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On Commutator and Power Subgroups of Some Coxeter Groups

Firat Ates [1], *I. Naci Cangul* [2,*], *Esra K. Cetinalp* [3], *A. Sinan Cevik* [4] *and Eylem Guzel Karpuz* [3]

[1] Faculty of Art and Science, Department of Mathematics, Balikesir University, 10100, Balikesir, Turkey
[2] Faculty of Art and Sciences, Department of Mathematics, Uludag University, 16059, Bursa-Turkey
[3] Kamil Özdag Science Faculty, Department of Mathematics, Karamanoglu Mehmetbey University, 70100, Karaman-Turkey
[4] Faculty of Science, Department of Mathematics, Selçuk University, 42075, Konya-Turkey

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Abstract: In this paper the commutator subgroups of the affine Weyl group of type $\widetilde{\mathbf{C}}_{n-1}$ $(n \geq 3)$ and the triangle Coxeter groups are studied. Also it is given all power subgroups of the affine Weyl group of type $\widetilde{\mathbf{A}}_{n-1}$ $(n \geq 3)$. We should note that, as in our knowledge, although the concept of this study seems in pure mathematics, it is known that affine Weyl groups have a direct relationship between discrete dynamical systems and Painlevé equations (cf. [16]).

Keywords: Coxeter group, commutator subgroup, power subgroup, Reidemeister-Schreier.

1 Introduction

A Coxeter group, named after H. S. M. Coxeter, is an abstract group that admits a formal description in terms of mirror symmetries. Coxeter groups were introduced in [8] as abstractions of reflection groups. These groups find applications in many areas of mathematics. Examples of finite Coxeter groups include the symmetry groups of regular polytopes, and Weyl groups of simple Lie algebras. The triangular groups corresponding to regular tessellations of the Euclidean plane and the hyperbolic plane, and the Weyl groups of infinite-dimensional Kac-Moody algebras can be given as examples of infinite Coxeter groups ([9]). Also it has been interested to obtain some solutions for the decision problems in Coxeter groups (cf. [13]).

In this paper we are interested in the affine Weyl groups of type $\widetilde{\mathbf{A}}_{n-1}$, $\widetilde{\mathbf{C}}_{n-1}$ $(n \geq 3)$ and the triangle Coxeter group. These groups have been studied extensively for many aspects in the literature. Affine Weyl groups, in particular, play a crucial role in the study of compact Lie groups ([4,5]). But, in here, we concern with these groups from the point of abstract group structure and find commutator subgroups of them and power subgroups of $\widetilde{\mathbf{A}}_{n-1}$. To obtain this kind of subgroups we use the Reidemeister-Schreier method (for more detail about this method, see [14]). This subgroups have been studied in detailed in [6,11,12] and [17] for Hecke and extended Hecke groups which are special Coxeter groups.

The *commutator subgroup* of a group G is denoted by G' and defined by $< [g,h]\ ;\ g,h \in G >$, where $[g,h] = ghg^{-1}h^{-1}$. Since G' is a normal subgroup of G, we can form the factor-group G/G' which is the smallest abelian quotient group of G. Now let k be a positive integer. Let us define G^k to be the subgroup generated by the k^{th} powers of all elements of the group G. So the group G^k is called the k^{th} *power subgroup* of G. As fully invariant subgroups, they are normal in G. In [18], the authors studied the commutator and the power subgroups of Hecke groups. Actually, our results in here can be thought as the generalization of the theories in reference [18].

At some part of the rest of this paper, for a good useage of space of the text, we will not use the notation $<>$ to define a presentation of related structers. We will give our results in seperate sections under the name of Affine Weyl group of type $\widetilde{\mathbf{A}}_{n-1}$, Affine Weyl group of type $\widetilde{\mathbf{C}}_{n-1}$ and Triangle Coxeter group.

* Corresponding author e-mail: cangul@uludag.edu.tr

2 Affine Weyl group of type $\widetilde{\mathbf{A}}_{n-1}$

The affine Weyl group of type $\widetilde{\mathbf{A}}_{n-1}$ $(n \geq 3)$ is actually an *irreducible Coxeter group* which the Coxeter graph is a polygon with n vertices [15]. A presentation (let us label it by **(1)**) for $\widetilde{\mathbf{A}}_{n-1}$ is defined by the generators $a_1, a_2, \cdots, a_n$ and the relators

$$a_i^2 = 1 \quad (1 \leq i \leq n),$$

$$(a_i a_{i+1})^3 = 1 \quad (1 \leq i \leq n-1), \quad (a_1 a_n)^3 = 1,$$

and

$$(a_i a_j)^2 = 1 \quad (1 \leq i < j-1 < n,\ (i,j) \neq (1,n)),$$

In [1], Albar showed that

$$\widetilde{\mathbf{A}}_{n-1} \cong \mathbb{Z}^{n-1} \rtimes S_n,$$

where S is the symmetric group of degree n. Then in [2], Albar et al. proved how $\widetilde{\mathbf{A}}_{n-1}$ appears naturally as a subgroup of the natural wreath product $W = \mathbb{Z}S_n$. Again in [1], the author pointed out the isomorphism

