Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100326

Generalized Order Statistics with Random Indices in a Stationary Gaussian Sequence

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Received: 24 Oct. 2015, Revised: 25 Jan. 2016, Accepted: 26 Jan. 2016
Published online: 1 May 2016

Abstract: In this paper we study the limit distributions of extreme, intermediate and central m-generalized order statistics (gos), as well as m-dual generalized order statistics (dgos), of a stationary Gaussian sequence (sGs) under equi-correlated set up, when the random sample size is assumed to converge weakly. Moreover, the result of extremes is extended to a wide subclass of gos (as well as dgos) which contains the most important models of ordered random variables (rv's).

Keywords: Gaussian sequences, generalized order statistics, dual generalized order statistics, random sample size

1 Introduction

In testing the strength of materials, reliability analysis, lifetime studies, etc., the realizations of experiments arise in nondecreasing order and therefore we need to consider several models of ascendingly ordered rv's. Kamps [17] introduced the concept of gos as a unification of several models of these ascendingly ordered rv's.

Theoretically, many of the models of ascendingly ordered rv's are contained in the gos model, such as ordinary order statistics (oos), order statistics with non-integral sample size, sequential order statistics (sos), record values, Pfeifer's record model and progressive type II censored order statistics (pos). These models can be applied in reliability theory. For instance, the rth extreme order statistic represents the life-length of some r-out-of-n system, whereas the sos model is an extension of the oos model and serves as a model describing certain dependencies or interactions among the system components caused by failures of components and the pos model is an important method of obtaining data in lifetime tests. Live units removed early on can be readily used in other tests, thereby saving cost to the experimenter.

The concept of gos enables that known results in submodels can be subsumed, generalized, and integrated within a general framework. In [17] gos were introduced via a distributional approach. Namely, the gos $X(1,n,\tilde{m},k)$, $X(2,n,\tilde{m},k),\ldots,$ $X(n,n,\tilde{m},k)$ are defined by their density function (pdf)

$$f^{(\tilde{m},k)}_{1,2,\ldots,n:n}(x_1,\ldots,x_n)$$

$$=k\Big(\prod_{i=1}^{n-1}\gamma_{i,n}\Big)\Big(\prod_{i=1}^{n-1}(1-F(x_i))^{m_i}\Big)(1-F(x_n))^{k-1}\Big(\prod_{i=1}^{n}f(x_i)\Big),$$

on the cone $\{(x_1,\ldots,x_n): x_0=F^{-1}(0)\le x_1\le\ldots\le x_n\le F^{-1}(1)=x^0\}$. The parameters $\gamma_{1,n},\ \ldots,\gamma_{n,n}$ are defined by $\gamma_{n,n}=k>0$ and $\gamma_{s,n}=k+n-s+M_s>0$, $s=1,2,\ldots,n-1$, where $\tilde{m}=(m_1,m_2\ ,\ldots,m_{n-1})$, $M_s=\sum_{j=s}^{n-1}m_j$ and $m_1,\ldots,m_n\in\Re$. If $m_1=m_2=\ldots=m_{n-1}=m$ (i.e., $\gamma_{s,n}=k+(n-s)(m+1)$, $s=1,2,\ldots,n-1$), we get a wide subclass of gos, which is called m-gos, and write $X(s,n,m,k)$ instead of $X(s,n,\tilde{m},k)$. The class of m-gos contains oos, $k-$records, sos, order statistics with non-integer sample size and pos, with special censoring schemes.

Nasri-Roudsari [18] (see, also [2]) has derived the marginal df of the sth $m-$gos, $m\neq-1$, in the form $\Phi^{(m,k)}_{s:n}(x)=I_{G_m(x)}(s,N-s+1)$, where $G_m(x)=1-(1-F(x))^{m+1}=1-\overline{F}^{m+1}(x)$, $N=\frac{k}{m+1}+n-1$ and $I_x(a,b)=\frac{1}{\beta(a,b)}\int_o^x t^{a-1}(1-t)^{b-1}dt$ is the incomplete beta ratio function. By using the well-known relation $I_x(a,b)=1-I_{\overline{x}}(b,a)$, where $\overline{x}=1-x$, the marginal df of the $(n-s+1)$th $m-$gos, $m\neq-1$, is given by $\Phi^{(m,k)}_{n-s+1:n}(x)=I_{G_m(x)}(N-R_s+1,R_s)$, where $R_s=\frac{k}{m+1}+s-1$. The possible non-degenerate limit df's

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and the convergence rate of the upper extreme m-gos, are discussed in [19]. The necessary and sufficient conditions of the weak convergence, as well as the form of the possible limit df's, of extreme, intermediate and central m-gos are derived in [2].

Burkschat et al. [8] introduced the concept of dgos to enable a common approach to descendingly ordered rv's like reversed order statistics and lower records models. The dgos $X_d(1,n,\tilde{m},k), X_d(2,n,\tilde{m},k), ..., X_d(n,n,\tilde{m},k)$ based on a df F are defined by their pdf

$$f^{d(\tilde{m},k)}_{1,2,...,n:n}(x_1,...,x_n)$$

$$= k(\prod_{i=1}^{n-1}\gamma_{i,n})(\prod_{i=1}^{n-1}(F(x_i))^{m_i})(F(x_n))^{k-1}(\prod_{i=1}^{n}f(x_i)),$$

where $x^0 = F^{-1}(1) > x_1 \geq x_2 \geq ... \geq x_n > F^{-1}(0) = x_0$. Moreover, we can write the df's of sth lower m-dgos $X(s,n,m,k)$ and the sth upper m-dgos $X(n-s+1,n,m,k)$ in the forms $\Phi^{d(m,k)}_{s:n}(x) = I_{T_m(x)}(N-s+1,s)$ and $\Phi^{d(m,k)}_{n-s+1:n}(x) = I_{T_m(x)}(R_s, N-R_s+1)$, respectively, where $T_m(x) = F^{m+1}(x)$.

Let $X_1, X_2, ..., X_n$ be a Gaussian sequence with zero expectation, unit variance and correlation $r_n = E(X_iX_j) \geq 0,\ i \neq j$. This sequence can be replaced, by the sequence $X_j = \sqrt{r_n}\,Y_0 + \sqrt{1-r_n}\,Y_j,\ 1 \leq j \leq n$, for the iid standard normal variates $Y_0, Y_1, ..., Y_n$. Moreover, $X_j = Y_j$, for $r_n = 0$. Therefore, for any $0 \leq s \leq n$, we get

$$X(s,n,m,k) = \sqrt{r_n}\,Y_0 + \sqrt{1-r_n}\,Y(s,n,m,k) \qquad (1)$$

and

$$X_d(s,n,m,k) = \sqrt{r_n}\,Y_0 + \sqrt{1-r_n}\,Y_d(s,n,m,k), \qquad (2)$$

where $X(s,n,m,k)$ (or $X_d(s,n,m,k)$) and $Y_{s:n}$ (or $Y_d(s,n,m,k)$) are the sth m-gos (or m-dgos) based on the sequences $\{X_j\}_{j=1}^{j=n}$ and $\{Y_j\}_{j=1}^{j=n}$, respectively.

A sequence $\{X(s_n,n,m,k)\}$ (or $\{X_d(s_n,n,m,k)\}$) is called a sequence of m-gos (or m-dgos) with variable rank if $1 \leq s_n \leq n$ and $s_n \to \infty$, as $n \to \infty$. Here, we have the following two distinct cases:

1-If $\frac{s_n}{n} \to 0$ (or $\frac{s_n}{n} \to 1$), as $n \to \infty$, then $X(s_n,n,m,k)$ and $X_d(s,n,m,k)$ are called lower intermediate m-gos and lower intermediate m-dgos (or upper intermediate m-gos and upper intermediate m-dgos), respectively.

2-If $\frac{s_n}{n} \to \lambda$ $(0 < \lambda < 1)$, as $n \to \infty$, then $X(s_n,n,m,k)$ and $X_d(s_n,n,m,k)$ are called central m-gos and central m-dgos, respectively. A remarkable example of the central order statistics is the pth sample quantile, where $s_n = [np], 0 < p < 1$, and $[x]$ denotes the largest integer not exceeding x (see [11]).

