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A Posteriori Error Estimates in $H^1(\Omega)$ Spaces for Parabolic Quasi-Variational Inequalities with Linear Source Terms Related to American Options Problem

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Abstract: In this paper, a posteriori error estimates for parabolic quasi-variational inequalities with linear source terms related to a black scholes model with constant coefficients using two different approaches are established using a theta time scheme combined with a finite element spatial approximation and the results of some numerical experiments are presented to support the theory.

Keywords: A posteriori error estimates, GODDM , DBC, Algorithm, HJB equations, PQVI.

1 Introduction

The Black–Scholes [29] or Black–Scholes–Merton model is a mathematical model of a financial market containing derivative investment instruments. From the model, one can deduce the Black–Scholes formula, which gives a theoretical estimate of the price of European-style options. The formula led to a boom in options trading and legitimised scientifically the activities of the Chicago Board Options Exchange and other options markets around the world [29] lt is widely used, although often with adjustments and corrections, by options market participants [29]. Many empirical tests have shown that the Black–Scholes price is ”fairly close” to the observed prices, although there are well-known discrepancies such as the ”option smile” [29].

The Black–Scholes model was first published by Fischer Black and Myron Scholes in their 1973 paper, ”The Pricing of Options and Corporate Liabilities”, published in the Journal of Political Economy. They derived a partial differential equation, now called the Black–Scholes equation, which estimates the price of the option over time. The key idea behind the model is to hedge the option by buying and selling the underlying asset in just the right way and, as a consequence, to eliminate risk. This type of hedging is called delta hedging and is the basis of more complicated hedging strategies such as those engaged in by investment banks and hedge funds.

Robert C. Merton was the first to publish a paper expanding the mathematical understanding of the options pricing model, and coined the term ”Black–Scholes options pricing model”. Merton and Scholes received the 1997 Nobel Memorial Prize in Economic Sciences for their work. Though ineligible for the prize because of his death in 1995, Black was mentioned as a contributor by the Swedish Academy [29]

The model’s assumptions have been relaxed and generalized in many directions, leading to a plethora of models that are currently used in derivative pricing and risk management. It is the insights of the model, as exemplified in the Black-Scholes formula, that are frequently used by market participants, as distinguished from the actual prices. These insights include no-arbitrage bounds and risk-neutral pricing. The Black-Scholes equation, a partial differential equation that governs the price of the option, is also important as it enables pricing when an explicit formula is not possible.

The American options problem in a black scholes model with constant coefficients and without dividend may be solved by considering the following Parabolic

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Quai-Variational Inequalities (PQVIs) with respect to the right-hand side as a linear source terms and an obstacle defined as an impulse control problem: find

$u(t,x) \in L^2(0,T,D(\Omega)) \cap C^2\left(0,T,H^{-1}(\Omega)\right)$

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au \leq f \text{ in } \Sigma, \\ u \leq Mu, \\ \left(\dfrac{\partial u}{\partial t} + Au - f\right)(u - Mu) = 0, \\ u(0,x) = u_0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \end{cases} \quad (1)$$

where Σ is a set in $\mathbb{R} \times \mathbb{R}^n$ defined as $\Sigma = \Omega \times [0,T]$ with $T^{\cdot} < +\infty$, and Ω is a smooth bounded domain of $\mathbb{R}^n$ with sufficiently smooth boundary Γ and A is an operator defined over $H^1(\Omega)$

$$A = -\Delta + a_0 \quad (2)$$

and $a_0 \in L^2(0,T,L^\infty(\Omega)) \cap C^0\left(0,T,H^{-1}(\Omega)\right)$ is sufficiently smooth functions and satisfy the following condition:

$$a_0(t,x) \geq \beta > 0,\ \beta \text{ is a constant}, \quad (3)$$

and $f(.)$ the right hand side satisfy

$$f \in L^2(0,T,L^\infty(\Omega)) \cap C^1\left(0,T,H^{-1}(\Omega)\right),\ f \geq 0, \quad (4)$$

M is an operator given by

$$Mu = k + \inf_{\xi \geq 0, x+\xi \in \overline{\Omega}} u(x+\xi), \quad (5)$$

where $k > 0$ and $\xi \geq 0$ and

$$Mu \in L^2\left(0,T,W^{2,\infty}(\Omega)\right). \quad (6)$$

Under [30] M is satisfying some proprieties as:
M is concavity that is to say, for $u,v \in C(\Omega)$

$$M(\delta u + (1-\delta)v) \geq \delta M(u) + (1-\delta)M(v) \quad (7)$$

and

$$\forall \eta \in \mathbb{R}, M(u+\eta) = M(u) + \eta. \quad (8)$$

The symbol $(.,.)_\Omega$ stands for the inner product in $L^2(\Omega)$.

