane \mathbb{C} containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 2. The fractional derivative of order α is defined, for a function $f(z)$, by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z f(\zeta)(z-\zeta)^{-\alpha} d\zeta; \quad (2)$$

where $0 \leq \alpha < 1$, and the function $f(z)$ is analytic in simply-connected region of the complex z -plane containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Remark. For the above two Definitions 1 and 2, we have

$$D_z^\alpha \{z^\nu\} = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} z^{\nu-\alpha}, \quad 0 < \alpha < 1; \nu > -1, \quad (3)$$

$$I_z^\alpha \{z^\nu\} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)} z^{\nu+\alpha}, \quad 0 < \alpha; \quad -1 < \nu, \quad (4)$$

* Corresponding author e-mail: akilic@upm.edu.my

where $z \neq 0$ and

$$D_z^\alpha f(z) = \frac{d}{dz} I_z^{1-\alpha} f(z). \quad (5)$$

For more information for those operators, can be found in [13] and [12]. In [18], Tremblay defined a fractional differential operator in complex z -plane \mathbb{C} and defined as the follows

$${}_z O_\beta^\alpha f(z) := \frac{\Gamma(\beta)}{\Gamma(\alpha)} z^{1-\beta} D_z^{\alpha-\beta} z^{\alpha-1} \quad (6)$$

where $D_z^{\alpha-\beta}$ is the fractional derivative Riemann-Liouville operator and $\beta \in \mathbb{N} \setminus \{0\}$, also (see [19], [14]). Note that in [20], we have

$$O_z^{\alpha,\beta} \{e^z\} = {}_1F_1(\alpha, \beta, z).$$

In [5] Ibrahim extended the operator (6) in unit disk \mathbb{U} and studied some univalence properties of this operator.

Definition 3. For all $z \in \mathbb{U}$, the Gauss hypergeometric function is denoted by ${}_2F_1(a, b, c; z)$ (or simply $F(a, b, c; z)$) and is defined as follows

$$\begin{aligned} {}_2F_1(a, b, c; z) &= \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!} \\ &= 1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \dots, \end{aligned} \quad (7)$$

where $a, b \in \mathbb{C}$, $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and $(a)_m$ is the Pochhammer symbol defined as:

$$(a)_0 = 1 \quad (a)_m = a(a+1) \cdots (a+m-1), \quad (m \in \mathbb{N}).$$

In particular, if $a = 1, b = c$, then the series in Eq. (7) takes the form

$$1 + z + z^2 + z^3 + \dots$$

In the following remark, we recall some properties of the Gauss hypergeometric function in unit disk which we need in the development of the work.

Remark. [3], [4] For all $z \in \mathbb{U}$ and $a, b \in \mathbb{C}, c \in \mathbb{C} \setminus \mathbb{Z}_0^{-1}$, then

(i) The differential of functions (7) defined as:

$$({}_2F_1(a, b, c; z))' = \frac{ab}{c} {}_2F_1(a+1, b+1, c+1; z).$$

(ii) The Euler integral representation for function (7) defined as

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt.$$

The aim of this article is to obtain a new fractional integral operator that involves the fractional integral of Srivastava-Owa operator and study some of properties in the open unit disk. We also consider the Gauss hypergeometric function, and modified this operator to stay in the univalent class and their subclasses.

2 Fractional calculus operators

In this section, we provide some definitions and give some related results in the present work. We prove some properties of the new fractional integral operator, for instance: the boundedness, compactness in the Bergman space and study two further examples.

The Bergman space $\mathfrak{A}^p(\mathbb{U})$ for $(0 < p < 1)$ is the set of functions f analytic in the open unit disk $\mathbb{U} := \{z : z \in \mathbb{C}; |z| < 1\}$ with the norm $\|f\|_{\mathfrak{A}^p}^p < \infty$ defined by

$$\|f\|_{\mathfrak{A}^p}^p = \frac{1}{\pi} \int_{\mathbb{U}} |f(z)|^p d\mathfrak{A} < \infty \quad z \in \mathbb{U},$$

where $d\mathfrak{A}$ is known as Lebesgue measure over \mathbb{U} .

Definition 4. Let $f(z)$ be analytic in a simple-connected region, for all $z \in \mathbb{U}$, containing the origin and $(0 < \alpha \leq 1)$, $(0 < \beta \leq 1)$ such that $(0 \leq \alpha - \beta < 1)$. Then the fractional integral operator $\mathfrak{L}_z^{\alpha,\beta}$ is given by:

$$\mathfrak{L}_z^{\alpha,\beta} f(z) := \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} z^{1-\alpha} \int_0^z \frac{t^{\beta-1} f(t)}{(z-t)^{1-\alpha+\beta}} dt, \quad (8)$$

where the multiplicity of $(z-t)^{\alpha-\beta-1}$ is removed by considering $\log(z-t)$ to be real when $(z-t) > 0$, and if $\alpha = \beta$, then we have