In many biological, agricultural and some quality control problems it is almost impossible to have a fixed sample size, because some observations always get lost for various reasons. Therefore, we often come across situations where the sample size n in $X(s,n,m,k)$ and $X_d(s,n,m,k)$ is a rv ν_n following a given distribution function (df). The rv's $X_{1:\nu_n} = X(1,\nu_n,0,1)$ and $X_{\nu_n:\nu_n} = X(\nu_n,\nu_n,0,1)$ arise naturally in reliability theory as the lifetimes of series and parallel systems, respectively, with ν_n identical components having lifetimes $X_1, X_2, ..., X_{\nu_n}$. Also, the rv $X_{1:\nu_n}$ arises naturally in transportation theory as the accident-free distance of a shipment of, say, explosives, where ν_n of them are defective, which may explode and cause an accident after $X_1, X_2, ..., X_{\nu_n}$ miles, respectively (cf. [20]). If one introduces the random sample size as an extension of a model (mainly for statistical inference), one can usually assume that it is independent of the underlying variables.

Many authors considered the limit theory of oos with random sample sizes when $r_n = 0$ (i.e., in the iid rv's case) and ν_n is independent of the basic rv's, where, the df of $\frac{\nu_n}{n}$ converges weakly to a non-degenerate df. Among those authors are [1,7,12,14,15]. Vasudeva and Moridani [21] studied the limit df of sth maxima of oos $X_{\nu_n:\nu_n}$ in the sGs (1), under a restrictive condition that the random correlation r_{ν_n} converges in probability to a positive constant or infinity. The most recent contribution relevant to this topic is [22], in which it is obtained the limit theorems for the maxima of stationary Gaussian process, with random index.

In Section 2, we study the upper (or lower) extreme m-gos $X(\acute{s}(\nu_n),\nu_n,\ m,k) = X(\nu_n-s+1,\nu_n,m,k)$ (or $X(s,\nu_n,m,k)$) and the upper (or lower) extreme m-dgos $X_d(\acute{s}(\nu_n),\nu_n,m,k) = X_d(\nu_n-s+1,\nu_n,m,k)$ (or $X_d(s,\nu_n,m,k)$) concerning the sequence (1) and (2), respectively, under mild conditions, where the restricted condition in [21] is got rid. Some of these results are extended to a wide subclass of gos, as well as dgos, when the parameters $\gamma_{1,n}, \gamma_{2,n}, ..., \gamma_{n,n}$ are assumed to be pairwise different. In Sections 3 and 4, we consider the parallel results for the central and intermediate m-gos and m-dgos, respectively.

Everywhere in what follows the symbols $\xrightarrow[n]{}$, $\xrightarrow[n]{w}$ and $\xrightarrow[n]{p}$ stand for convergence, converge weakly and converge in probability, as $n \to \infty$, respectively. Moreover, for every $s, x \geq 0$, $\Gamma_s(x) = \frac{1}{\Gamma(s)}\int_0^x t^{s-1}e^{-t}dt$ stands for the incomplete gamma ratio function, while $\bar{\Gamma}_s(x) = 1 - \Gamma_s(x)$ denotes its survivor function. Finally, $\mathcal{N}(x)$ denotes the standard normal df.

2 Extreme m-gos (dgos) with random indices in a sGs

The weak convergence of the sequences $\{\frac{X(\acute{s}(\nu_n),\nu_n,m,k)-a_{n,m}}{b_{n,m}}\}$ and $\{\frac{X_d(\acute{s}(\nu_n),\nu_n,m,k)-a_{n,m}}{b_{n,m}}\}$, are investigated in Theorems (2.1) and (2.2), respectively, where $a_{n,m} = \frac{1}{b_{n,m}} - \frac{1}{2}b_{n,m}\ (\log\log n^{\frac{1}{m+1}} + \log 4\pi)$ and $b_{n,m} = (\frac{2}{m+1}\log n)^{\frac{-1}{2}}$. Moreover, Theorems (2.3) and

(2.4) give the corresponding results concerning $\{\frac{X(s,\nu_n,m,k)-\acute{a}_{n,m}}{\acute{b}_{n,m}}\}$ and $\{\frac{X_d(s,\nu_n,m,k)-\acute{a}_{n,m}}{\acute{b}_{n,m}}\}$, respectively, where $\acute{a}_{n,m}=\frac{1}{\acute{b}_{n,m}}-\frac{1}{2}\acute{b}_{n,m}(\log\log n(m+1)+\log 4\pi)$ and $\acute{b}_{n,m}=(2\log n(m+1))^{\frac{-1}{2}}$.

Theorem 2.1. Let ν_n be a sequence of integer valued rv's independent of $X_1,...,X_n$ and $P(\nu_n<x)=A_n(x)$. Furthermore,

(A):let $A_n(nx)\xrightarrow[n]{w}A(x)$, where $A(+0)=0$ and $A(x)$ is a non-degenerate df. Then

(B$_1$):$P\left(\frac{X(s'(\nu_n),\nu_n,m,k)-a_{n,m}}{b_{n,m}}<x\right)\xrightarrow[n]{w}\Psi(x)$ $=\int_0^\infty H^{(m,k)}(x;\tau,z)dA(z)$, if

(C$_1$):$r_n\log n\xrightarrow[n]{}\tau\geq 0$, where

$$H^{(m,k)}(x;\tau,z)$$

$$=\begin{cases}\bar{\Gamma}_{R_s}(ze^{-(m+1)x-\tau})*\mathscr{N}(\sqrt{\frac{m+1}{2\tau}}x), & \tau>0.\\ \bar{\Gamma}_{R_s}(ze^{-(m+1)x}), & \tau=0,\end{cases}$$

and (*) stands for the convolution operation. Moreover,

(B$_2$):$P\left(\frac{X(s'(\nu_n),\nu_n,m,k)-a_{n,m}}{\sqrt{r_n}}<x\right)\xrightarrow[n]{w}\mathscr{N}(x)$, if

(C$_2$):$r_n\log n\xrightarrow[n]{}\infty$ and r_n is slowly varying function of n (see, [16]), i.e., for every $\theta>0$, we get $\frac{r_{n\theta}}{r_n}\xrightarrow[n]{}\theta$.

Conversely, if (B$_1$) and (C$_1$) (with $\tau=0$) hold, then the relation (A) will be satisfied.

Proof. Let $P_{nq}=P(\nu_n=q)$. Then, by the total probability rule, we get

$$M^{(m,k)}_{s'(\nu_n):\nu_n}(a_{n,m}+b_{n,m}x)=P(X(s'(\nu_n),\nu_n,m,k)<a_{n,m}+b_{n,m}x)$$

$$=\sum_{q=s}^{\infty}M^{(m,k)}_{s'(q):q}(a_{n,m}+b_{n,m}x)P_{nq}. \quad (3)$$

Assume that $z=\frac{q}{n}$, thus the sum in (3) is a Riemann sum of the integral

$$M^{(m,k)}_{s'(\nu_n):\nu_n}(a_{n,m}+b_{n,m}x)$$

$$=\int_0^\infty M^{(m,k)}_{s'(nz):nz}(a_{n,m}+b_{n,m}x)dA_n(nz). \quad (4)$$

Now, consider the condition (A) with (C$_1$), by using (1), we get $\frac{X(\acute{s}(nz),nz,m,k)-a_{n,m}}{b_{n,m}}=U^{(m,k)}_{nz}+V^{(m,k)}_{nz}$, where $U^{(m,k)}_{nz}=\frac{\sqrt{r_{nz}}}{b_{n,m}}Y_0$ and $V^{(m,k)}_{nz}=\frac{\sqrt{1-r_{nz}}}{b_{n,m}}\left[Y(\acute{s}(nz),nz,m,k)-a_{n,m}(1-r_{nz})^{-\frac{1}{2}}\right]$. Moreover, $U^{(m,k)}_{nz}$ and $V^{(m,k)}_{nz}$ are independent. Therefore,

$$\left.\begin{array}{ll}P(U^{(m,k)}_{nz}<x)\xrightarrow[n]{w}\mathscr{N}(\sqrt{\frac{m+1}{2\tau}}x), & \text{if } \tau>0,\\ U^{(m,k)}_{nz}\xrightarrow[n]{p}0, & \text{if } \tau=0.\end{array}\right\} \quad (5)$$