The stationary and evolutionary free boundary problems are encountered in several applications; for example, in stochastic control, the solution of (1) characterizes the infimum of the cost function associated to an optimally controlled stochastic switching process without costs for switching and for the calculus of quasi-stationary state for the simulation of petroleum or gaseous deposit.. (cf., e.g., [1]). From the mathematical analysis point of view, problem (1) was intensively studied in the late 1980s (see [11], [12]). On the numerical and computational side ([6], [8]-[11]). However, as far as finite element approximation is concerned, only few works are known in the literature ([3]-[6], [8]-[11]).

The Schwarz alternating method can be used to solve elliptic boundary value problems on domains which consist of two or more overlapping subdomains. It was invented by Herman Amandus Schwarz in 1890. This method has been used for solving the stationary or evolutionary boundary value problems on domains which consist of two or more overlapping subdomains (see [1]–[6], [9], [10], [17]–[19], [8], [20]–[28]). The solution to these qualitative problems is approximated by an infinite sequence of functions resulting from solving a sequence of stationary or evolutionary boundary value problems in each of the subdomains. An extensive analysis of Schwarz alternating method for nonlinear elliptic boundary value problems can be found in [12]–[14], [16], [20]. Also the effectiveness of Schwarz methods for these problems, especially those in fluid mechanics, has been demonstrated in many papers. See the proceedings of the annual domain decomposition conference [15] and [21]–[23], [25]–[26], [24]. Moreover, a priori estimates of the errors for stationary problems is given in several papers; see for instance [22], [23] where a variational formulation of the classical Schwarz method is derived. In [21], geometry-related convergence results are obtained. In [16,17,18], an accelerated version of the GODDM has been treated. In addition, in [16], convergence for simple rectangular or circular geometries has been studied. However, a criterion to stop the iterative process has not been given. All these results can also be found in the recent books on domain decomposition methods [9], [8]. Recently in [17], [18], an improved version of the Schwarz method for highly heterogeneous media has been presented. The method uses new optimized boundary conditions specially designed to take into account the heterogeneity between the subdomains on the boundaries. A recent overview of the current state of the art on domain decomposition methods can be found in [1], [24].

In general, the a priori estimate for stationary problems is not suitable for assessing the quality of the approximate solutions on subdomains, since it depends mainly on the exact solution itself, which is unknown. An alternative approach is to use an approximate solution itself in order to find such an estimate. This approach, known as a posteriori estimate, became very popular in the 1990s with finite element methods; see the monographs [1], [29]. In [29], an algorithm for a nonoverlapping domain decomposition has been given. An a posteriori error analysis for the elliptic case has also been used by [1] to determine an optimal value of the penalty parameter for penalty domain decomposition methods for constructing fast solvers.

Quite a few works on maximum norm error analysis of overlapping nonmatching grids methods for elliptic problems are known in the literature (cf., e.g., [14]–[15]). To prove the main result of this paper, we proceed as in [4]. More precisely, we develop an approach which combines a geometrical convergence result, due to [10], and a lemma which consists of an error estimation in the maximum norm between the continuous and discrete Schwarz iterate.

In [4], the authors derived a posteriori error estimates for the generalized overlapping domain decomposition method (GODDM) with Robin boundary conditions on the boundaries for second order boundary value problems; they have shown that the error estimate in the continuous case depends on the differences of the traces of the subdomain solutions on the boundaries after a discretization of the domain by finite elements method. Also they used the techniques of the residual a posteriori error analysis to get an a posteriori error estimate for the discrete solutions on subdomains.

A numerical study of stationary and evolutionary free boundary problems of the finite element, combined with a finite difference, methods has been achieved in [4], [10]–[18], [27] and using the domain decomposition method combined with finite element method, has been treated in [8]-[11]. Moreover, in a recent research [3], we have treated the overlapping domain decomposition method combined with a finite element approximation for elliptic quasi-variational inequalities related to impulse control problem with respect to the mixed boundary conditions for Laplace operator Δ, where a maximum norm analysis of an overlapping Schwarz method on nonmatching grids has been used. Then, in [9] we have extended the last result to the parabolic quasi variational inequalities with the similar conditions, and using the theta time scheme combined with a finite element spatial approximation, we have proved that the discretization on every subdomain converges in uniform norm. Furthermore, a result of asymptotic behavior in uniform norm has been given.