$$\widetilde{\mathbf{A}}_{n-1} / \widetilde{\mathbf{A}}'_{n-1} = < a_1; a_1^2 = 1 > \cong C_2$$

and so the index $\left| \widetilde{\mathbf{A}}_{n-1} : \widetilde{\mathbf{A}}'_{n-1} \right| = 2$. By taking $\{1, a_1\}$ is a Schreier transversal for $\widetilde{\mathbf{A}}'_{n-1}$ and then applying the Reidemeister-Schreier process, the following result is obtained.

Theorem 1.[1] *The commutator subgroup of the affine Weyl group $\widetilde{\mathbf{A}}_{n-1}$ $(n \geq 3)$, say $\widetilde{\mathbf{A}}'_{n-1}$, is presented by*

$$\begin{aligned} &< b_1, b_2, \cdots, b_{n-1}; b_1^3 = b_i^2 = b_{n-1}^3 \,(1 \leq i \leq n-2), \\ &\qquad (b_i b_{i+1}^{-1})^3 = 1 \,(1 \leq i \leq n-2), \\ &\qquad (b_i b_j^{-1})^2 = 1 \,(1 \leq i \leq j-1 < n-1) > . \end{aligned}$$

As a generalization of this above result, we will find a presentation for the quotient $\widetilde{\mathbf{A}}_{n-1} / \widetilde{\mathbf{A}}^t_{n-1}$ $(t \in \mathbb{Z}^+)$ by adding relations $R^t = 1$ to the presentation of $\widetilde{\mathbf{A}}_{n-1}$ given in **(1)** for all relations R in $\widetilde{\mathbf{A}}_{n-1}$.

Theorem 2.*Let $\widetilde{\mathbf{A}}^t_{n-1}$ $(n \geq 3)$ be the power subgroup of $\widetilde{\mathbf{A}}_{n-1}$. Then*

$$\widetilde{\mathbf{A}}^t_{n-1} = \begin{cases} \{1\} & ;\ t = 6s_1,\ s_1 \geq 1, \\ \widetilde{\mathbf{A}}'_{n-1}; & t = 6s_2 + 2 \text{ or } t = 6s_2 + 4,\ s_2 \geq 0, \\ \widetilde{\mathbf{A}}_{n-1}; & \text{otherwise}. \end{cases}$$

*Proof.*Let us first assume that $t = 6s_1$, where $s_1 \geq 1$. Then, for the group $\widetilde{\mathbf{A}}_{n-1} / \widetilde{\mathbf{A}}^{6s_1}_{n-1}$, we get the generators $a_1, a_2, \cdots, a_n$ while the relators

$$\begin{aligned} a_i^2 &= 1 \,(1 \leq i \leq n), \\ (a_i a_{i+1})^3 = 1 \,(1 \leq i \leq n-1), \quad (a_1 a_n)^3 &= 1, \\ (a_i a_j)^2 = 1 \,(1 \leq i < j-1 < n \quad \text{and} \quad (i,j) &\neq (1,n)), \\ a_i^{6s_1} &= 1 \,(1 \leq i \leq n), \\ (a_i a_{i+1})^{6s_1} &= 1 \,(1 \leq i \leq n-1) \end{aligned}$$

On account of the power of relations in **(1)**, it is easily seen that $\widetilde{\mathbf{A}}_{n-1} / \widetilde{\mathbf{A}}^{6s_1}_{n-1} = \widetilde{\mathbf{A}}_{n-1}$ and thus $\widetilde{\mathbf{A}}^{6s_1}_{n-1} = \{1\}$.

Now assume $t = 6s_2 + 2$, $s_2 \geq 0$ and consider the following presentation for the group $\widetilde{\mathbf{A}}_{n-1} / \widetilde{\mathbf{A}}^{6s_2+2}_{n-1}$. As previously the generators are $a_1, a_2, \cdots, a_n$ while the relators are

$$\begin{aligned} (a_i a_{i+1})^3 = 1 \,(1 \leq i \leq n-1), \quad (a_1 a_n)^3 &= 1, \\ (a_i a_j)^2 = 1 \,(1 \leq i < j-1 < n,\ (i,j) &\neq (1,n)), \\ a_i^{6s_2+2} &= 1 \,(1 \leq i \leq n), \\ (a_i a_{i+1})^{6s_2+2} &= 1 \,(1 \leq i \leq n-1). \end{aligned}$$

Since $(a_i a_{i+1})^{6s_2+2} = (a_i a_{i+1})^3 = 1$ for all $1 \leq i \leq n-1$, we have $a_i = a_{i+1}$ $(1 \leq i \leq n-1)$. Hence we get

$$\widetilde{\mathbf{A}}_{n-1} / \widetilde{\mathbf{A}}^{6s_2+2}_{n-1} = < a_1; a_1^2 = 1 > \cong C_2.$$

So by considering Theorem 1, we deduce that $\widetilde{\mathbf{A}}^{6s_2+2}_{n-1} = \widetilde{\mathbf{A}}'_{n-1}$. Similarly, one can apply same progress for $t = 6s_2 + 4$, $s_2 \geq 0$, and so obtain $\widetilde{\mathbf{A}}^{6s_2+4}_{n-1} = \widetilde{\mathbf{A}}'_{n-1}$.