On the other hand, we can write $P(V^{(m,k)}_{nz}<x)=P(Y(\acute{s}(nz),nz,m,k)<A_{nz,m}+B_{nz,m}x)$, where $A_{nz,m}=(1-r_{nz})^{-\frac{1}{2}}a_{n,m}$ and $B_{n,m}=(1-r_{nz})^{-\frac{1}{2}}b_{n,m}$. By using Theorem 2.1 of [2], we get $P(Y(\acute{s}(nz),nz,m,k)<a_{nz,m}+b_{nz,m}x)\xrightarrow[n]{w}\bar{\Gamma}_{R_s}(e^{-(m+1)x})$. Moreover, it is easy to verify that $\frac{A_{nz,m}-a_{nz,m}}{b_{nz,m}}\xrightarrow[n]{}\frac{1}{m+1}(\tau-\log z)$ and $\frac{B_{nz,m}}{b_{nz,m}}\xrightarrow[n]{}1$. The latter is evident from the assumption $r_n\log n\xrightarrow[n]{}\tau\geq 0$ and thus $r_n\xrightarrow[n]{}0$ (i.e., $r_{nz}\xrightarrow[n]{}0$). Hence, only the first relation needs proof. Applying that $(1-r_{nz})^{-\frac{1}{2}}=1+\frac{1}{2}r_{nz}(1+\circ(1))$, $(\frac{2}{m+1}\log nz)^{\frac{1}{2}}=\sqrt{\frac{2}{m+1}\log n}+\frac{\log z}{\sqrt{2(m+1)\log n}}(1+\circ(1))$ and $\log\log(nz)^{\frac{1}{m+1}}=\log\log n^{\frac{1}{m+1}}+\frac{1}{m+1}\log(1+\frac{\log z}{\log n})$ and bearing in mind that $\frac{\log\log n}{\log n}\xrightarrow[n]{}0$, we get

$$\frac{A_{nz,m}-a_{nz,m}}{b_{nz,m}}=\left(1+\frac{r_{nz}}{2}(1+o(1))\right)$$

$$\left[\frac{2}{m+1}\log n+(1+o(1))\frac{\log z}{m+1}-\frac{1}{2}(\log\log n^{\frac{1}{m+1}}+\log 4\pi)\right.$$

$$\left.-\frac{\log z}{4\log n}(\log\log n^{\frac{1}{m+1}}+\log 4\pi)(1+o(1))\right]-\frac{2}{m+1}\log n$$

$$-\frac{2\log z}{m+1}+\frac{1}{2}\left[\log\log n^{\frac{1}{m+1}}+\log(1+\frac{\log z}{\log n})^{\frac{1}{m+1}}+\log 4\pi\right]$$

$$\xrightarrow[n]{}\frac{1}{m+1}(-\log z+\tau).$$

Thus, in view of Khinchin's type theorem, we get

$$P(V_{nz}<x)\xrightarrow[n]{w}\bar{\Gamma}_{R_s}(ze^{-(m+1)x-\tau}). \quad (6)$$

By combining (5) and (6), Lemma 2.2.1 in [13] thus yields

$$M^{(m,k)}_{s'(nz):nz}(a_{n,m}+b_{n,m}x)\xrightarrow[n]{w}H^{(m,k)}(x;\tau,z), \quad (7)$$

uniformly with respect to x over any finite interval of z (the continuity of the limit in x, implies that the convergence is uniform). Now, let c be a continuity point of $A(x)$ such that $1-A(c)<\varepsilon$. Then

$$\int_c^\infty H^{(m,k)}(x;\tau,z)dA(z)\leq 1-A(c)<\varepsilon. \quad (8)$$

Moreover, for sufficiently large n, in view of condition (A), we get

$$\int_c^\infty M^{(m,k)}_{s'(nz):nz}(a_{n,m}+b_{n,m}x)dA_n(nz)\leq 1-A_n(nc)<2\varepsilon. \quad (9)$$

For estimating the difference $M^{(m,k)}_{s'(\nu_n):\nu_n}(a_{n,m}+b_{n,m}x)-\Psi(x)$, we first estimate $\int_0^c M^{(m,k)}_{s'(nz):nz}(a_{n,m}+b_{n,m}x)dA_n(nz)-\int_0^c H^{(m,k)}(x;\tau,z)dA(z)$. By using the triangle inequality

$$\left|\int_0^c M^{(m,k)}_{s'(nz):nz}(a_{n,m}+b_{n,m}x)dA_n(nz)-\int_0^c H^{(m,k)}(x;\tau,z)dA(z)\right|$$

$$\leq\left|\int_0^c M^{(m,k)}_{s'(nz):nz}(a_{n,m}+b_{n,m}x)dA_n(nz)\right.$$

$$-\int_0^c H^{(m,k)}(x;\tau,z)dA_n(nz)\Big|$$

$$+\left|\int_0^c H^{(m,k)}(x;\tau,z)dA_n(nz)-\int_0^c H^{(m,k)}(x;\tau,z)dA(z)\right|. \quad (10)$$

Since, the convergence in (7) is uniform over the finite interval $[0,c]$. Therefore, for any arbitrary $\varepsilon>0$ and for sufficiently large n, we get

$$\left|\int_0^c [M^{(m,k)}_{s'(nz):nz}(a_{n,m}+b_{n,m}x)-H^{(m,k)}(x;\tau,z)]dA_n(nz)\right|$$

$$\leq \varepsilon(A_n(nc)-A_n(0))\leq\varepsilon. \quad (11)$$

The third difference in (10) can be estimated by constructing Riemann sums, which are close to the integral there. Namely, let n_0 be a fixed number, and let $0=c_0<c_1<...<c_{n_0}=c$ be the continuity points of $A(x)$. Furthermore, let n_0 and c_i be such that $|\int_0^c H^{(m,k)}(x;\tau,z)dA_n(nz)-\sum_{i=0}^{n_0} H^{(m,k)}(x;\tau,c_i)(A_n(nc_i)-A_n(nc_{i-1}))|<\varepsilon$, and $|\int_0^c H^{(m,k)}(x;\tau,z)dA(z)-\sum_{i=0}^{n_0} H^{(m,k)}(x;\tau,c_i)(A(c_i)-A(c_{i-1}))|<\varepsilon$. Since, by assumption $A_n(nc_i)\xrightarrow[n]{w}A(c_i)$, $0\leq i\leq n_0$, the two Riemann sums are closer to each other than ε for all sufficiently large n. Thus, once again by the triangle inequality, the absolute value of the difference of the integrals is smaller than 3ε. Combining this fact with (11), the left hand side term of (10) becomes smaller than 4ε for all large n. Thus, in view of (4), (8) and (9), we get

$$\left|M^{(m,k)}_{s'(\nu_n):\nu_n}(a_{n,m}+b_{n,m}x)-\Psi(x)\right|$$
$$=\Big|\int_0^c M^{(m,k)}_{s'(nz):nz}(a_{n,m}+b_{n,m}x)dA_n(nz)$$
$$+\int_c^\infty M^{(m,k)}_{s'(nz):nz}(a_{n,m}+b_{n,m}x)dA_n(nz)$$
$$-\int_0^c H^{(m,k)}(x;\tau,z)dA(z)-\int_c^\infty H^{(m,k)}(x;\tau,z)dA(z)\Big|$$
$$<\Big|\int_0^c M^{(m,k)}_{s'(nz):nz}(a_{n,m}+b_{n,m}x)dA_n(nz)$$
$$-\int_0^c H^{(m,k)}(x;\tau,z)dA(z)\Big|$$
$$+\left|\int_c^\infty M^{(m,k)}_{s'(nz):nz}(a_{n,m}+b_{n,m}x)dA_n(nz)\right|$$
$$+\left|\int_c^\infty H^{(m,k)}(x;\tau,z)dA(z)\right|=7\varepsilon.$$

This completes the proof of the first part of the theorem.