In this paper, we prove an a posteriori error estimates for the generalized overlapping domain decomposition method with Dirichlet boundary conditions on the boundaries for the discrete solutions on subdomains of PQVI with linear source terms using the theta time scheme combined with a finite element spatial approximation, similar to that in [4], which investigated Laplace equation. Moreover, an Furthermore, the results of some numerical experiments are presented to support the theory.

The outline of the paper is as follows: In section 2, we introduce some necessary notations, then we give the variational formulation of our model. In section 3 and 4, a posteriori error estimate for both continuous and discrete cases are proposed for the convergence of the discrete solution using the theta time scheme combined with a finite element method on subdomains. Finally, in section 4 the results of some numerical experiments are presented to support the theory.

2 Parabolic quasi-variational inequalities

The problem (1) can be transformed into the following continuous parabolic quasi-variational inequalities: find $u \in L^2(0,T,D(\Omega)) \cap C^2(0,T,H^{-1}(\Omega))$ solution to

$$\begin{cases} \left(\dfrac{\partial u}{\partial t}, v-u\right) + a(u,v-u) \geq (f,v-u), \\ u \leq Mu,\ v \leq Mu, \\ u(0,x) = u_0 \text{ in } \Omega,\ \dfrac{\partial u}{\partial \eta} = 0 \text{ on } \partial\Omega \end{cases}, \tag{9}$$

where $a(.,.)$ is the bilinear form defined as

$$a(u,u) = (\nabla u, \nabla u) - (a_0 u, u) \tag{10}$$

2.1 The spatial discretization

Let Ω be decomposed into triangles and τ_h denote the set of all those elements $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_l, $l = \{1,...,m(h)\}$ defined by $\varphi_l(M_s) = \delta_{ls}$ where M_s is a vertex of the considered triangulation. We introduce the following discrete spaces V^h of finite element

$$V^h = \left\{ \begin{array}{l} u \in L^2\left(0,T,H_0^1(\Omega)\right) \cap C\left(0,T,H_0^1(\bar{\Omega})\right), \text{ such that} \\ u\,|_K \in P_1,\ K_i \in \tau_h, \text{ and } u(.,0) = u_0 \text{ in } \Omega,\ \dfrac{\partial u_h^k}{\partial \eta} \text{ in } \partial\Omega \end{array} \right\}, \tag{11}$$

where r_h is the usual interpolation operator defined by

$$v \in L^2\left(0,T,H_0^1(\Omega)\right) \cap C\left(0,T,H_0^1(\bar{\Omega})\right),\ r_h v = \sum_{i=1}^{m(h)} v(M_i)\,\varphi_i(x). \tag{12}$$

In the sequel of the paper, we shall make use of the discrete maximum principle assumption (dmp). In other words, we shall assume that the matrices $(A)_{ps} = a(\varphi_p, \varphi_s)$ is M-matrices [12].

We discretize in space the problem (9), we get the following semi-discrete PQVIs

$$\begin{cases} \left(\dfrac{\partial u_h}{\partial t}, v_h - u_h\right) + a(u_h, v_h - u_h) \geq (f(t), v_h - u_h),\ \ v_h \in V^h, \\ u_h \leq r_h M,\ v_h \leq r_h M, \\ u_h(0) = u_{h0},\ \dfrac{\partial u_h}{\partial \eta} = 0 \text{ on } \partial\Omega \end{cases}. \tag{13}$$

Now, we need to prove the following main theorem, so we use it in identifying the result on the time energy behavior.