Until now we have investigated even power subgroups of $\widetilde{\mathbf{A}}_{n-1}$. On the other hand the odd power subgroups of $\widetilde{\mathbf{A}}_{n-1}$ can be classified as in the following.

Let $t = 2s_3 + 1$, $s_3 \geq 1$. With respect to this case, we obtain the following presentation (having generators $a_1, a_2, \cdots, a_n$) for the group $\widetilde{\mathbf{A}}_{n-1} / \widetilde{\mathbf{A}}^{2s_3+1}_{n-1}$:

$$\begin{aligned} a_i^2 &= 1 \,(1 \leq i \leq n), \\ (a_i a_{i+1})^3 = 1 \,(1 \leq i \leq n-1), \quad (a_1 a_n)^3 &= 1, \\ (a_i a_j)^2 = 1 \,(1 \leq i < j-1 < n,\ (i,j) &\neq (1,n)), \\ a_i^{2s_3+1} &= 1 \,(1 \leq i \leq n), \\ (a_i a_{i+1})^{2s_3+1} &= 1 \,(1 \leq i \leq n-1). \end{aligned}$$

Since $a_i^{2s_3+1} = a_i^2 = 1$, for all $1 \leq i \leq n$, we clearly have $a_i = 1$. Hence we obtain $\widetilde{\mathbf{A}}_{n-1} / \widetilde{\mathbf{A}}^{2s_3+1}_{n-1} = \{1\}$ and so $\widetilde{\mathbf{A}}^{2s_3+1}_{n-1} = \widetilde{\mathbf{A}}_{n-1}$.

Hence the result.

3 Affine Weyl group of type $\widetilde{\mathbf{C}}_{n-1}$

The affine Weyl group of type $\widetilde{\mathbf{C}}_{n-1}$ $(n \geq 3)$ is another infinite *irreducible Coxeter group* and, according to the [3], it has the following presentation:

$$\begin{aligned} \widetilde{\mathbf{C}}_{n-1} = &< y_1, y_2, \ \cdots, y_n; y_i^2 = 1 \,(1 \leq i \leq n), \\ &\quad (y_i y_j)^2 = 1 \,(1 \leq i < j-1 \leq n-1), \\ &\quad (y_i y_{i+1})^3 = 1 \,(2 \leq i \leq n-1), \\ &\quad (y_1 y_2)^4 = (y_{n-1} y_n)^4 = 1 > . \end{aligned}$$

Let us label this above presentation by **(2)**.

A simple calculation shows that $\widetilde{\mathbf{C}}_2$ is the triangle group $\nabla(2,4,4)$ which is one of the Euclidean triangle groups. In [3], the authors proved that

$$\widetilde{\mathbf{C}}_{n-1} \cong D^{n-1}_{\mathscr{I}} \rtimes S_{n-1},$$

where $\mathscr{I}$ denotes the infinity, $D_{\mathscr{I}}$ is the infinite dihedral group and S_{n-1} is the symmetric group of degree $n-1$.

The main result of this section is as follows:

Theorem 3. *The commutator subgroup of the affine Weyl group $\widetilde{\mathbf{C}}_{n-1}$ $(n \geq 3)$, say $\widetilde{\mathbf{C}}'_{n-1}$, is the free product of four cyclic groups of order 2. In other words,*

$$\widetilde{\mathbf{C}}'_{n-1} = C_2 * C_2 * C_2 * C_2.$$

Proof. We adjoin the commutator relations $y_k y_l = y_l y_k$ $(1 \leq k < l \leq n)$ to the presentation **(2)**. This gives us a presentation for $\widetilde{\mathbf{C}}_{n-1}/\widetilde{\mathbf{C}}'_{n-1}$ of which order gives the index. Then we have

$$\begin{aligned}\widetilde{\mathbf{C}}_{n-1}/\widetilde{\mathbf{C}}'_{n-1} = <y_1, y_2, \ \cdots, y_n\,;\, & y_i^2 = 1\,(1 \leq i \leq n),\\ & (y_i y_{i+1})^3 = 1\,(2 \leq i \leq n-1),\\ & (y_i y_j)^2 = 1\,(1 \leq i < j-1 < n),\\ & (y_1 y_2)^4 = (y_{n-1} y_n)^4 = 1,\\ & (y_k y_l)^2 = 1\,(1 \leq k < l \leq n) > .\end{aligned}$$