Turning to the condition (A) with (C_2), starting with the relation (4), we notice that

$$\frac{X(s'(nz),nz,m,k)-a_{n,m}}{\sqrt{r_n}}=\sqrt{\frac{r_{nz}}{r_n}}Y_0+T^{(m,k)}_{nz}, \quad (12)$$

where $T^{(m,k)}_{nz}=\sqrt{\frac{1-r_{nz}}{r_n}}[Y(s'(nz),nz,m,k)-(1-r_{nz})^{-\frac{1}{2}}a_{n,m}]\leq\sqrt{\frac{1-r_{nz}}{r_n}}[Y(s'(nz),nz,\ m,k)-a_{n,m}]$, for large n, since $0\leq r_{nz}\leq 1$ and $a_{n,m}>0$, for large n. Therefore, $|T^{(m,k)}_{nz}|\leq|Y(s'(nz),nz,m,k)-a_{n,m}|r_n^{-\frac{1}{2}}$, since $0\leq r_{nz}\leq 1$. Applying the condition that r_n is slowly varying, then, for every finite value z, we have $\sqrt{\frac{r_{nz}}{r_n}}\xrightarrow[n]{}1$. Therefore,

$$P\left(\sqrt{\frac{r_{nz}}{r_n}}Y_0<x\right)\xrightarrow[n]{w}\mathcal{N}(x). \quad (13)$$

On the other hand, for every $\varepsilon>0$, we get

$$P\left(|T^{(m,k)}_{nz}|\geq\varepsilon\right)$$
$$\leq P\left(\frac{|Y(s'(nz),nz,m,k)-a_{nz,m}|}{b_{nz,m}}\times r_n^{-\frac{1}{2}}b_{nz,m}+L_{n,m}\geq\varepsilon\right)$$
$$=P\Bigg(\frac{|Y(s'(nz),nz,m,k)-a_{nz,m}|}{b_{nz,m}}$$
$$\geq(\varepsilon-L_{n,m})\frac{\sqrt{r_{nz}}}{b_{nz,m}}\times\sqrt{\frac{r_n}{r_{nz}}}\Bigg), \quad (14)$$

where

$$L_{n,m}=\frac{a_{nz,m}-a_{n,m}}{\sqrt{r_n}}=\frac{1}{\sqrt{r_n}}\Bigg[\Big(\frac{1}{b_{nz,m}}-\frac{1}{b_{n,m}}\Big)$$
$$-\frac{1}{2}\Big(b_{nz,m}(\log\log(nz)^{\frac{1}{m+1}}+\log 4\pi)$$
$$-b_{n,m}(\log\log n^{\frac{1}{m+1}}+\log 4\pi)\Big)\Bigg].$$

Applying, $\frac{1}{\sqrt{r_n}}(\frac{1}{b_{nz,m}}-\frac{1}{b_{n,m}})=\frac{\log z}{\sqrt{2(m+1)r_n\log n}}(1+\circ(1))\xrightarrow[n]{}0$, we get

$$\lim_{n\to\infty}L_{n,m}=\lim_{n\to\infty}\frac{-1}{2\sqrt{r_n}}\Bigg[\frac{\log\frac{1}{m+1}(\log n+\log z)+\log 4\pi}{\sqrt{2\log(nz)^{\frac{1}{m+1}}}}$$
$$-\frac{\log\log n^{\frac{1}{m+1}}+\log 4\pi}{\sqrt{2\log n^{\frac{1}{m+1}}}}\Bigg]$$
$$=\lim_{n\to\infty}\frac{-1}{2\sqrt{\frac{2r_n}{m+1}\log n}}\Bigg[\log(1+\frac{\log z}{\log n})^{\frac{1}{m+1}}-\Bigg[\frac{(\log\log n)\log z}{2\log n}$$
$$+\frac{(\log(1+\frac{\log z}{\log n}))\log z}{2\log n}+\frac{(\log 4\pi)\log z}{2\log n}\Bigg](1+\circ(1))\Bigg]=0.$$

Since $\frac{r_{nz}}{r_n}\xrightarrow[n]{}1$ and $\frac{r_{nz}}{b_{nz,m}}=\sqrt{\frac{2r_{nz}}{m+1}\log nz}\xrightarrow[n]{}\infty$, the relation (14) implies

$$P\left(|T^{(m,k)}_{nz}|\geq\varepsilon\right)\xrightarrow[n]{}0. \quad (15)$$

Combining (12), (13), (14) and (15), Lemma 2.2.1 in [13] thus yields $M^{(m,k)}_{s'(nz):nz}(a_{n,m}+b_{n,m}x)\xrightarrow[n]{w}\mathcal{N}(x)$. The remaining part of this case follows exactly as the proof of the case $r_n\log n\xrightarrow[n]{}\tau$.

Turning now to prove the converse part that (B$_1$) and (C$_1$) imply (A). Starting with the relation (4), by the compactness of df's, we can select a subsequence $n^\star$, such that $A_{n^\star}(n^\star z)\xrightarrow[n^\star]{w}A^\star(z)$, where $A^\star(z)$ is an extended df (i.e., $A^\star(\infty)-A^\star(0)\le 1$). Therefore, by repeating the first part of theorem (when $\tau=0$) for the subsequence $n^\star$, with the exception that the point c is chosen such that $A^\star(\infty)-A^\star(c)\le\varepsilon$, we get $M^{(m,k)}_{s'(v_{n^\star}):v_{n^\star}}(a_{n^\star,m}+b_{n^\star,m}x)\xrightarrow[n^\star]{w}\Psi(x)=\int_0^\infty\bar{\Gamma}_{R_s}(ze^{-(m+1)x})\,dA^\star(z)$. Since, the two limits $\Psi(x)$ and $\bar{\Gamma}_{R_s}(ze^{-(m+1)x})$ are df's, then $\Psi(\infty)=1=\int_0^\infty dA^\star(z)=A^\star(\infty)-A^\star(0)$. Thus, $A^\star$ is a df. Now, if $A_n(nz)$ did not converge weakly, then we could select two subsequences n_1 and n_2 such that $A_{n_i}(zn_i)\xrightarrow[n_1]{w}A_i(z), i=1,2$. This implies that

$$\Psi(x)=\int_0^\infty\bar{\Gamma}_{R_s}(ze^{-(m+1)x})dA_1(z)$$

$$=\int_0^\infty\bar{\Gamma}_{R_s}(ze^{-(m+1)x})dA_2(z). \tag{16}$$

Let $G_i(t)=\int_0^\infty\bar{\Gamma}_{R_s}(tz)dA_i(z), i=1,2$. Clearly, $G_1(t)$ and $G_2(t)$ are analytic functions in the region $D=\{t:0<|t|<\infty\}$. Moreover, in view of (16), G_1 and G_2 coincide on some interval contained in D. for all real values of x. Thus by the uniqueness theory of analytic functions, $G_1(t)$ and $G_2(t)$ coincide on the region D, which means that $A_1(z)=A_2(z)$. This completes the proof of the theorem.□

Theorem 2.2. Let v_n be a sequence of integer valued rv's independent of $X_1,...,X_n$ and $P(v_n<x)=A_n(x)$. Furthermore,

(A):let $A_n(nx)\xrightarrow[n]{w}A(x)$, where $A(+0)=0$ and $A(x)$ is a non-degenerate df. Then

(B$_1$):$P\left(\frac{X_d(s'(v_n),v_n,m,k)+a_{n,m}}{b_{n,m}}<x\right)\xrightarrow[n]{w}\Psi(x)$
$=\int_0^\infty H^{d(m,k)}(x;\tau,z)dA(z)$, if

(C$_1$):$r_n\log n\xrightarrow[n]{}\tau\ge 0$, where

$$H^{d(m,k)}(x;\tau,z)$$

$$=\begin{cases}\bar{\Gamma}_{R_s}(ze^{(m+1)x-\tau})*\mathcal{N}\left(\sqrt{\frac{m+1}{2\tau}}x\right), & \tau>0,\\ \bar{\Gamma}_{R_s}(ze^{(m+1)x}), & \tau=0.\end{cases}$$

Moreover,

(B$_2$):$P\left(\frac{X_d(s'(v_n),v_n,m,k)+a_{n,m}}{\sqrt{r_n}}<x\right)\xrightarrow[n]{w}\mathcal{N}(x)$, if

(C$_2$):$r_n\log n\xrightarrow[n]{}\infty$ and r_n is slowly varying function of n.

Conversely, if (B$_1$) and (C$_1$) (with $\tau=0$) hold, then the relation (A) will be satisfied.

Proof. By representation (2) and by using Theorem 1.1 of [5], it is easy to see that the proof of Theorem 2.2 is similar to the proof of Theorem 2.1, with only the exception of obvious changes.□

Theorem 2.3. Let v_n be a sequence of integer valued rv's independent of $X_1,...,X_n$ and $P(v_n<x)=A_n(x)$. Furthermore,

(A):let $A_n(nx)\xrightarrow[n]{w}A(x)$, where $A(+0)=0$ and $A(x)$ is a non-degenerate df. Then

(B$_1$):$P\left(\frac{X(s,v_n,m,k)+a'_{n,m}}{b'_{n,m}}<x\right)\xrightarrow[n]{w}\Psi(x)$
$=\int_0^\infty H^{(m,k)}(x;\tau,z)dA(z)$, if

(C$_1$):$r_n\log n\xrightarrow[n]{}\tau\ge 0$, where

$$H^{(m,k)}(x;\tau,z)=\begin{cases}\Gamma_s(ze^{x-\tau})*\mathcal{N}\left(\frac{x}{\sqrt{2\tau}}\right), & \tau>0,\\ \Gamma_s(ze^{x}), & \tau=0.\end{cases}$$

Moreover,

(B$_2$):$P\left(\frac{X(s,v_n,m,k)+a'_{n,m}}{\sqrt{r_n}}<x\right)\xrightarrow[n]{w}\mathcal{N}(x)$, if

(C$_2$):$r_n\log n\xrightarrow[n]{}\infty$ and r_n is slowly varying function of n.