$$\mathfrak{L}_z^{\alpha,\alpha} f(z) = f(z).$$

Definition 5. Let $f(z)$ be an analytic function in a simple-connected region, for all $z \in \mathbb{U}$, containing the origin and $(0 < \alpha \leq 1)$, $(0 < \beta \leq 1)$ such that $(0 \leq \alpha - \beta < 1)$. Then we define the fractional differential operator $\mathfrak{T}_z^{\alpha,\beta}$ as follows:

$$\mathfrak{T}_z^{\alpha,\beta} f(z) := \frac{\Gamma(\beta) z^{1-\beta}}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} \frac{d}{dz} \int_0^z \frac{t^{\alpha-1} f(t)}{(z-t)^{\alpha-\beta}} dt \quad (9)$$

similar to the definition 4 the multiplicity of $(z-t)^{\beta-\alpha}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$. In particular if $\alpha = \beta$, we have

$$\mathfrak{T}_z^{\alpha,\alpha} f(z) = f(z).$$

In the present work, we apply the definition 1 and definition 2 to define the operators in (8) and (9); respectively. In the following theorem 1, we consider to show that the operator (8) is bounded in the space $\mathfrak{A}^p(\mathbb{U})$.

Theorem 1. (Boundedness) Let $f \in \mathcal{A}$ on unit disk \mathbb{U} . Then the operator $\mathfrak{L}_z^{\alpha,\beta} : \mathfrak{A}^p \rightarrow \mathfrak{A}^p$ is a bounded operator and

$$\|\mathfrak{L}_z^{\alpha,\beta} f(z)\|_{\mathfrak{A}^p}^p \leq \|f(z)\|_{\mathfrak{A}^p}^p \quad (f \in \mathcal{A})$$

for all $z \in \mathbb{U}$.

Proof. Suppose that $f(z) \in \mathfrak{A}^2$, then it follows that

$$\begin{aligned} \|\mathfrak{L}_z^{\alpha,\beta} f(z)\|_{\mathfrak{A}^p}^p &= \frac{1}{\pi} \int_{\mathbb{U}} |\mathfrak{L}_z^{\alpha,\beta} f(z)|^p d\mathfrak{A} \\ &= \frac{1}{\pi} \int_{\mathbb{U}} \left| \frac{z^{1-\alpha}\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^z (z-t)^{\alpha-\beta-1} t^{\beta-1} f(t) dt \right|^p d\mathfrak{A} \\ &= \frac{1}{\pi} \int_{\mathbb{U}} \left| \frac{z^{1-\alpha}\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \right. \\ &\quad \left. \times \int_0^z (1-\frac{t}{z})^{\alpha-\beta-1} z^{\alpha-\beta-1} t^{\beta-1} f(t) dt \right|^p d\mathfrak{A}. \end{aligned}$$

By setting $u = \frac{t}{z}$ and using Beta function

$$B(a, c) = \int_0^1 (1-t)^{a-1} t^{c-1} dt = \frac{\Gamma(a)\Gamma(c)}{\Gamma(a+c)},$$

then we obtain

$$\begin{aligned} \|\mathfrak{L}_z^{\alpha,\beta} f(z)\|_{\mathfrak{A}^p}^p &= \frac{1}{\pi} \int_{\mathbb{U}} \left| \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^1 (1-u)^{\alpha-\beta-1} u^{\beta-1} f(uz) du \right|^p d\mathfrak{A} \\ &\leq \frac{1}{\pi} \int_{\mathbb{U}} \left| \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^z (1-u)^{\alpha-\beta-1} u^{\beta-1} f(uz) du \right|^p d\mathfrak{A} \\ &\leq \frac{1}{\pi} \int_{\mathbb{U}} |f(uz)|^p d\mathfrak{A}. \end{aligned}$$

This completes the proof of the Theorem 1.

Theorem 2.(Compactness) Let $f(z) \in \mathfrak{A}$ on \mathbb{U} . Then $\mathfrak{L}_z^{\alpha,\beta} : \mathfrak{A}^p \rightarrow \mathfrak{A}^p$ is compact.

Proof. Assume that the function $(f_n)_{n \in \mathbb{N}}$ be a sequence in \mathfrak{A}^p and that $f_n \rightarrow 0$ uniformly on $\overline{\mathbb{U}}$ as $n \rightarrow \infty$. Then

$$\begin{aligned} \|\mathfrak{L}_z^{\alpha,\beta} f_n\|_{\mathfrak{A}^p}^p &= \sup_{z \in \overline{\mathbb{U}}} \left\{ \frac{1}{\pi} \int_{\mathbb{U}} |\mathfrak{L}_z^{\alpha,\beta} f_n|^p d\mathfrak{A} \right\} \\ &= \sup_{z \in \overline{\mathbb{U}}} \left\{ \frac{1}{\pi} \int_{\mathbb{U}} \left| \frac{\Gamma(\alpha)z^{1-\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)} \right. \right. \end{aligned} \tag{10}$$

$$\begin{aligned} &\quad \left. \times \int_0^z (z-t)^{\alpha-\beta-1} t^{\beta-1} f_n(t) dt \right|^p d\mathfrak{A} \Big\} \\ &\leq \sup_{z \in \overline{\mathbb{U}}} \left\{ \frac{1}{\pi} \int_{\mathbb{U}} |f_n(\xi)|^p d\mathfrak{A} \right\} \\ &\leq \|f_n\|_{\mathfrak{A}^p}^p. \end{aligned} \tag{11}$$