Theorem 1.[30]*Let us assume that the discrete bilinear form $a(.,.)$ is weakly coercive in $V^h \subset H_0^1(\Omega)$, there exist two constants $\alpha > 0$ and $\lambda > 0$ such that*

$$a(u_h, u_h) + \lambda\, u_h\|_2 \geq \alpha \|u_h\|_1,$$

where

$$\lambda = \left(\frac{\|b_k\|_\infty^2}{2\gamma} + \frac{\gamma}{2} + \|a_0\|_\infty\right), \alpha = \frac{\gamma}{2}.$$

2.2 A priory estimates for the semi-discrete PQVIs

In [30], it can be identified the energy behavior

$$E_h(t) = \int_\Omega (u_h)^2\, dx$$

to the following result

$$E_h(t) \leq e^{-2(\eta-\varepsilon)t} E_h(0) + \frac{1}{2\varepsilon}\int_0^t \left[e^{2(\eta-\varepsilon)(s-t)}\left(\int_\Omega \left(f^i\right)^2 dx\right)\right] ds. \quad (9)$$

2.3 The time discretization

Now, we discretize the problem (13) with respect to time by using the theta-scheme. Therefore, we search a sequence of elements $u_h^k \in V^h$ which approaches $u_h(t_k)$, $t_k = k\Delta t$, with initial data $u_h^0 = u_{0h}$.
Thus we have, for any $\theta \in [0,1]$ and $k = 1,\ldots,n$

$$\left(u_h^k - u_h^{k-1}, v_h - u_h^{\theta k}\right) + \Delta t.a\left(u_h^{\theta k}, v_h - u_h^{\theta k}\right) \geq \Delta t.\left(f^{\theta,k}, v_h - u_h^{\theta k}\right), \quad (10)$$

where

$$u_h^{\theta k} = \theta u_h^k + (1-\theta)u_h^{k-1} \quad (11)$$

and

$$f^{\theta,k} = \theta f^k + (1-\theta) f^{k-1}. \quad (12)$$

By multiplying and dividing by θ and then by adding $\left(\frac{u_h^{k-1}}{\theta\Delta t}, v_h - u_h^{\theta k}\right)$ to both side of the inequalities (10), we get

$$\left(\frac{u_h^{\theta k}}{\theta\Delta t}, v_h - u_h^{\theta k}\right) + a\left(u_h^{\theta k}, v_h - u_h^{\theta k}\right) \geq \left(f^{\theta,k} + \frac{u_h^{\theta k-1}}{\theta\Delta t}, v_h - u_h^{\theta k}\right),\ v_h \in V_h. \quad (13)$$

Then, the problem (13) can be reformulated into the following coercive discrete system of elliptic quasi-variational inequalities

$$c\left(u_h^{\theta k}, v_h - u_h^{\theta k}\right) \geq \left(f^{\theta,k} + \mu u_h^{k-1}, v_h - u_h^{\theta k}\right),\ v_h,\ u_h^{\theta k} \in V_h \quad (19)$$

such that

$$\begin{cases} c\left(u_h^{\theta k}, v_h - u_h^{\theta k}\right) = \mu\left(u_h^{\theta k}, v_h - u_h^{\theta k}\right) \\ \quad + a\left(u_h^{\theta k}, v_h - u_h^{\theta k}\right),\ v_h,\ u_h^{\theta k} \in V_h, \\ \mu = \dfrac{1}{\theta\Delta t},\ k = 1,\ldots,n. \end{cases} \quad (20)$$

Using the properties of the M in [31], we have

$$u_h^{\theta k} = \theta r_h u_h^k + (1-\theta) r_h u_h^{k-1} \leq \theta r_h\left(Mu_h^k\right) + (1-\theta) r_h\left(Mu_h^{k-1}\right)$$

$$\leq r_h\left(\theta u_h^k + (1-\theta) u_h^{k-1}\right)$$

$$\leq r_h M u_h^{\theta k},$$

thus

$$u_h^{\theta k} \leq r_h M u_h^{\theta k}. \quad (21)$$

2.4 Stability analysis for the PQVIs

In [30], we proved that, if $\theta \geq \frac{1}{2}$ the theta-scheme way is stable unconditionally, and if $0 \leq \theta < \frac{1}{2}$ the theta scheme is stable unless

$$\Delta t < \frac{2C}{(1-2\theta)} h^2, \quad (22)$$

where λ_{sh}^h are the eigenvalues of the operator A.