Conversely, if (B$_1$) and (C$_1$) (with $\tau=0$) hold, then the relation (A) will be satisfied.

Proof. By representation (1) and by using Theorem 1.1 of [5], it is easy to see that the proof of Theorem 2.3 is similar to the proof of Theorem 2.1, with only the exception of obvious changes.□

Theorem 2.4. Let v_n be a sequence of integer valued rv's independent of $X_1,...,X_n$ and $P(v_n<x)=A_n(x)$. Furthermore,

(A):let $A_n(nx)\xrightarrow[n]{w}A(x)$, where $A(+0)=0$ and $A(x)$ is a non-degenerate df. Then

(B$_1$):$P\left(\frac{X_d(s,v_n,m,k)-a'_{n,m}}{b'_{n,m}}<x\right)\xrightarrow[n]{w}\Psi(x)$
$=\int_0^\infty H^{d(m,k)}(x;\tau,z)dA(z)$, if

(C$_1$):$r_n\log n\xrightarrow[n]{}\tau\ge 0$, where

$$H^{d(m,k)}(x;\tau,z)=\begin{cases}\bar{\Gamma}_{R_s}(ze^{-(x+\tau)})*\mathcal{N}\left(\frac{x}{\sqrt{2\tau}}\right), & \tau>0,\\ \bar{\Gamma}_{R_s}(ze^{-x}), & \tau=0.\end{cases}$$

Moreover,

(B$_2$):$P\left(\frac{X_d(s,v_n,m,k)-a'_{n,m}}{\sqrt{r_n}}<x\right)\xrightarrow[n]{w}\mathcal{N}(x)$, if

(C$_2$):$r_n\log n\xrightarrow[n]{}\infty$ and r_n is slowly varying function of n.

Conversely, if (B$_1$) and (C$_1$) (with $\tau=0$) hold, then the relation (A) will be satisfied.

Proof. By representation (2) and by using Theorem 1.2 of [5], it is easy to see that the proof of Theorem 2.4 is similar to the proof of Theorem 2.1, with only the exception of obvious changes.□

Although, the above theorems provide a set-up, which includes many interesting models such as oos, sos and pos, with censoring scheme $(R,...,R)\in\mathcal{N}^M$, a number of models of gos are excluded in this set-up, e.g., pos with general censoring scheme $(R_1,...,R_M)$. The following two theorems extend Theorems 2.3 and 2.4 to a very wide subclass of gos in which the vector $\tilde{m} = (m_1, m_2, ..., m_{n-1})$ is arbitrarily chosen such that $m_i > -1, i = 1, 2, ..., n-1$, and the parameters $\gamma_{1,n}, \gamma_{2,n}, ..., \gamma_{n,n}$ are pairwise different, i.e., $\gamma_{i,n} \neq \gamma_{j,n}, i \neq j$, for all $i, j \in \{1, ..., n\}$. For instance, this assumption is no restriction on pos with general censoring scheme $(R_1, ..., R_M)$.

Theorem 2.5. Let $\acute{a}_{\gamma_{1,n}} = \frac{1}{\acute{b}_{\gamma_{1,n}}} - \frac{1}{2}\acute{b}_{\gamma_{1,n}}(\log\log\gamma_{1,n} + \log 4\pi)$, $\acute{b}_{\gamma_{1,n}} = (2\log\gamma_{1,n})^{\frac{-1}{2}}$ and $\gamma_{1,n} \xrightarrow[n]{} \infty$. Furthermore, let ν_n be a sequence of integer valued rv's independent of $X_1, ..., X_n$ and $P(\nu_n < x) = A_n(x)$. Furthermore,

(A):let $A_n(nx) \xrightarrow[n]{w} A(x)$, where $A(+0) = 0$ and $A(x)$ is a non-degenerate df. Then

(B$_1$):$P\left(\frac{X(s,\nu_n,\tilde{m},k)+a'_{\gamma_{1,n}}}{b'_{\gamma_{1,n}}} < x\right) \xrightarrow[n]{w} \Psi(x)$
$= \int_0^\infty H^{(\tilde{m},k)}(x;\tau,z)dA(z)$, if

(C$_1$):$r_n \log\gamma_{1,n} \xrightarrow[n]{} \tau \geq 0$, where

$$H^{(\tilde{m},k)}(x;\tau,z) = \begin{cases} \Gamma_s(ze^{x-\tau}) * \mathcal{N}(\frac{x}{\sqrt{2\tau}}), & \tau > 0, \\ \Gamma_s(ze^x), & \tau = 0. \end{cases}$$

Moreover,

(B$_2$):$P\left(\frac{X(s,\nu_n,\tilde{m},k)+a'_{\gamma_{1,n}}}{\sqrt{r_n}} < x\right) \xrightarrow[n]{w} \mathcal{N}(x)$, if

(C$_2$):$r_n \log\gamma_{1,n} \xrightarrow[n]{} \infty$ and r_n is slowly varying function of n.

Conversely, if (B$_1$) and (C$_1$) (with $\tau = 0$) hold, then the relation (A) will be satisfied.

Proof. By representation (1) and by using Theorem 2.1 of [4], it is easy to see that the proof of Theorem 2.5 is similar to the proof of Theorem 2.3, with only the exception of obvious changes.□

Theorem 2.6. Let ν_n be a sequence of integer valued rv's independent of $X_1, ..., X_n$ and $P(\nu_n < x) = A_n(x)$. Furthermore,

(A):let $A_n(nx) \xrightarrow[n]{w} A(x)$, where $A(+0) = 0$ and $A(x)$ is a non-degenerate df. Then

(B$_1$):$P\left(\frac{X_d(s,\nu_n,\tilde{m},k)-a'_{\gamma_{1,n}}}{b'_{\gamma_{1,n}}} < x\right) \xrightarrow[n]{w} \Psi(x)$
$= \int_0^\infty H^{d(\tilde{m},k)}(x;\tau,z)dA(z)$, if

(C$_1$):$r_n \log\gamma_{1,n} \xrightarrow[n]{} \tau \geq 0$, where

$$H^{d(\tilde{m},k)}(x;\tau,z) = \begin{cases} \bar{\Gamma}_{R_s}(ze^{-(x+\tau)}) * \mathcal{N}(\frac{x}{\sqrt{2\tau}}), & \tau > 0, \\ \bar{\Gamma}_{R_s}(ze^{-x}), & \tau = 0. \end{cases}$$

Moreover,

(B$_2$):$P\left(\frac{X_d(s,\nu_n,\tilde{m},k)-a'_{\gamma_{1,n}}}{\sqrt{r_n}} < x\right) \xrightarrow[n]{w} \mathcal{N}(x)$, if

(C$_2$):$r_n \log\gamma_{1,n} \xrightarrow[n]{} \infty$ and r_n is slowly varying function of n.

Conversely, if (B$_1$) and (C$_1$) (with $\tau = 0$) hold, then the relation (A) will be satisfied.

Proof. By representation (2) and by using Corollary 2.1 of [3], it is easy to see that the proof of Theorem 2.6 is similar to the proof of Theorem 2.4, with only the exception of obvious changes.□

3 Central *m*-gos (dgos) with random indices in a sGs

Let $0 < \lambda < 1$ and x_0 be such that $\mathcal{N}(x_0) = \lambda$. Moreover, let s_n be a central rank sequence such that $\sqrt{n}(\frac{s_n}{n} - \lambda) \xrightarrow[n]{} 0$. It is known that (c.f. Theorem 2.2 of [2])

$$P(\frac{Y(s_n,n,m,k)-x_0}{c_n} < x), P(\frac{Y_d(s_n,n,m,k)-x_0}{c_n} < x)$$

$$\xrightarrow[n]{w} \mathcal{N}(\frac{c^*_{\lambda(m)}}{c^*_\lambda}(m+1)x),$$

where $c_n = \frac{\sqrt{\lambda(1-\lambda)}}{\sqrt{n}\phi(x_0)}$, $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ is the pdf of the standard normal distribution, $c_\lambda = \sqrt{\lambda(1-\lambda)}$, $\lambda(m) = 1-(1-\lambda)^{\frac{1}{m+1}}$ and $c^*_\lambda = \frac{c_\lambda}{\lambda}$. Under the above conditions concerning λ and s_n, the following theorem gives the limit df of the s_nth central m-gos and m-dgos of sGs's (1) and (2), respectively.