Since $(y_i y_{i+1})^3 = 1$ $(2 \leq i \leq n-1)$ and $(y_k y_l)^2 = 1$ $(1 \leq k < l \leq n)$ we have $(y_i y_{i+1})^3 = (y_i y_{i+1})^2 = 1$ for $2 \leq i \leq n-1$. This implies that $y_i = y_{i+1}$ $(2 \leq i \leq n-1)$. Therefore

$$\widetilde{\mathbf{C}}_{n-1}/\widetilde{\mathbf{C}}'_{n-1} = <y_1, y_2; y_1^2, y_2^2, (y_1 y_2)^2> \cong C_2 \times C_2.$$

Thus $\left|\widetilde{\mathbf{C}}_{n-1} : \widetilde{\mathbf{C}}'_{n-1}\right| = 4$. Let $\{1, y_1, y_2, y_1 y_2\}$ be a Schreier transversal for $\widetilde{\mathbf{C}}'_{n-1}$. Applying the Reidemeister-Schreier process we obtain all possible products as follows:

$$\begin{aligned}
&S_{1y_1} = y_1.y_1 = 1, && S_{y_1y_1} = y_1^2.y_1^2 = 1,\\
&S_{1y_2} = y_2.y_2 = 1, && S_{y_1y_2} = y_1y_2.y_2y_1 = 1,\\
&S_{1y_i} = y_i.1 = y_i, && S_{y_1y_i} = y_1y_iy_1,\\
&S_{y_2y_1} = y_2y_1.y_1y_2 = 1, && S_{y_1y_2y_1} = y_1y_2y_1.y_1y_2y_1 = 1,\\
&S_{y_2y_2} = y_2^2.y_2^2 = 1, && S_{y_1y_2y_2} = y_1y_2^2.y_2^2y_1 = 1,\\
&S_{y_2y_i} = y_2y_iy_2, && S_{y_1y_2y_i} = y_1y_2y_iy_2y_1,
\end{aligned}$$

where $3 \leq i \leq n$. For convenience, let us label the generators obtained in above as in the following:

$$\begin{aligned}
&y_3 = x_1, \quad y_4 = x_2, \cdots, y_n = x_{n-2},\\
&y_1y_3y_1 = z_1, \quad y_1y_4y_1 = z_2, \cdots, y_1y_ny_1 = z_{n-2},\\
&y_2y_3y_2 = t_1, \quad y_2y_4y_2 = t_2, \cdots, y_2y_ny_2 = t_{n-2},\\
&y_1y_2y_3y_2y_1 = m_1, \quad y_1y_2y_4y_2y_1 = m_2, \cdots\\
&\cdots, y_1y_2y_ny_2y_1 = m_{n-2}.
\end{aligned}$$

Then by using Reidemeister rewriting process we get the defining relations as follows:

$$\begin{aligned}
\tau(y_iy_i) &= S_{1y_i}S_{1y_i} = y_i^2 = x_{i-2}^2 \ (3 \leq i \leq n),\\
\tau(y_iy_{i+1}y_iy_{i+1}y_iy_{i+1}) &= S_{1y_i}S_{1y_{i+1}}S_{1y_i}S_{1y_{i+1}}S_{1y_i}S_{1y_{i+1}}\\
&= (y_iy_{i+1})^3 = (x_{i-2}x_{i-1})^3\\
&\quad (3 \leq i \leq n-1),\\
\tau(y_iy_jy_iy_j) &= S_{1y_i}S_{1y_j}S_{1y_i}S_{1y_j}\\
&= (y_iy_j)^2 = (x_{i-2}x_{j-2})^2\\
&\quad (3 \leq i < j-1 \leq n-1),
\end{aligned}$$

$$\begin{aligned}
\tau(y_{n-1}y_ny_{n-1}y_ny_{n-1}y_ny_{n-1}y_n) &= S_{1y_{n-1}}S_{1y_n}S_{1y_{n-1}}S_{1y_n}\\
&\quad S_{1y_{n-1}}S_{1y_n}S_{1y_{n-1}}S_{1y_n}\\
&= (y_{n-1}y_n)^4 = (x_{n-3}x_{n-2})^4,
\end{aligned}$$

$$\tau(y_1y_iy_iy_1) = S_{y_1y_i}S_{y_1y_i}S_{y_1y_1} = (y_1y_iy_1)^2 = z_{i-2}^2 \ (3 \leq i \leq n),$$