Since $f_n \rightarrow 0$ on $\overline{\mathbb{U}}$, we obtain $\|f_n\|_{\mathfrak{A}^p} \rightarrow 0$, and that $\varepsilon > 0$, by putting $n \rightarrow \infty$ in Eq.(10), we have that $\lim_{n \rightarrow \infty} \|\mathfrak{L}_z^{\alpha,\beta} f_n\|_{\mathfrak{A}^p}^p = 0$. Hence, the compactness of the operator $\mathfrak{L}_z^{\alpha,\beta}$ follows.

Now, we are ready to prove some properties of fractional integral operator $\mathfrak{L}_z^{\alpha,\beta}$ in open unit disk \mathbb{U} .

Proposition 1. Let $f(z), g(z) \in \mathfrak{A}$ and $a, b \in \mathbb{C}$, then

$$\mathfrak{L}_z^{\alpha,\beta}(af + bg) = a\mathfrak{L}_z^{\alpha,\beta} f + b\mathfrak{L}_z^{\alpha,\beta} g$$

for all $z \in \mathbb{U}$.

Proposition 2. Let $f(z) \in \mathfrak{A}$, $0 < \alpha \leq 1$ and $0 < \beta \leq 1$, then

$$\mathfrak{I}_z^{\alpha,\beta} \mathfrak{L}_z^{\alpha,\beta} f(z) = f(z) \tag{12}$$

is hold true for all $z \in \mathbb{U}$.

Proof. For proof Eq.(12), we use Dirichlet formula, we obtain

$$\begin{aligned} &(\mathfrak{I}_z^{\alpha,\beta} \mathfrak{L}_z^{\alpha,\beta}) f(z) \\ &= \frac{\Gamma(\beta)z^{1-\beta}}{\Gamma(\alpha)\Gamma(\alpha-\beta)} \frac{d}{dz} \left\{ \int_0^z (z-t)^{\beta-\alpha} t^{\alpha-1} \mathfrak{L}_z^{\alpha,\beta} f(t) dt \right\}, \\ &= \frac{z^{1-\beta}}{\Gamma(1-\alpha+\beta)\Gamma(\alpha-\beta)} \times \\ &\quad \frac{d}{dz} \left\{ \int_0^z (z-t)^{\beta-\alpha} \int_{\xi}^t (t-\xi)^{\alpha-\beta-1} \xi^{\beta-1} f(\xi) d\xi dt \right\}, \\ &= \frac{z^{1-\beta}}{\Gamma(1-\alpha+\beta)\Gamma(\alpha-\beta)} \times \\ &\quad \frac{d}{dz} \left\{ \int_0^z \xi^{\beta-1} f(\xi) d\xi \cdot \Gamma(1-\alpha+\beta)\Gamma(\alpha-\beta) \right\}, \\ &= f(z). \end{aligned}$$

Note, the inner integral is assessed by the same methods as in Theorem 1.

Proposition 3. Let $f(z) \in \mathfrak{A}$, then

$$\mathfrak{L}_z^{\alpha,\beta} f(z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} z^{1-\beta} \mathcal{D}_z^{1-\alpha+\beta} z^{\beta-1} f(z)$$

where $\mathcal{D} = \frac{d}{dz}$.

Proof. For proofing, we consider Eq.(5) in Remark 1.

Proposition 4. Let $f(z) \in \mathfrak{A}$. For all $z \in \mathbb{U}$ and for some $0 < \alpha \leq 1, 0 < \beta \leq 1$, we have

$$\mathfrak{L}_z^{\alpha,\beta} \mathfrak{L}_z^{\beta,\alpha} f(z) = f(z) \tag{13}$$

and

$$\mathfrak{L}_z^{\alpha,\beta} \mathfrak{L}_z^{\beta,\mu} f(z) = \mathfrak{L}_z^{\alpha,\mu} f(z). \tag{14}$$

Proof. The proof is straightforward by following the method as in the proposition 2.