Proposition 1.[30] *We assume that the coerciveness condition $a(.,.)$of is satisfied with $\lambda = 0$ for each $k = 1,\ldots,n$ we find*

$$\left\|u_h^k\right\|_2^2 + 2\Delta t \sum_{k=1}^n a\left(u_h^{\theta k}, u_h^{\theta k}\right) \leq C(n)\left(\|u_{0h}\|_2^2 + \sum_{k=1}^n \Delta t \left\|f^k\right\|_2^2\right). \quad (23)$$

3 The space continuous for generalized overlapping domain decomposition

Let Ω be a bounded domain in $\mathbb{R}^2$ with a piecewise C^1 boundary Γ. We split the domain Ω into two overlapping subdomains Ω_1 and Ω_2 such that

$$\Omega_1 \cap \Omega_2 = \Omega_{12},\ \partial\Omega_s \cap \Omega_t = \Gamma_s,\ s \neq t \text{ and } s,t = 1,2.$$

We need the spaces

$$V_s = H^1(\Omega) \cap H^1(\Omega_s) = \left\{ v \in H^1(\Omega_i) :\ v_{\partial\Omega_i \cap \partial\Omega} = 0 \right\}$$

and

$$W_s = H_0^{\frac{1}{2}}(\Gamma_s) = \{ v_{\Gamma_s},\ v \in V_s \text{ and } v = 0 \text{ on } \partial\Omega_s \backslash \Gamma_s \}, \tag{14}$$

which is a subspace of

$$H^{\frac{1}{2}}(\Gamma_s) = \left\{ \psi \in L^2(\Gamma_s) : \psi = \varphi_{\Gamma_s} \text{ for } \varphi \in V_s,\ s = 1,2 \right\},$$

equipped with the norm

$$\|\varphi\|_{W_s} = \inf_{v \in V_s v = \varphi \text{ on } \Gamma_s} \|v\|_{1,\Omega}. \tag{25}$$

We define the continuous counterparts of the continuous Schwarz sequences defined in (19), respectively by $u_1^{k,m+1} \in H_0^1(\Omega)$, $m = 0,1,2,...$, solution of

$$\begin{cases} c\left(u_1^{\theta,k,m+1}, v - u_1^{\theta,k,m+1}\right) \\ \geq \left(F^\theta\left(u_1^{\theta,k-1,m+1}\right), v - u_1^{\theta,k,m+1}\right)_{\Omega_1}, \\ u_1^{\theta,k,m+1} = 0,\ \text{ on } \partial\Omega_1 \cap \partial\Omega = \partial\Omega_1 - \Gamma_1, \\ \dfrac{\partial u_1^{\theta,k,m+1}}{\partial\eta_1} + \alpha_1 u_1^{\theta,k,m+1} = \dfrac{\partial u_2^{\theta,k,m}}{\partial\eta_1} + \alpha_1 u_1^{\theta,k,m} \text{ on } \Gamma_1 \end{cases}$$

and $u_2^{\theta,k,m+1} \in H_0^1(\Omega)$ solution of

$$\begin{cases} c\left(u_2^{\theta,k,m+1}, v^i - u_2^{\theta,k,m+1}\right) \\ \geq \left(F\left(u_2^{\theta,k-1,m+1}\right), v^i - u_2^{\theta,k,m+1}\right)_{\Omega_2},\quad m = 0,1,2,.., \\ u_2^{\theta,k,m+1} = 0,\ \text{ on } \partial\Omega_2 \cap \partial\Omega = \partial\Omega_2 - \Gamma_2, \\ \dfrac{\partial u_2^{\theta,k,m+1}}{\partial\eta_2} + \alpha_2 u_2^{\theta,k,m+1} = \dfrac{\partial u_1^{\theta,k,m}}{\partial\eta_2} + \alpha_2 u_2^{\theta,k,m},\ \text{ on } \Gamma_2, \end{cases} \tag{27}$$

where η_s is the exterior normal to Ω_s and α_s is a real parameter, $s = 1,2$.

In the next section, our main interest is to obtain an a posteriori error estimate, we need for stopping the iterative process as soon as the required global precision is reached. Namely, by applying Green formula in Laplace operator with the new boundary conditions of generalized Schwarz alternating method, we get

$$\begin{aligned} &\left(-\Delta\ u_1^{\theta,k,m+1}, v_1 - u_1^{\theta,k,m+1}\right)_{\Omega_1} \\ &= \left(\nabla u_1^{\theta,k,m+1}, \nabla\left(v_1 - u_1^{\theta,k,m+1}\right)\right)_{\Omega_1} \\ &- \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial\eta_1}, v_1 - u_1^{\theta,k,m+1}\right)_{\partial\Omega_1 - \Gamma_1} \\ &+ \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial\eta_1}, v_1 - u_1^{\theta,k,m+1}\right)_{\Gamma_1} \\ &= \left(\nabla u_1^{\theta,k,m+1}, \nabla\left(v_1^i - u_1^{\theta,k,m+1}\right)\right)_{\Omega_1} \\ &- \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial\eta_1}, v_1^i - u_1^{\theta,k,m+1}\right)_{\Gamma_1} \end{aligned}$$