$$\begin{aligned}
\tau(y_1y_iy_{i+1}y_iy_{i+1}y_iy_{i+1}y_1) &= S_{y_1y_i}S_{y_1y_{i+1}}S_{y_1y_i}\\
&\quad S_{y_1y_{i+1}}S_{y_1y_i}S_{y_1y_{i+1}}S_{y_1y_1}\\
&= (y_1y_iy_1.y_1y_{i+1}y_1)^3\\
&= (z_{i-2}z_{i-1})^3\\
&\quad (3 \leq i < j-1 \leq n-1),
\end{aligned}$$

$$\begin{aligned}
\tau(y_1y_iy_jy_iy_jy_1) &= S_{y_1y_i}S_{y_1y_j}S_{y_1y_i}S_{y_1y_j}S_{y_1y_1}\\
&= (y_1y_iy_1.y_1y_jy_1)^2\\
&= (z_{i-2}z_{j-2})^2 \ (3 \leq i < j-1 \leq n-1),
\end{aligned}$$

$$\begin{aligned}
\tau(y_1y_{n-1}y_ny_{n-1}y_ny_{n-1}y_ny_{n-1}y_ny_1) &= S_{y_1y_{n-1}}S_{y_1y_n}S_{y_1y_{n-1}}S_{y_1y_n}\\
&\quad S_{y_1y_{n-1}}S_{y_1y_n}S_{y_1y_{n-1}}\\
&\quad S_{y_1y_n}S_{y_1y_1}\\
&= (y_1y_{n-1}y_1.y_1y_ny_1)^4\\
&= (z_{n-3}z_{n-2})^4,
\end{aligned}$$

$$\tau(y_2y_iy_iy_2) = S_{y_2y_i}S_{y_2y_i}S_{y_2y_2} = (y_2y_iy_2)^2 = t_{i-2}^2 \ (3 \leq i \leq n),$$

$$\begin{aligned}
\tau(y_2y_iy_{i+1}y_iy_{i+1}y_iy_{i+1}y_2) &= S_{y_2y_i}S_{y_2y_{i+1}}S_{y_2y_i}S_{y_2y_{i+1}}S_{y_2y_i}\\
&\quad S_{y_2y_{i+1}}S_{y_2y_2}\\
&= (y_2y_iy_2.y_2y_{i+1}y_2)^3\\
&= (t_{i-2}t_{i-1})^3\\
&\quad (3 \leq i < j-1 \leq n-1),\\
\tau(y_2y_iy_jy_iy_jy_2) &= S_{y_2y_i}S_{y_2y_j}S_{y_2y_i}S_{y_2y_j}S_{y_2y_2}\\
&= (y_2y_iy_2y_2y_jy_2)^2\\
&= (t_{i-2}t_{j-2})^2\\
&\quad (3 \leq i < j-1 \leq n-1),
\end{aligned}$$

$$\begin{aligned}
\tau(y_2y_{n-1}y_ny_{n-1}y_ny_{n-1}y_ny_{n-1}y_ny_2) &= S_{y_2y_{n-1}}S_{y_2y_n}S_{y_2y_{n-1}}S_{y_2y_n}\\
&\quad S_{y_2y_{n-1}}S_{y_2y_n}S_{y_2y_{n-1}}\\
&\quad S_{y_2y_n}S_{y_2y_2}\\
&= (y_2y_{n-1}y_2.y_2y_ny_2)^4\\
&= (t_{n-3}t_{n-2})^4,
\end{aligned}$$

$$\begin{aligned}
\tau(y_1y_2y_iy_iy_2y_1) &= S_{y_1y_2y_i}S_{y_1y_2y_i}S_{y_1y_2y_2}S_{y_1y_1}\\
&= (y_1y_2y_iy_2y_1)^2 = m_{i-2}^2 \ (3 \leq i \leq n),
\end{aligned}$$

$$\begin{aligned}
\tau(y_1y_2y_iy_{i+1}y_iy_{i+1}y_iy_{i+1}y_2y_1) &= S_{y_1y_2y_i}S_{y_1y_2y_{i+1}}S_{y_1y_2y_i}S_{y_1y_2y_{i+1}}\\
&\quad S_{y_1y_2y_i}S_{y_1y_2y_{i+1}}S_{y_1y_2y_2}S_{y_1y_1}\\
&= (y_1y_2y_iy_2.y_1y_1y_2y_{i+1}y_2y_1)^3\\
&= (m_{i-2}m_{i-1})^3\\
&\quad (3 \leq i < j-1 \leq n-1),
\end{aligned}$$

$$\begin{aligned}\tau(y_1y_2y_iy_jy_iy_jy_2y_1) &= S_{y_1y_2y_i}S_{y_1y_2y_j}S_{y_1y_2y_i} \\ &\quad S_{y_1y_2y_j}S_{y_1y_2y_2}S_{y_1y_1} \\ &= (y_1y_2y_iy_2y_1.y_1y_2y_jy_2y_1)^2 \\ &= (m_{i-2}m_{j-2})^2 \\ &\quad (3 \le i < j-1 \le n-1),\end{aligned}$$

$$\begin{aligned}\tau(y_1y_2y_{n-1}y_ny_{n-1}y_n \quad y_{n-1}y_ny_{n-1}y_ny_2y_1) &= S_{y_1y_2y_{n-1}}S_{y_1y_2y_n} \\ &\quad \cdots S_{y_1y_2y_{n-1}}S_{y_1y_2y_n} \\ &\quad S_{y_1y_2y_2}S_{y_1y_1} \\ &= (y_1y_2y_{n-1}y_2y_1.y_1y_2y_ny_2y_1)^4 \\ &= (m_{n-3}m_{n-2})^4.\end{aligned}$$