3 New operator and special functions

In this section we show that the operator $\mathfrak{L}_z^{\alpha,\beta}$ represents some special functions. In the next we consider one special functions in geometric function theory, that is also known as *Gauss hypergeometric function* and study some their properties in the unit disk \mathbb{U} . First of all, let the class of all analytic functions $f(z)$ denoted by \mathfrak{A} and *normalized* as the form:

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad z \in \mathbb{U} \tag{15}$$

and satisfying $f(0) = 0$ and $f'(0) = 1$. Further, let us denote the subclasses of \mathcal{A} by \mathcal{S} and \mathcal{K} which are respectively, univalent and convex functions in \mathbb{U} . In [2], we know that if $f(z)$ is given by (15) which is a member in the class of univalent functions \mathcal{S} , then $|a_m| \leq m, m = \{2, 3, \dots\}$. Furthermore, if $f(z)$ given by (15) is in the class of convex functions \mathcal{K} , then $|a_m| \leq 1, m = \{1, 2, 3, \dots\}$.

Now, we consider to find the upper bounded of the operator (8) of univalent and convex functions.

Theorem 3. For extension the operator (8) in unit disk, let $f(z) = \sum_{m=0}^{\infty} a_m z^m$ belongs to class of analytic functions \mathcal{A} , then

$$\begin{aligned} \mathfrak{I}_z^{\alpha, \beta} f(z) &= \sum_{m=0}^{\infty} \mathfrak{I}_z^{\alpha, \beta} \{a_m z^m\} \\ &= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha - \beta)} \sum_{m=0}^{\infty} B(\alpha - \beta, \beta + m) a_m z^m. \end{aligned}$$

Proof. For all $z \in \mathbb{U}$, we obtain

$$\begin{aligned} \mathfrak{I}_z^{\alpha, \beta} f(z) &= \sum_{m=0}^{\infty} \mathfrak{I}_z^{\alpha, \beta} \{a_m z^m\} \\ &= \frac{\Gamma(\alpha) z^{1-\alpha}}{\Gamma(\beta)\Gamma(\alpha - \beta)} \int_0^{\infty} (z-s)^{\alpha-\beta-1} s^{\beta-1} \sum_{m=0}^{\infty} a_m s^m ds \\ &= \frac{\Gamma(\alpha) z^{1-\alpha}}{\Gamma(\beta)\Gamma(\alpha - \beta)} \int_0^{\infty} \left(1 - \frac{s}{z}\right)^{\alpha-\beta-1} s^{\beta-1} \sum_{m=0}^{\infty} a_m s^m ds \\ &= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha - \beta)} \int_0^1 (1-u)^{\alpha-\beta-1} (u)^{\beta-1} \times \\ &\quad \sum_{m=0}^{\infty} a_m (zu)^m du \end{aligned}$$

where we can change the order of integration and summation since the series $\sum_{m=0}^{\infty} a_m z^m u^m$ is uniformly convergent in open unit disk \mathbb{U} for $0 \leq u \leq 1$ and the integral $\int_0^1 |(1-u)^{\alpha-\beta-1} (u)^{\beta-1}| du$ is convergent as long as $0 < \alpha \leq 1$ and $0 < \beta \leq 1$, then we have

$$\begin{aligned} \mathfrak{I}_z^{\alpha, \beta} f(z) &= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha - \beta)} \sum_{m=0}^{\infty} a_m z^m \times \\ &\quad \int_0^1 (1-u)^{\alpha-\beta-1} (u)^{m+\beta-1} du \\ &= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha - \beta)} \sum_{m=0}^{\infty} B(\alpha - \beta, m + \beta) a_m z^m. \end{aligned}$$

Hence, we arrive at the desired result.

Theorem 4. (Univalence) Let $f(z) \in \mathcal{S}$. Then

$$|\mathfrak{I}_z^{\alpha, \beta} f(z)| \leq r ({}_2F_1(1, \beta, \alpha; r))'.$$

Proof. By assuming that the function $f(z)$ given by (15) in the class \mathcal{S} , then by using Example 1, we have

$$\begin{aligned} \mathfrak{I}_z^{\alpha, \beta} f(z) &= \frac{\Gamma(\alpha)}{\Gamma(\beta)} z^{1-\alpha} I_z^{\alpha-\beta} \left\{ \sum_{m=1}^{\infty} a_m z^{m+\beta-1} \right\}, a_1 = 1 \\ &= \frac{\Gamma(\alpha)}{\Gamma(\beta)} \sum_{m=0}^{\infty} \frac{\Gamma(m + \beta + 1)}{\Gamma(m + \alpha + 1)} a_{m+1} z^{m+1}, \\ &= \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(\beta + 1)_m}{\Gamma(\alpha + 1)_m} a_{m+1} z^{m+1}, \end{aligned}$$

Subsequently,

$$\begin{aligned} |\mathfrak{I}_z^{\alpha, \beta} f(z)| &\leq r \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(\beta + 1)_m}{\Gamma(\alpha + 1)_m} (m + 1) r^m \quad |z| = r \\ &= r \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(2)_m (\beta + 1)_m}{\Gamma(\alpha + 1)_m} \frac{r^m}{m!} \\ &= r \frac{\beta}{\alpha} \left\{ {}_2F(1 + 1, \beta + 1, \alpha + 1; r) \right\}. \quad (16) \end{aligned}$$

Remark. We note that, the series in Eq. (16) is absolutely convergent for all $z \in \mathbb{U}$, so that represented as the analytic functions and holds true for property of Gauss hypergeometric function in open unit disk \mathbb{U} (see [3]; p.28; Ch.1; Eq.(1.6.11)).