Theorem 3.1. Let the condition (A) in Theorems 2.1-2.6 satisfied. Then,

(B$^\star_1$):$P(\frac{X(s_{\nu_n},\nu_n,m,k)-x_0}{c_n} < x) \xrightarrow[n]{w} \Phi(x)$
$= \int_0^\infty L(x;\tau,z)dA(z)$, if

(C$^\star_1$): $nr_n \xrightarrow[n]{} \tau \geq 0$, where $L(x;\tau,z) = \mathcal{N}\left((\sqrt{z\frac{\lambda(1-\lambda)}{\tau\phi^2(x_0)+\lambda(1-\lambda)}}\frac{c^*_{\lambda(m)}}{c^*_\lambda}(m+1)x\right)$. Moreover,

(B$^\star_2$): $P(\frac{X(s_{\nu_n},\nu_n,m,k)-x_0}{\sqrt{r_n}} < x) \xrightarrow[n]{w} \mathcal{N}(x)$, if $x_0 \geq 0$ and

(C$^\star_2$): $nr_n \xrightarrow[n]{} \infty$ and r_n is slowly varying function of n.

Conversely, if (B$^\star_1$) and (C$^\star_1$) hold, then the relation (A) will be satisfied.

Proof. Proceeding exactly as the proof of (3), we get

$$M^{(m,k)}_{s_{\nu_n}:\nu_n}(x_0+c_nx) = P(X(s_{\nu_n},\nu_n,m,k) < x_0+c_nx)$$

$$= \int_0^\infty M^{(m,k)}_{s_{nz}:nz}(x;z,\tau)dA_n(nz), \tag{17}$$

where $M^{(m,k)}_{s_{nz}:nz}(x;z,\tau) = P(X(s_{nz},nz,m,k) < x_0+c_nx)$. First, consider the condition (A) with (C$^\star_1$), by using (1), we get

$$\frac{X(s_{zn},nz,m,k)-x_0}{c_n} = \frac{\sqrt{r_{nz}}}{c_n}Y_0$$

$$+\frac{\sqrt{1-r_{nz}}Y(s_{zn},nz,m,k)-x_0}{c_n} = U^{(m,k)}_{nz} + V^{(m,k)}_{nz}, \tag{18}$$

$U_{nz}^{(m,k)}=\frac{\sqrt{nzr_{nz}}\phi(x_0)}{\sqrt{z\lambda(1-\lambda)}}Y_0$ and $V_{nz}^{(m,k)}=\frac{\sqrt{1-r_{nz}}Y(s_{nz},nz,m,k)-x_0}{c_n}$. Moreover, $U_{nz}^{(m,k)}$ and $V_{nz}^{(m,k)}$ are independent. If $nr_n \xrightarrow[n]{} \tau,\ 0\leq\tau<\infty$, then

$$\left.\begin{array}{l} P(U_{nz}^{(m,k)}<x)\xrightarrow[n]{w}\mathscr{N}(\sqrt{z\frac{\lambda(1-\lambda)}{\tau\phi^2(x_0)}}\frac{c^*_{\lambda(m)}}{c^*_\lambda}(m+1)x), \\ \quad \text{if } \tau>0, \\ U_{nz}^{(m,k)}\xrightarrow[n]{p}0, \text{ if } \tau=0. \end{array}\right\} \quad (19)$$

On the other hand, we have

$$P(V_{nz}^{(m,k)}<x)=P(Y(s_{nz},nz,m,k)<A_{nz,m}+B_{nz,m}x), \quad (20)$$

where $A_{nz,m}=\frac{x_0}{\sqrt{1-r_{nz}}}$ and $B_{nz,m}=\frac{c_n}{\sqrt{1-r_{nz}}}$. Now, if $nr_n\xrightarrow[n]{}\tau\geq 0$, we get

$$\frac{A_{nz,m}-x_0}{c_{nz}}=\frac{(1+\frac{r_{nz}}{2}(1+o(1)))x_0-x_0}{\sqrt{\lambda(1-\lambda)}}\sqrt{nz}\phi(x_0)$$

$$\sim\frac{\sqrt{r_{nz}}\sqrt{nzr_{nz}}x_0\phi(x_0)}{2\sqrt{\lambda(1-\lambda)}}\xrightarrow[n]{}0.$$

Moreover, $\frac{B_{nz,m}}{c_{nz}}=(1+\frac{r_{nz}}{2}(1+o(1)))\xrightarrow[n]{}\sqrt{z}$. Therefore, an application of Khinchin's type theorem yields

$$P(V_{nz}^{(m,k)}\leq x)\xrightarrow[n]{w}\mathscr{N}(\sqrt{z}\frac{c^*_{\lambda(m)}}{c^*_\lambda}(m+1)x). \quad (21)$$

By combining (18),(19), (20) and (21), we get

$$M_{s_{nz}:nz}^{(m,k)}(x_0+c_nx)\xrightarrow[n]{w}L(x;\tau,z)$$

$$=\mathscr{N}(\sqrt{z\frac{\lambda(1-\lambda)}{\tau\phi^2(x_0)+\lambda(1-\lambda)}}\frac{c^*_{\lambda(m)}}{c^*_\lambda}(m+1)x),\tau\geq 0, \quad (22)$$

uniformly with respect to x over any finite interval of z (the continuity of the limit in x, implies that the convergence is uniform). The remaining part of the proof of the theorem, under the condition $\infty>\tau\geq 0$, follows now by using the relations (17) and (22) exactly as the proof of Theorem 2.1, under the same condition (i.e., $r_n\log n\xrightarrow[n]{}\tau\geq 0$).

Turning now to the proof of the condition (A) with $(C_2^\star)$, by using (1), we get $\frac{X(s_{nz},nz,m,k)-x_0\sqrt{1-r_{nz}}}{\sqrt{r_n}}=\sqrt{\frac{r_{nz}}{r_n}}Y_0+S_{nz}^{(m,k)}$, where $|S_{nz}^{(m,k)}|=\frac{\sqrt{1-r_{nz}}}{\sqrt{r_n}}|Y(s_{nz},nz,m,k)-x_0|\leq\frac{|Y(s_{nz},nz,m,k)-x_0|}{\sqrt{r_n}}$. Thus,

$$P(|S_{nz}^{(m,k)}|\geq\varepsilon)\leq P\left(\frac{|Y(s_{nz},nz,m,k)-x_0|}{\sqrt{r_n}}\geq\varepsilon\right)$$

$$=P\left(\frac{|Y(s_{nz},nz,m,k)-x_0|}{c_{nz}}\geq\frac{\sqrt{nzr_n}\phi(x_0)}{\sqrt{\lambda(1-\lambda)}}\varepsilon\right)\xrightarrow[n]{w}0.$$

Lemma 2.2.1 in [13] thus yields $M_{s_{nz}:nz}^{(m,k)}(x_0+c_nx)\xrightarrow[n]{w}\mathscr{N}(\frac{c^*_{\lambda(m)}}{c^*_\lambda}(m+1)x)$. The derivation of the limit df's of central m-dgos of sGs (2) is proceeded exactly as the same as those of central m-gos of the sequence (1). The remaining part of the proof of the theorem, under the condition $\tau=\infty$, follows now by using the relations (17) and the last relation (i.e., $M_{s_{nz}:nz}^{(m,k)}(x_0+c_nx)\xrightarrow[n]{w}\mathscr{N}(\frac{c^*_{\lambda(m)}}{c^*_\lambda}(m+1)x)$. uniformly with respect to x) exactly as the proof of Theorem 2.1, under the condition $r_n\log n\xrightarrow[n]{}\infty$.