Hence we obtain the following presentation for the subgroup $\widetilde{\mathbf{C}}'_{n-1}$: The generators are

$$x_p, z_p, t_p, m_p,$$

and the relators are

$$\begin{aligned}&x_p^2 = z_p^2 = t_p^2 = m_p^2 \, (1 \le p \le n-2), \\ &(x_px_{p+1})^3 = (z_pz_{p+1})^3 = (t_pt_{p+1})^3 = \\ &(m_pm_{p+1})^3 \, (1 \le p \le n-3), \\ &(x_px_q)^2 = (z_pz_q)^2 = (t_pt_q)^2 = (m_pm_q)^2 \\ &(1 \le p < q-1 \le n-3), \\ &(x_{n-3}x_{n-2})^4 = (z_{n-3}z_{n-2})^4 = \\ &(t_{n-3}t_{n-2})^4 = (m_{n-3}m_{n-2})^4.\end{aligned}$$

Let us take $p = n-3$ for the relation $(x_px_{p+1})^3$. Then we get $(x_{n-3}x_{n-2})^4 = (x_{n-3}x_{n-2})^3$. Hence $x_{n-3}x_{n-2} = 1$ and so $x_{n-3} = x_{n-2}$. Since the relation $(x_px_q)^2 = 1$ holds for $1 \le p < q-1 \le n-3$ and $x_{n-3} = x_{n-2}$, we definitely have $(x_px_{p+1})^2 = 1$ for some $1 \le p \le n-3$. So we get $(x_px_{p+1})^3 = (x_px_{p+1})^2 = 1$ and thus $x_p = x_{p+1}$ for $1 \le p \le n-3$. Similarly we obtain $z_p = z_{p+1}$, $t_p = t_{p+1}$ and $m_p = m_{p+1}$. Therefore we get

$$\widetilde{\mathbf{C}}'_{n-1} = < x_1, z_1, t_1, m_1; x_1^2 = z_1^2 = t_1^2 = m_1^2 = 1 >$$

and after labeling $x_1 = x$, $z_1 = z$, $t_1 = t$ and $m_1 = m$ in above presentation, it is easy to see that

$$\widetilde{\mathbf{C}}'_{n-1} \cong C_2 * C_2 * C_2 * C_2.$$

Consequently, the commutator subgroup of $\widetilde{\mathbf{C}}_{n-1}$ $(n \ge 3)$ is free product of four cyclic groups of order 2.

These complete the proof.

4 Triangle Coxeter group

Let us consider the Coxeter group, say G, having three generators $\{a,b,c\}$, and relations

$$a^2 = 1,\, b^2 = 1,\, c^2 = 1,\, (ab)^p = 1,\, (bc)^q = 1,\, (ca)^r = 1,$$

where $p,q,r \in \mathbb{Z}$, $p,q,r \ge 2$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Let us label this presentation by **(3)**. This group is called *triangular Coxeter group* (see [10] for the details about triangle groups).

Theorem 4. *The commutator subgroup of triangular Coxeter group G given in presentation* **(3)** *is defined by*

$$G' = \begin{cases} G_1 & ;\ p,q,r \text{ even} \\ G_2 & ;\ p,q,r \text{ odd and} \\ & \ \ p \text{ even}, q,r \text{ odd} \\ C_2 * C_2 * C_2 * C_2 & ;\ p,q \text{ even}, r \text{ odd} \end{cases}$$

where G_1 is the free product of the groups $< (ab)^2 >$, $< (ca)^2 >$, $< (bc)^2 >$, $< bcacba >$ and $< a(cb)^2a >$, and moreover G_2 is the free product of two $(2,2,q)$-generated groups.