Next, we are interested to find the upper bound for inequality involving the hypergeometric function, which is given in the following Theorem.

Theorem 5. (Convexity) Let the function $f(z)$ belongs to class of convex functions \mathcal{K} . Then

$$|\mathfrak{I}_z^{\alpha, \beta} f(z)| \leq \frac{r\beta}{\alpha} \left\{ {}_2F(1, \beta + 1, \alpha + 1; r) \right\}.$$

Proof. By imposing that $f(z) \in \mathcal{K}$, we obtain

$$\begin{aligned} |\mathfrak{I}_z^{\alpha, \beta} f(z)| &\leq r \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(\beta + 1)_m}{\Gamma(\alpha + 1)_m} r^m, \quad |z| = r \\ &= r \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(1)_m (\beta + 1)_m}{\Gamma(\alpha + 1)_m} \frac{r^m}{m!} \\ &= r \frac{\beta}{\alpha} \left\{ {}_2F(1, \beta + 1, \alpha + 1; r) \right\} \quad (17) \end{aligned}$$

for all $z \in \mathbb{U}$.

Theorem 6. Let $f(z) \in \mathcal{K}$, then

$$\mathfrak{I}_z^{\alpha, \beta} f(z) \leq \frac{r}{B(\beta, \alpha - \beta)} \int_0^1 s^{\beta} (1-s)^{\alpha-\beta-1} (1-rs)^{-1} ds.$$

Proof. Suppose that, $f(z) \in \mathcal{K}$ on \mathbb{U} , then we have

$$\begin{aligned} |\mathfrak{L}_z^{\alpha,\beta} f(z)| &= \frac{\Gamma(\alpha)}{\Gamma(\beta)} z^{1-\alpha} I_z^{\alpha-\beta} z^{\beta-1} f(z) \\ &\leq \frac{\beta}{\alpha} \sum_{m=0}^{\infty} \frac{(1)_m (\beta+1)_m}{(\alpha+1)_m} \cdot \frac{r^{m+1}}{m!} \\ &= \frac{\beta}{\alpha} \sum_{m=0}^{\infty} (1)_m \frac{\Gamma(m+\beta+1)\Gamma(\alpha+1)}{\Gamma(m+\alpha+1)\Gamma(\beta+1)} \cdot \frac{r^{m+1}}{m!} \\ &= \frac{\Gamma(\alpha)}{\Gamma(\beta)} \sum_{m=0}^{\infty} (1)_m \frac{\Gamma(m+\beta+1)}{\Gamma(m+\alpha+1)} \cdot \frac{r^{m+1}}{m!} \\ &= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \sum_{m=0}^{\infty} (1)_m \frac{\Gamma(m+\beta+1)\Gamma(\alpha-\beta)}{\Gamma(\beta+m+\alpha+1-\beta)} \cdot \frac{r^{m+1}}{m!} \end{aligned}$$

Multiplying and dividing by $\Gamma(\alpha-\beta)$,

$$\begin{aligned} &= \frac{r\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \sum_{m=0}^{\infty} (1)_m \\ &\quad \times \left\{ \int_0^1 s^{m+\beta} (1-s)^{\alpha-\beta-1} ds \right\} \frac{r^m}{m!} \end{aligned}$$

where $0 < \alpha - \beta, 0 < \beta + m$, so $\alpha > \beta > 0$ and since

$$\frac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)} = B(q,p) = \int_0^1 s^{q-1}(1-s)^{p-1} ds$$

then it follows that

$$\begin{aligned} |\mathfrak{L}_z^{\alpha,\beta} f(z)| &\leq \frac{r\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \\ &\quad \times \int_0^1 s^\beta (1-s)^{\alpha-\beta-1} \left\{ \sum_{m=0}^{\infty} (1)_m \frac{(rs)^m}{m!} \right\} ds \end{aligned} \quad (18)$$

and the substitution

$$\frac{1}{B(\beta, \alpha - \beta)} = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha - \beta)},$$

yields that we have

$$\begin{aligned} |\mathfrak{L}_z^{\alpha,\beta} f(z)| &\leq \frac{r}{B(\beta, \alpha - \beta)} \int_0^1 s^\beta (1-s)^{\alpha-\beta-1} (1-rs)^{-1} ds. \end{aligned}$$

Remark. From Theorem 5 and Theorem 4, we can easily conclude that the new operator in (8) is a generalization of Carlson-Shaffer operator (see [1]) and it can be written in the following form, if $\beta/\alpha = 1$ then we have:

$$\mathfrak{L}_z^{\alpha,\beta} f(z) = \mathcal{L}(\beta+1, \alpha+1)f(z), \quad f \in \mathcal{A}, z \in \mathbb{U}, \quad (19)$$

where

$$\mathcal{L}(\beta+1, \alpha+1)f(z) = \phi(\beta+1, \alpha+1; z) * f(z)$$

and $*$ stands for the convolution (or Hadamard product) of two functions which is given by (15),

$$\begin{aligned} \phi(\beta+1, \alpha+1; z) &= z + \sum_{m=2}^{\infty} \frac{(\beta+1)_{m-1}}{(\alpha+1)_{m-1}} z^m \quad z \in \mathbb{U} \\ &= {}_2F_1(1, \beta+1, \alpha+1; z). \end{aligned}$$

In the next we provide some examples.