thus we can deduce

$$\begin{aligned} &\left(-\Delta u_1^{\theta,k,m+1}, v_1 - u_1^{\theta,k,m+1}\right)_{\Omega_1} \\ &= \left(\nabla u_1^{\theta,k,m+1}, \nabla\left(v_1 - u_1^{\theta,k,m+1}\right)\right)_{\Omega_1} \\ &- \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial\eta_1}, v_1 - u_1^{\theta,k,m+1}\right)_{\partial\Omega_1 - \Gamma_1} \\ &+ \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial\eta_1}, v_1 - u_1^{\theta,k,m+1}\right)_{\Gamma_1} \\ &= \left(\nabla u_1^{\theta,k,m+1}, \nabla\left(v_1 - u_1^{\theta,k,m+1}\right)\right)_{\Omega_1} \\ &- \left(\frac{\partial u_2^{\theta,k,m+1}}{\partial\eta_2} + \alpha_1\ u_2^{\theta,k,m} - \alpha_1 u_1^{\theta,k,m+1}, v_1 - u_1^{\theta,k,m+1}\right)_{\Gamma_1} \\ &= \left(\nabla u_1^{\theta,k,m+1}, \nabla\left(v_1 - u_1^{\theta,k,m+1}\right)\right)_{\Omega_1} \\ &+ \left(\alpha_1 u_1^{\theta,k,m+1}, v_1^i - u_1^{\theta,k,m+1}\right)_{\Gamma_1} \\ &= \left(\nabla u_1^{\theta,k,m+1}, \nabla\left(v_1 - u_1^{\theta,k,m+1}\right)\right)_{\Omega_1} \\ &+ \left(\alpha_1 u_1^{\theta,k,m+1}, v_1 - u_1^{\theta,k,m+1}\right)_{\Gamma_1} \\ &- \left(\frac{\partial u_2^{\theta,k,m+1}}{\partial\eta_1} + \alpha_1 u_2^{\theta,k,m}, v_1 - u_1^{\theta,k,m+1}\right)_{\Gamma_1}, \end{aligned}$$

thus the problem 15 equivalent to; find $u_1^{\theta,k,m+1} \in V_1$ such that

$$c(u_1^{\theta,k,m+1}, v_1 - u_1^{\theta,k,m+1}) + \left(\alpha_1 u_1^{\theta,k,m}, v_1 - u_1^{\theta,k,m+1}\right)_{\Gamma_1} \geq \left(F^{\theta}(u_1^{\theta,k-1,m+1}), v_1 - u_1^{\theta,k,m+1}\right)_{\Omega_1} + \left(\frac{\partial u_2^{\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_2^{\theta,k,m}, v_1 - u_1^{\theta,k,m+1}\right)_{\Gamma_1}, \forall v_1 \in V_1$$

and for (27), $u_2^{\theta,k,m+1} \in V_2$, we have

$$c(u_2^{\theta,k,m+1}, v_2 - u_2^{\theta,k,m+1}) + \left(\alpha_2 u_2^{\theta,k,m+1}, v_2 - u_2^{\theta,k,m+1}\right)_{\Gamma_2} \geq \left(F(u_2^{\theta,k-1,m+1}), v_2 - u_2^{\theta,k,m+1}\right)_{\Omega_2} + \left(\frac{\partial u_1^{\theta,k,m+1}}{\partial \eta_2} + \alpha_2 u_1^{\theta,k,m}, v_2^i - u_2^{\theta,k,m+1}\right)_{\Gamma_2}.$$

4 A posteriori error estimate in continuous case

Since it is numerically easier to compare the subdomain solutions on the interfaces Γ_1 and Γ_2 rather than on the overlap Ω_{12}, thus we need to introduce two auxiliary problems defined on nonoverlapping subdomains of Ω. This idea allows us to obtain the a posteriori error estimate by following the steps of Otto and Lube [24]. We define these auxiliary problems by coupling each one of the problems 15 and 27 with another problem in a nonoverlapping way over Ω. These auxiliary problems are needed for the analysis and not for the computation section.