Proof. Let us consider the presentation of triangular Coxeter group G given in **(3)**. If we adjoin the relations $(ab)^2 = (bc)^2 = (ca)^2 = 1$ to presentation **(3)**, then we get

$$\begin{aligned}G/G' =< a,b,c\,;\, &a^2 = 1, b^2 = 1, c^2 = 1 (ab)^p = 1 \\ &(bc)^q = 1, (ca)^r = 1, (ab)^2 = 1, \\ &(bc)^2 = 1, (ca)^2 = 1 > .\end{aligned}$$

We need to investigate our aim as in the following cases:

Case (i) p,q,r *even* : Assume that $p,q,r > 2$. Then we get

$$G/G' \cong C_2 \times C_2 \times C_2,$$

and so $\left|G : G'\right| = 8$. Now let $\{1,a,b,c,ab,bc,ac,abc\}$ be a Schreier transversal for G'. Applying the Reidemeister-Schreier process, we get all possible products as in the following:

$$\begin{array}{ll} S_{1a} = a.a = 1, & S_{1b} = b.b = 1, \\ S_{aa} = a^2.1 = 1, & S_{ab} = ab.ba = 1, \\ S_{ba} = ba.ba = (ba)^2, & S_{bb} = b^2.1 = 1, \\ S_{ca} = ca.ca = (ca)^2, & S_{cb} = cb.cb = (cb)^2, \\ S_{aba} = aba.b = (ab)^2, & S_{abb} = abb.a = 1, \\ S_{aca} = aca.c = (ac)^2, & S_{acb} = acb.cba = acbcba, \\ S_{bca} = bca.cba = bcacba, & S_{bcb} = bcb.c = (bc)^2, \\ S_{abca} = abca.cb = abcacb, & S_{abcb} = abcb.ca = abcbca, \end{array}$$

$$\begin{array}{ll} S_{1c} = c.c = 1, & S_{abc} = abc.cba = 1, \\ S_{ac} = ac.ca = 1, & S_{acc} = acc.a = 1, \\ S_{bc} = bc.cb = 1, & S_{bcc} = bcc.b = 1, \\ S_{cc} = c^2.1 = 1, & S_{abcc} = abcc.ba = 1. \end{array}$$

Since $(ba)^2 = (ab)^{-2}$, $(ac)^2 = (ca)^{-2}$, $(cb)^2 = (bc)^{-2}$, $abcacb = (bcacba)^{-1}$ and $abcbca = (acbcba)^{-1}$, the generators of G' are $(ab)^2$, $(ca)^2$, $(bc)^2$, $bcacba$ and $a(cb)^2a$. Therefore G' is defined as the free product of groups $<(ab)^2>, <(ca)^2>, <(bc)^2>, <bcacba>$ and $<a(cb)^2a>$.

We note that if we take $p=q=r=2$, then it is easily seen that $G/G^{'}=G\cong C_2\times C_2\times C_2$ and thus $G^{'}=\{1\}$.

Case (ii) p,q,r *odd* : Since $(ab)^p=(ab)^2=1$ this gives us $(ab)^{p-2}=1$. Further, since $(ab)^{p-2}=(ab)^2=1$ we have $(ab)^{p-4}=1$. By continuing on this process, since p is odd we get $ab=1$ and so $a=b$. Similarly we obtain $b=c$ and $c=a$ since q and r are odd numbers as well. Therefore we have $a=b=c$ and hence

$$G/G^{'}=<a;a^2=1>\cong C_2.$$

So $\left|G:G^{'}\right|=2$. Now let $\{1,a\}$ be a Schreier transversal for $G^{'}$. Applying the Reidemeister-Schreier process we get all possible products as follows:

$$\begin{aligned} S_{1a}&=a.a=1, & S_{aa}&=a^2.a^2=1,\\ S_{1b}&=b.1=b, & S_{ab}&=aba,\\ S_{1c}&=c.1=c, & S_{ac}&=aca. \end{aligned}$$

Here we take $b=x$, $c=y$, $aba=z$ and $aca=t$ as generators for $G^{'}$. Using Reidemeister rewriting process we get the following relations.

$$\begin{aligned} &\tau(bb)=S_{1b}S_{1b}=b.b=x^2,\\ &\tau(cc)=S_{1c}S_{1c}=c.c=y^2,\\ &\tau(bcbc\cdots bc)=S_{1b}S_{1c}S_{1b}S_{1c}\cdots S_{1b}S_{1c}=bc.bc.\cdots bc\\ &=(xy)^q,\\ &\tau(abba)=S_{ab}S_{ab}S_{aa}=aba.aba.1=z^2,\\ &\tau(acca)=S_{ac}S_{ac}S_{aa}=aca.aca.1=t^2,\\ &\tau(abcbc\cdots bca)=S_{ab}S_{ac}S_{ab}S_{ac}\cdots S_{ab}S_{ac}S_{aa}\\ &=aba.aca.aba.aca.\cdots aba.aca.1=(zt)^q. \end{aligned}$$

Thus we obtain

$$G^{'}=<x,y,z,t;x^2=y^2=z^2=t^2=(xy)^q=(zt)^q=1>$$

which is clearly isomorphic to free prodcut of two $(2,2,q)$-generated groups.