Example 1. Let $f(z) = z^\nu$, for all $z \in U$, then

$$\mathfrak{L}_z^{\alpha,\beta} \{z^\nu\} = \frac{\Gamma(\alpha)\Gamma(\nu+\beta)}{\Gamma(\beta)\Gamma(\nu+\alpha)} z^\nu. \quad (20)$$

Solution. By consider the Equation (8), we obtain

$$\mathfrak{L}_z^{\alpha,\beta} \{z^\nu\} = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} z^{1-\alpha} \int_0^z \frac{t^{\beta-1}}{(z-t)^{1-\alpha+\beta}} t^\nu dt$$

where $0 < \alpha \leq 1, 0 < \beta \leq 1$ and $0 < \alpha - \beta < 1$ and $z \in \mathbb{U}, \nu \in \mathbb{R}$. By using substitution $w = \frac{t}{z}$, we compute the inner integral to get

$$\int_0^z \frac{t^{\nu+\beta-1}}{(z-t)^{1-\alpha+\beta}} dt = z^{\nu+\alpha-1} B(\alpha-\beta, \nu+\beta)$$

where $B(\cdot, \cdot)$ is the Beta function. Thus, we have

$$\mathfrak{L}_z^{\alpha,\beta} \{z^\nu\} = \frac{\Gamma(\alpha)\Gamma(\nu+\beta)}{\Gamma(\beta)\Gamma(\nu+\alpha)} z^\nu. \quad (21)$$

Example 2. Let $f(z) = e^z = \sum_{v=0}^{\infty} \frac{1}{v!} z^v$ for all $z \in \mathbb{U}$, then

$$\begin{aligned} \mathfrak{L}_z^{\alpha,\beta} \{e^z\} &= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^1 (1-u)^{1-\alpha+\beta} u^{\beta-1} e^{zu} du, \\ &= {}_1F_1(\beta, \alpha; z), \quad (0 < \beta < \alpha). \end{aligned}$$

Solution. In this example we follow same as methods in Example 1 and we obtain

$$\begin{aligned} \mathfrak{L}_z^{\alpha,\beta} \{e^z\} &= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} z^{1-\alpha} \int_0^z (z-t)^{\alpha-\beta-1} t^{\beta-1} e^t dt \\ &= \frac{\Gamma(\alpha)z^{1-\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^1 (1-u)^{1-\alpha+\beta} z^{\alpha-\beta} (uz)^{\beta-1} e^{zu} du \\ &= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^1 (1-u)^{1-\alpha+\beta} u^{\beta-1} e^{zu} du \\ &= {}_1F_1(\beta, \alpha; z). \end{aligned}$$

Remark. We note that ${}_1F_1(\beta, \alpha; z)$ is confluent hypergeometric function (or Kummer function), for any $z \in \mathbb{C}$ this function is convergent and it has an integral representation (see [3], p.29, Eq.1.6.15). In any case, may be can represent the operator $\mathfrak{L}_z^{\alpha,\beta}$ in Example 2 as the following integral

$$\mathfrak{L}_z^{\alpha,\beta} \{e^z\} = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^1 (1-u)^{1-\alpha+\beta} u^{\beta-1} e^{zu} du.$$

Or, by using the fact of the beta function $B(\beta, \alpha - \beta) = \frac{\Gamma(\beta)\Gamma(\alpha-\beta)}{\Gamma(\alpha)}$ yields

$$\begin{aligned} \mathfrak{L}_z^{\alpha,\beta} \{e^z\} &= \\ &= \frac{1}{B(\beta, \alpha - \beta)} \int_0^1 u^{\beta-1} (1-u)^{\alpha-\beta-1} e^{zu} du, \quad (0 < \beta < \alpha). \end{aligned}$$

Example 3. Let $0 < \alpha \leq 1$, $0 < \beta \leq 1$ and $|z| < 1$, then

$$\mathfrak{L}_z^{\alpha, \beta} \{(1-z)^{-\nu}\} = {}_2F_1(\nu, \beta, \alpha; z).$$

where

$$(1-z)^{-\nu} = 1 + \nu z + \frac{\nu(\nu+1)}{2!} z^2 + \frac{\nu(\nu+1)(\nu+2)}{3!} z^3 + \dots$$