Turning now to prove the converse part that $(B_1^\star)$ and $(C_1^\star)$ imply (A). Starting with the relation (17), by the compactness of df's, we can select a subsequence $n^\star$, such that $A_{n^\star}(n^\star z)\xrightarrow[n^\star]{w}A^\star(z)$, where $A^\star(z)$ is an extended df (i.e., $A^\star(\infty)-A^\star(0)\leq 1$). Therefore, by repeating the first part of theorem for the subsequence $n^\star$, with the exception that the point c is chosen such that $A^\star(\infty)-A^\star(c)\leq\varepsilon$, we get $M_{s_{\nu_{n^\star}}:\nu_{n^\star}}^{(m,k)}(x_0+c_{n^\star}x)\xrightarrow[n^\star]{w}\Phi(x)=\int_0^\infty L(x;\tau,z)dA^\star(z)$. Since, the two limits $\Phi(x)$ and $L(x;\tau,z)$ are df's, then $\Phi(\infty)=1=\int_0^\infty dA^\star(z)=A^\star(\infty)-A^\star(0)$. Thus, $A^\star$ is a df. Now, if $A_n(nz)$ did not converge weakly, then we could select two subsequences n_1 and n_2 such that $A_{n_i}(zn_i)\xrightarrow[n_1]{w}A_i(z),vi=1,2$. In this case, we have

$$\Phi(x)=\int_0^\infty\mathscr{N}\left(\sigma\sqrt{z}x\right)dA_1(z)$$

$$=\int_0^\infty\mathscr{N}\left(\sigma\sqrt{z}x\right)dA_2(z), \quad (23)$$

where $\sigma=\sqrt{\frac{\lambda(1-\lambda)}{\tau\phi^2(x_0)+\lambda(1-\lambda)}}\frac{c^*_{\lambda(m)}}{c^*_\lambda}(m+1)$. Let $G_i(t)=\int_0^\infty\mathscr{N}(t\sqrt{z})dA_i(z),i=1,2$. If the functions $G_1(t)$ and $G_2(t)$ are determined in an interval $t_1<t<t_2$, then in this interval both of them will be analytic. By differentiating $G_1(t)$ and $G_2(t)$ with respect to t, in view of (23), we get $\int_0^\infty e^{-\frac{\sigma zt^2}{2}}\sqrt{z}dA_1(z)=\int_0^\infty e^{-\frac{\sigma zt^2}{2}}\sqrt{z}A_2(z)$. Put $\sigma^\star=\frac{\sigma t^2}{2}$, we get $\int_0^\infty e^{-\sigma^\star z}\sqrt{z}dA_1(z)=\int_0^\infty e^{-\sigma^\star z}\sqrt{z}\,dA_2(z)$. Since, the Laplace transformations with respect to the measures $\sqrt{z}A_1(z)$ and $\sqrt{z}A_2(z)$ coincide, we deduce that $A_1(z)=A_2(z)$. This completes the proof of the theorem.□

The derivation of the limit df's of central m-dgos of Gaussian sequence (2) is proceeded exactly as the same as those of central m-gos of the sequence (1).

4 Intermediate *m*-gos (dgos) with random indices in a sGs

In this section we consider a general nondecreasing intermediate rank sequence $s_n=\circ(n)\xrightarrow[n]{}0$, for which $\frac{\log s_n}{\log n}\xrightarrow[n]{}\beta,\ 0\leq\beta\leq 1$. Actually, the latter condition is very wide, e.g., it is easily to verify that this condition is satisfied, with $\beta=\alpha$, when $s_n\sim\ell^2n^\alpha,\ 0<\alpha<1$ (the Chibisov rank sequence, see [10]). Also, this condition will be satisfied, with $\beta=0$, when $s_n=\log n$ (i.e., when the rank sequence is slowly varying function of n). Finally, it will be satisfied, with $\beta=1$, when $s_n=\frac{n}{\log n}$ (i.e., when the rank sequence is rapidly varying function of n). The following results is indispensable for the study the intermediate m-gos $X(s^\star(n),n,m,k)$, and m-dgos $X_d(s_n,n,m,k)$, where $s^\star(n)=n-s_n+1$.

Lemma 4.1. Let $Y_1,Y_2,...,Y_n$, be iid rv's, with common df $\mathcal{N}(x)$. Furthermore, let $Y(1,n,m,k)\le Y(2,n,m,k)\le ... \le Y(n,n,m,k)$ be the corresponding m-gos. Then, $P(\frac{Y(s^\star(n),n,m,k)-x_{\omega_n}}{d_{\omega_n}}<x)\xrightarrow[n]{w}\mathcal{N}((m+1)x)$, where $d_n=\frac{1}{x_n\sqrt{s_n}}$, $1-\mathcal{N}(x_n)=\frac{s_n}{n}$, $x_n\sim\sqrt{2\log\frac{n}{s_n}}$, as $n\to\infty$, and $\omega_n=s_N(\frac{s_N}{n})^{\frac{-1}{m+1}}$ (remember that $N=\frac{k}{m+1}+n-1\sim n$, thus $\omega_n=\circ(n)\xrightarrow[n]{}\infty$).

Lemma 4.2. Let $Y_d(1,n,m,k)\ge Y_d(2,n,m,k)\ge ... \ge Y_d(n,n,m,k)$ be m-dgos corresponding to the rv's $Y_1,Y_2,...,Y_n$. Then, $P(\frac{Y_d(s_n,n,m,k)-x_{\sigma_n}}{d_{\sigma_n}}<x)\xrightarrow[n]{w}\mathcal{N}(x)$, where $d_n=-\frac{1}{x_n\sqrt{s_n}}$, $\mathcal{N}(x_n)=\frac{s_n}{n}$, $x_n\sim-\sqrt{2\log\frac{n}{s_n}}$, as $n\to\infty$, and $\sigma_n=(m+1)N$.

Theorem 4.1. Let the condition (A) in the Theorems 2.1-2.6 satisfied. Then,

($B_1^{\star\star}$): $P(\frac{X(s^\star(\nu_n),\nu_n,m,k)-x_{\omega_{\nu_n}}}{d_{\omega_n}}<x)\xrightarrow[n]{w}\mathcal{N}(\frac{(m+1)z^{\frac{\eta}{2}}}{\sqrt{1+2\nu\tau z^{\eta}(m+1)(m\beta+1)(1-\beta)}}x)$, $0\le\nu,\eta<\infty$, if

($C_1^{\star\star}$): $r_ns_n\log n\xrightarrow[n]{}\tau\ge0$, $\frac{s_{\omega_n}}{s_n}\xrightarrow[n]{}\nu$ and $\frac{s_{\omega_{nz}}}{s_{\omega_n}}\xrightarrow[n]{}z^{\eta}$. Moreover,

($B_2^{\star\star}$): $P(\frac{X(s^\star(\nu_n),\nu_n,m,k)-x_{\omega_{\nu_n}}\sqrt{1-r_{\nu_n}}}{\sqrt{r_n}}<x)\xrightarrow[n]{w}\mathcal{N}(x)$, if

($C_2^{\star\star}$): $r_ns_n\log n\xrightarrow[n]{}\infty$ and r_n is slowly varying function of n.

Remark 4.1. Actually, the condition $\frac{s_{\omega_n}}{s_n}\xrightarrow[n]{}\nu$ is not restrictive for being that the sequence $\frac{s_{\omega_n}}{s_n}$ is bounded, as $n\to\infty$. This fact can be easily seen, since s_n is nondecreasing and $\omega_n=\circ(n)\le n$, for large n. Moreover, it is easily to see that $\nu=0$, for the two cases $s_n\sim\ell^2n^{\alpha}$ and $s_n\sim\frac{n}{\log n}$. Moreover, $\nu=\frac{1}{m+1}$, if $s_n\sim\log n$.

Remark 4.2. It is easily to see that $\eta=\frac{\alpha}{m+1}$, for $s_n\sim\ell^2n^{\alpha}$ and $\eta=\frac{1}{m+1}$, for $s_n\sim\frac{n}{\log n}$. On the other hand, $\eta=0$, when $s_n\sim\log n$.