Case (iii) p,q *even*, r *odd* : Since p and q are even we have $(ab)^2=(bc)^2=1$ for the smallest power of ab and bc. But since r is odd and $(ca)^r=(ca)^2=1$, we get $(ca)^{r-2}=1$ and so $ca=1$. Hence we obtain $a=c$. Therefore we have

$$G/G^{'}=<a,b;a^2=b^2=(ab)^2=1>\cong C_2\times C_2.$$

Thus $\left|G:G^{'}\right|=4$. Now let $\{1,a,b,ab\}$ be a Schreier transversal for $G^{'}$ and we apply the Reidemeister-Schreier process to get all possible products as follows:

$$\begin{aligned} S_{1a}&=a.a=1, & S_{aa}&=a^2.a^2=1,\\ S_{1b}&=b.b=1, & S_{ab}&=ab.ba=1,\\ S_{1c}&=c.1=c, & S_{ac}&=aca,\\ S_{ba}&=ba.ab=1, & S_{aba}&=aba.aba=1,\\ S_{bb}&=b^2.b^2=1, & S_{abb}&=ab^2.b^2a=1,\\ S_{bc}&=bcb, & S_{abc}&=abcba. \end{aligned}$$

We take $c=x$, $aca=y$, $bcb=z$ and $abcba=t$ as generators for $G^{'}$. Then by using Reidemeister rewriting process we get the relations as follows:

$$\begin{aligned} &\tau(cc)=S_{1c}S_{1c}=c.c=x^2,\\ &\tau(acca)=S_{ac}S_{ac}S_{aa}=aca.aca.1=y^2,\\ &\tau(bccb)=S_{bc}S_{bc}S_{bb}=bcb.bcb.1=z^2,\\ &\tau(abccba)=S_{abc}S_{abc}S_{abb}S_{aa}=abcba.abcba.1.1=t^2. \end{aligned}$$

Thus we obtain

$$G^{'}=<x,y,z,t;x^2=y^2=z^2=t^2=1>\cong C_2*C_2*C_2*C_2.$$

Case (iv) p *even*, q,r *odd* : Since $(bc)^q=(bc)^2=1$ and $(ca)^r=(ca)^2=1$ we have $(bc)^{q-2}=1$ and $(ca)^{r-2}=1$. Since q and r are odd by the finite number of steps we deduce that $a=b=c$. This gives us $G/G^{'}=<a;a^2=1>\cong C_2$. So similarly to case (b) we conclude that $G^{'}$ is free product of two $(2,2,q)$-generated groups.

Hence the result.

The result given below follows from Theorem 3 and 4.

Corollary 1.*Let us consider the group G given in **(3)**. If p,q are even and r is odd, then the commutator subgroup of the triangle group G is isomorphic to the commutator subgroup of the affine Weyl group $\widetilde{\mathbf{C}}_{n-1}$ $(n\geq 3)$.*

5 Conclusion

The main subject in here is the Coxeter groups which have so many applications in both pure and applied mathematics ([13]). However the other part Affine Weyl Groups $\widetilde{\mathbf{A}}_n$ taken so much interest in the meaning of solvability of word problems and so in the meaning of special algorithmic problems ([7]). For a future project, one can study to make a connection between Grobner bases and power (or commutator) subgroups. Because if a positive solution can be obtained for that project, then this will be directly implied the signal process in computer science.

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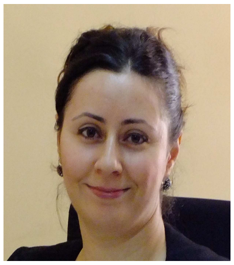
Firat Ates He is full time Professor in Balikesir University. After he graduated from Ege University, Department of Mathematics in 2000, he obtained his Ms.C (2004) and Ph.D. (2007) from Balikesir University under the supervision of Prof. Dr. Ahmet Sinan Cevik. His main studying area is the combinatorial group theory.

Ismail Naci Cangul graduated from the Mathematics Department, Faculty of Arts and Sciences, Uludag University in 1987, to which he was admitted in 1983. He completed his first Ms.C at Uludag University in 1989, second MSc at Warwick University in 1990 and Ph.D at Southampton University in 1994. He is a full-time professor at Uludag University. His complete CV can be found in the web site www.ismailnacicangul.com.

Esra Kirmizi Cetinalp Esra K. Cetinalp is a reserch assistant in the Department of Mathematics at Karamanoglu Mehmetbey University. She is interested in combinatorial group and semigroup theory. Her supervisor is Prof. Dr. Eylem Guzel Karpuz.

Ahmet Sinan Cevik received his graduate degree in Hacettepe University while his Ph.D degree in Mathematics from Glasgow University under the supervision Prof. Steve Pride. His research interests are combinatorial group theory, algebraic and spectral graph theory. His academical CV can be found in his web site www.ahmetsinancevik.com.



Eylem Guzel Karpuz Eylem G. Karpuz received the PhD degree in Department of Mathematics at Balikesir University. She is Associate Professor in the Department of Mathematics at Karamanoglu Mehmetbey University. Her research interests are combinatorial group and semigroup theory, automata theory.