Solution. By direct calculations, we obtain

$$\begin{aligned} & \mathfrak{L}_z^{\alpha, \beta} \{(1-z)^{-\nu}\} \\ &= \frac{\Gamma(\alpha) z^{1-\alpha}}{\Gamma(\beta) \Gamma(\alpha-\beta)} \int_0^z s^{\beta-1} (1-s)^{-\nu} (z-s)^{\alpha-\beta-1} ds \\ &= \frac{\Gamma(\alpha) z^{1-\alpha}}{\Gamma(\beta) \Gamma(\alpha-\beta)} \int_0^z s^{\beta-1} (1-s)^{-\nu} \left(1 - \frac{s}{z}\right)^{\alpha-\beta-1} ds \\ &= \frac{\Gamma(\alpha)}{\Gamma(\beta) \Gamma(\alpha-\beta)} \int_0^1 u^{\beta-1} (1-u)^{\alpha-\beta-1} (1-uz)^{-\nu} du \\ &= {}_2F_1(\nu, \beta, \alpha; z). \end{aligned}$$

Therefore the proof is complete.

4 Some Applications of operator $\mathfrak{L}_z^{\alpha, \beta}$ in \mathbb{U}

In this section, we modify the operator $\mathfrak{L}_z^{\alpha, \beta}$ in unit disk, in order to keep and ensure the existence in class of univalent functions and their subclasses (see [16], [15]). Further, we also discuss the distortion inequalities for the modified operator (8).

Definition 6. For all $z \in \mathbb{U}$ and some $0 < \alpha \leq 1$, $0 < \beta \leq 1$, such that $0 \leq \alpha - \beta < 1$. We define the modification of fractional integral operator (8) as follows:

$$L_z^{\alpha, \beta} f(z) : \mathcal{A} \rightarrow \mathcal{A}$$

$$\begin{aligned} L_z^{\alpha, \beta} f(z) &= \left(\frac{\alpha}{\beta}\right) \mathfrak{L}_z^{\alpha, \beta} f(z), \quad (f \in \mathcal{A}) \\ &= z + \sum_{m=2}^{\infty} \frac{\Gamma(\alpha+1)\Gamma(m+\beta)}{\Gamma(\beta+1)\Gamma(m+\alpha)} a_m z^m \\ &= z + \sum_{m=2}^{\infty} \Theta(m) a_m z^m \end{aligned}$$

where

$$\Theta(m) = \frac{\Gamma(\alpha+1)\Gamma(m+\beta)}{\Gamma(\beta+1)\Gamma(m+\alpha)}.$$

Note that if $\alpha = \beta$ then $L_z^{\alpha, \alpha} f(z) = f(z)$.

Remark. For all $|z| < 1$; $z \in \mathbb{C}$, then we have

$$L_z^{\alpha, \beta} f(z) \Big|_{z=0} = 0$$

and

$$(L_z^{\alpha, \beta} f(z))' \Big|_{z=0} = 1,$$

where $0 < \alpha \leq 1$ and $0 < \beta \leq 1$.

Theorem 7. If $f(z) \in \mathcal{A}$, then for $0 < \alpha \leq 1$, $0 < \beta \leq 1$ and $0 \leq \alpha - \beta < 1$

$$|L_z^{\beta, \alpha} f(z) - z| \leq M, \quad |z| < 1$$

$$\text{where } M := \frac{(2-r)(\beta+1)}{(1-r)^2(\alpha+1)}.$$

Proof. By assuming that the $f \in \mathcal{S}$, we obtain

$$\begin{aligned} |L_z^{\alpha, \beta} f(z) - z| &= \left| \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)} z^{1-\alpha} I_z^{\alpha-\beta} z^{\beta-1} f(z) - z \right| \\ &= \left| \sum_{m=2}^{\infty} \frac{\Gamma(\alpha+1)\Gamma(m+\beta)}{\Gamma(\beta+1)\Gamma(m+\alpha)} a_m z^m \right| \\ &= \left| \sum_{m=2}^{\infty} \Theta(m) a_m z^m \right| \end{aligned}$$

$$\text{where } \Theta(m) = \frac{\Gamma(\alpha+1)\Gamma(m+\beta)}{\Gamma(\beta+1)\Gamma(m+\alpha)} \quad (m > 2).$$

Note that,

$$\Theta(m) \leq \Theta(2) = \frac{\beta+1}{\alpha+1},$$

$$\begin{aligned} |L_z^{\alpha, \beta} f(z) - z| &= \left| \sum_{m=2}^{\infty} \frac{\beta+1}{\alpha+1} a_m z^m \right| \\ &\leq \frac{\beta+1}{\alpha+1} \sum_{m=2}^{\infty} |a_m| |z|^m \\ &\leq \frac{\beta+1}{\alpha+1} |z|^2 \sum_{m=2}^{\infty} m r^{m-2} \quad (22) \\ &= \frac{(2-r)(\beta+1)}{(1-r)^2(\alpha+1)} |z|^2. \end{aligned}$$

where the Eq.(22) used $|z| \leq r$ and $|a_m| \leq m$.