Proof. Proceeding exactly as the proof of (4), we get

$$M^{(m,k)}_{s^\star_{\nu_n},\nu_n}(x_{\nu_n}+b_nx)=P(X(s^\star(\nu_n),\nu_n,m,k)<x_{\omega_{\nu_n}}+d_{\omega_n}x)$$

$$=\int_0^{\infty}M^{(m,k)}_{s^\star(zn):nz}(x;z,\tau)dA_n(nz), \tag{24}$$

where $M^{(m,k)}_{s^\star_{zn}:nz}(x;z,\tau)=P(X(s^\star(nz),nz,m,k)<x_{\omega_{nz}}+d_{\omega_n}x)$. First, consider the condition (A) with ($C_1^{\star\star}$), by using (1), we get

$$\frac{X(s^\star(nz),nz,m,k)-x_{\omega_{nz}}}{d_{\omega_n}}=\frac{\sqrt{r_{nz}}}{d_{\omega_n}}Y_0$$

$$+\frac{\sqrt{1-r_{nz}}Y(s^\star(nz),nz,m,k)-x_{\omega_{nz}}}{d_{\omega_n}}=U^{(m,k)}_{nz}+V^{(m,k)}_{nz}. \tag{25}$$

Moreover, $U^{(m,k)}_{nz}$ and $V^{(m,k)}_{nz}$ are independent. Then, if $s_nr_n\log n\xrightarrow[n]{}\tau$, $0\le\tau<\infty$, and $\frac{\log s_n}{\log n}\xrightarrow[n]{}\beta, 0\le\beta\le1$, it is easily to check that $s_nr_n\log\omega_n\xrightarrow[n]{}\frac{\tau}{m+1}(m\beta+1)$. Consequently, we get

$$x_{\omega_n}\sqrt{s_{\omega_n}r_{nz}}\sim\sqrt{2s_nr_n\log\omega_n}\sqrt{\frac{r_{nz}}{r_n}\frac{s_{\omega_n}}{s_n}}\sqrt{1-\frac{\log s_{\omega_n}}{\log\omega_n}}$$

$$\xrightarrow[n]{}\sqrt{\frac{2\nu\tau}{m+1}(m\beta+1)(1-\beta)}.$$

Thus,

$$\left.\begin{array}{l}P(U^{(m,k)}_{nz}<x)\xrightarrow[n]{w}\mathcal{N}(\sqrt{\frac{m+1}{2\nu\tau(m\beta+1)(1-\beta)}}x),\\ \qquad\text{if }\tau>0,\beta\neq1,\nu>0,\\ U^{(m,k)}_{nz}\xrightarrow[n]{p}0,\text{ if }\tau=0,\text{ or }\beta=1,\text{ or }\nu=0.\end{array}\right\} \tag{26}$$

On the other hand, we have

$$P(V^{(m,k)}_{nz}<x)=P(Y(s^\star(nz),nz,m,k)<A_{\omega_{nz}}+B_{\omega_{nz}}x), \tag{27}$$

where $A_{\omega_{nz}}=\frac{x_{\omega_{nz}}}{\sqrt{1-r_{nz}}}$ and $B_{\omega_{nz}}=\frac{d_{\omega_n}}{\sqrt{1-r_{nz}}}$. It is clear that $\frac{B_{\omega_{nz}}}{d_{\omega_{nz}}}=\frac{(1-r_{nz})^{-\frac{1}{2}}d_{\omega_n}}{d_{\omega_{nz}}}=(1+\frac{r_{nz}}{2}(1+o(1)))\frac{x_{\omega_{nz}}}{x_{\omega_n}}\sqrt{\frac{s_{\omega_{nz}}}{s_{\omega_n}}}\xrightarrow[n]{}(\sqrt{z})^{\eta}$, for $r_{nz}\xrightarrow[n]{}0$ and $\frac{x_{\omega_{nz}}}{x_{\omega_n}}\xrightarrow[n]{}1$. Moreover, $\frac{A_{\omega_{nz}}-x_{\omega_{nz}}}{d_{\omega_{nz}}}\sim\frac{1}{2}(x_{\omega_{nz}}\sqrt{s_{nz}r_{nz}})(\sqrt{\frac{s_{\omega_{nz}}}{s_{nz}}})(\sqrt{r_{nz}}x_{\omega_{nz}})\xrightarrow[n]{}0$, since $\sqrt{r_{nz}}x_{\omega_{nz}}\xrightarrow[n]{}0$ and $\frac{s_{\omega_{nz}}}{s_{nz}}\le1$, for large n. Therefore, in view of Lemma 4.1 and by applying Khinchin's type theorem, we get

$$P(V^{(m,k)}_{n}<x)\xrightarrow[n]{w}\mathcal{N}((m+1)z^{\frac{\eta}{2}}x). \tag{28}$$

Combining (25),(26), (27) and (28), we get ($B_1^{\star\star}$).

Turning now to the proof of the condition (A) with ($C_2^{\star\star}$), by using (1), we get $\frac{X(s^\star(nz),nz,m,k)-x_{\omega_{nz}}\sqrt{1-r_{nz}}}{\sqrt{r_n}}=\sqrt{\frac{r_{nz}}{r_n}}Y_0+T^{(m,k)}_{nz}$, where $|T^{(m,k)}_{nz}|=\frac{\sqrt{1-r_{nz}}}{\sqrt{r_n}}|Y(s^\star(nz),nz,m,k)-x_{\omega_{nz}}|\le\frac{|Y(s^\star(nz),nz,m,k)-x_{\omega_{nz}}|}{\sqrt{r_n}}$ and $\sqrt{\frac{r_{nz}}{r_n}}\xrightarrow[n]{}1$. Thus,

$$P(|T^{(m,k)}_{nz}|\ge\varepsilon)\le P\left(\frac{|Y(s^\star(nz),nz,m,k)-x_{\omega_{nz}}|}{\sqrt{r_n}}\ge\varepsilon\right)$$

$$=P\left(\frac{|Y(s^\star(nz),nz,m,k)-x_{\omega_{nz}}|}{d_{\omega_{nz}}}\right.$$

$$\left.\ge\sqrt{s_{nz}r_{nz}}x_{\omega_{nz}}\sqrt{\frac{s_{\omega_{nz}}}{s_{nz}}}\sqrt{\frac{r_n}{r_{nz}}}\varepsilon\right)\xrightarrow[n]{}0.$$

Therefore, Lemma 2.2.1 in [13] implies ($B_2^{\star\star}$) The remaining part of the proof of the theorem, under the condition $\tau=\infty$, follows now by using the relations (24) and the relation $M^{(m,k)}_{s^\star_{nz}:nz}(x_{\omega_{nz}}+d_{\omega_n}x)\xrightarrow[n]{w}\mathcal{N}(x)$ exactly as the proof of Theorem 2.1, under the condition $s_n r_n \log n \xrightarrow[n]{} \infty$. This completes the proof of the theorem.□

Theorem 4.2. Let the condition (A) in Theorems 2.1-2.6 satisfied. Then,

($B_1^{\star\star}$): $P\left(\frac{X_d(s(\nu_n),\nu_n,m,k)-x_{\sigma\nu_n}}{d_{\sigma_n}} < x\right) \xrightarrow[n]{w} \mathcal{N}\left(\frac{z^{\frac{\xi}{2}}}{\sqrt{1+2\mu\tau z^{\xi}(1-\beta)}}x\right)$, $0 \le \mu, \xi < \infty$, if

($C_1^{\star\star}$): $r_n s_n \log n \xrightarrow[n]{} \tau \ge 0$, $\frac{s_{\sigma_n}}{s_n} \xrightarrow[n]{} \mu$ and $\frac{s_{\sigma_{nz}}}{s_{\sigma_n}} \xrightarrow[n]{} z^{\xi}$. Moreover,

($B_2^{\star\star}$): $P\left(\frac{X_d(s(\nu_n),\nu_n,m,k)-x_{\sigma\nu_n}\sqrt{1-r_{\nu_n}}}{\sqrt{r_n}} < x\right) \xrightarrow[n]{w} \mathcal{N}(x)$, if

($C_2^{\star\star}$): $r_n s_n \log n \xrightarrow[n]{} \infty$ and r_n is slowly varying function of n.

Proof. The proof of theorem follows along the same line of the proof of Theorem 4.1, by combining the result of [9] with the result of Lemma 3.1 of [6]. Note that in this case $\mu = (m+1)^{\alpha}$ and $\xi = \alpha$ when $s_n \sim \ell^2 n^{\alpha}$, while $\mu = m+1$ and $\xi = 1$ when $s_n \sim \frac{n}{\log n}$. Moreover, $\mu = 1$ and $\xi = 0$ when $s_n \sim \log n$.□

5 Conclusion

Often the realizations of experiments in reliability analysis and lifetime studies are stationary and arise in nondecreasing (or nonincreasing) order, where in such experiments it is almost impossible to have a fixed sample size, because some observations always get lost for various reasons. Hence, the need arises to establish the limiting distributions of several models of ascendingly (or descendingly) ordered stationary random variables with random number. In this paper we obtained the limit distributions of extreme, intermediate and central generalized order statistics, as well as dual generalized order statistics, of a stationary Gaussian sequence of random variables under equi-correlated set up, when the sample size is itself a random variable.

Acknowledgement

The authors would like to thank the anonymous referees and the editor-in-chief for their efforts and the constructive suggestions that improved the representation substantially.

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