5 Conclusion

In the present work we defined a new fractional integral operator in a complex domain \mathbb{C} and then we prove that this operator is in the class of univalent functions by using some modifications. In addition, we also studied some properties of this new operator and its representation by Gauss hypergeometric function in the open unit disk \mathbb{U} . The topological properties of related operators are also considered such as the boundedness and compactness.

References

- [1] Carlson, B. C., Shaffer, D. B., Starlike and prestarlike hypergeometric functions. *SIAM J. Math. Anal.*, **15**, 737–745 (1984).
- [2] Duren, P. L., *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band **259**, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo (1983).

- [3] Kilbas, A. Anatoliy, Srivastava, H. M., Trujillo, J. J., Theory and Applications of Fractional Differential Equations, Elsevier Science Limited, USA, **204**, (2006).
- [4] Kim, Y. Ch., Rønning, F., Integral transforms of certain subclasses of analytic functions, *Jour. Math. Anal. Appl.*, **258**(2), 466–489, (2001).
- [5] Ibrahim, R. W., Jahangiri, J. M., Boundary fractional differential equation in a complex domain, *Boundary Value Problems*, **66**(1) (2014).
- [6] J. Krzyz, Z. Lewandowski, On the integral of univalent functions, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom.*, **11**, 447–448 (1963).
- [7] R. J. Libera, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.*, 755–758 (1965).
- [8] M. Biernacki, Sur l'intégrale des fonctions univalentes, *Bull. Acad. Polon. Sci.*, **8**, 29–34 (1960).
- [9] Owa, S., On the distortion theorems. I, *Kyungpook Math. J.*, **18**, 53–59 (1978).
- [10] Owa, S. Srivastava, H. M., Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.*, **39**, 1057–1077 (1987).
- [11] S. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.*, 109–115 (1975).
- [12] Srivastava, H., An application of the fractional derivative. *Math. Japon.*, **29**, 383–389 (1984).
- [13] Srivastava, H. M., Saigo, M., Owa, S., A class of distortion theorems involving certain operators of fractional calculus. *Jour. Math. Anal. Appl.*, **131**(2), 412–420 (1988).
- [14] Gaboury, S., Tremblay, R., A note on some new series of special functions, *Integral Transforms and Special Functions*, **25**(5), 336–343 (2014).
- [15] Kiryakova. V. S., The operators of generalized fractional calculus and their action in classes of univalent functions. In: *Proc. Inter. Symp. on Geometric Function Theory and Applications 2010*, Sofia, August 27–31 (2010).
- [16] Kiryakova, V.S, Saigo, M, and Srivastava. H. M., Some criteria for univalence of analytic functions involving generalized fractional calculus operators. *Fract. Calc. Appl. Anal.*, **1**, 79–104 (1998).
- [17] Noor, K.I., Noor, M. A., On certain classes of analytic functions defined by noor integral operator. *Jour. Math. Anal. Appl.*, **281**(1), 244–252 (2003).
- [18] Tremblay, R., Une contribution a la theorie de la derivee fractionnaire. PhD. Thesis, Laval University, Canada (1974).
- [19] Tremblay, R., Some Operational Formulas Involving the Operators xD , $x\Delta$ and Fractional Derivatives, *SIAM Journal on Mathematical Analysis*, **10**(5), 933–943 (1979).
- [20] Tremblay, R and B. J. Fugere, The use of fractional derivatives to expand analytical functions in terms of quadratic functions with applications to special functions, *Applied mathematics and computation*, **187**(1), 507–529 (2007).



Adem Kılıçman is a full Professor at the Department of Mathematics, Faculty of Science, University Putra Malaysia. He received his B.Sc. and M.Sc. degrees from Department of Mathematics, Hacettepe University, Turkey and Ph.D from Leicester University, UK. He has joined University Putra Malaysia in 1997, since then working in Faculty of Science. He is also an active member of Institute for Mathematical Research, University Putra Malaysia. His research areas includes Functional Analysis and Topology.



Rabha W. Ibrahim received her Ph.D. degrees in Mathematics from UKM, Malaysia. She was a senior lecturer at the IMS, University of Malaya, Malaysia. Currently, she is a senior researcher at Faculty of Computer Science and Information Technology, University Malaya, Malaysia. Her research interest includes fractional calculus, real and complex. Her research work has been published in several international journals and conference proceedings.



Zainab Abdunaby received the B.Sc. in Mathematics, (2002), University of Baghdad and M.Sc. in Mathematics, (2009), Al-Mustansiriyah University, Baghdad, Iraq. She is a Lecturer of Mathematics in Al-Mustansiriyah University, College of Science, Department of Mathematics, Baghdad, Iraq. Her research interests are in the areas of Functional analysis, Complex domain, Geometric function theory, fractional calculus operator and applications. Currently she is proceeding for her PhD in University Putra Malaysia (UPM).