To define these auxiliary problems we need to split the domain Ω into two sets of disjoint subdomains : (Ω_1, Ω_3) and (Ω_2, Ω_4) such that

$$\Omega = \Omega_1 \cup \Omega_3, \text{ with } \Omega_1 \cap \Omega_3 = \varnothing \Omega = \Omega_2 \cup \Omega_4, \quad \text{with } \Omega_2 \cap \Omega_4 = \varnothing.$$

Let $(u_1^{k,m}, u_2^{k,m})$ be the solution of problems 15 and 27, we define the couple $(u_1^{k,m}, u_3^{k,m})$ over (Ω_1, Ω_3) to be the solution of the following nonoverlapping problems

$$\begin{cases} \dfrac{u_1^{k,m+1} - u_1^{k-1,m+1}}{\Delta t} - \Delta\, u_1^{\theta,k,m+1} + a_0^k u_1^{\theta,k,m+1} \\ \geq F^{\ \theta}\left(u_1^{\theta,k-1,m+1}\right) \text{ in } \Omega_1, \\ u_1^{\theta,k,m+1} = 0, \text{ on } \partial\Omega_1 \cap \partial\Omega,\ k = 1,...,n, \\ \dfrac{\partial u_1^{\theta,k,m+1}}{\partial \eta_1} + \alpha u_1^{\theta,k,m} = \dfrac{\partial u_2^{\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_2^{\theta,k,m}, \text{ on } \Gamma_1 \end{cases}$$

and

$$\begin{cases} \dfrac{u_3^{k,m+1} - u_3^{k-1,m+1}}{\Delta t} - \Delta\, u_3^{\theta,k,m+1} + a_0^k u_3^{\theta,k,m+1} \\ \geq F^{\theta}\left(u_3^{\theta,k-1,m+1}\right) \text{ in } \Omega_3, \\ u_3^{\theta,k,m+1} = 0, \text{ on } \partial\Omega_3 \cap \partial\Omega, \\ \dfrac{\partial u_3^{\theta,k,m+1}}{\partial \eta_3} + \alpha_3 u_3^{\theta,k,m} = \dfrac{\partial u_1^{\theta,k,m+1}}{\partial \eta_3} + \alpha_3 u_1^{\theta,k,m}, \text{ on } \Gamma_1. \end{cases}$$

It can be taken $\varepsilon_1^{\theta,k,m} = u_2^{\theta,k,m+1} - u_3^{\theta,k,m+1}$ on Γ_1, the difference between the overlapping and the nonoverlapping solutions $u_2^{\theta,k,m+1}$ and $u_3^{\theta,k,m+1}$ of the problems (15) and (27) and (resp.,15 and 15) in Ω_3. Because both overlapping and the nonoverlapping problems converge see [26] that is, $u_2^{\theta,k,m+1}$ and $u_3^{\theta,k,m+1}$ tend to $u_3^{\theta,k}$ (resp. $u_3^{\theta,k}$), then $\varepsilon_1^{\theta,k,m}$ should tend to naught when m tends to infinity in V_2 .

By taking

$$\Lambda_3^{k,m} = \frac{\partial u_2^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_2^{\theta,k,m}.$$

$$\Lambda_1^{k,m} = \frac{\partial u_1^{\theta,k,m}}{\partial \eta_3} + \alpha_3 u_1^{\theta,k,m}. \tag{32}$$

$$\Lambda_3^{k,m} = \frac{\partial u_3^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{\theta,k,m} + \frac{\partial \varepsilon_1^{\theta,k,m}}{\partial \eta_1} + \alpha_1 \varepsilon_1^{\theta,k,m}.$$

$$\Lambda_1^{k,m} = \frac{\partial u_1^{\theta,k,m}}{\partial \eta_3} + \alpha_3 u_1^{\theta,k,m}.$$

Using Green formula, (15) and (15) can be reformulated to the following system of elliptic variational equations

$$c(u_1^{\theta,k,m+1}, v_1 - u_1^{\theta,k,m+1}) + \left(\alpha_1 u_1^{\theta,k,m}, v_1 - u_1^{\theta,k,m+1}\right)_{\Gamma_1} \geq \left(F^{\theta}(u_1^{\theta,k-1,m+1}), v_1 - u_1^{\theta,k,m+1}\right)_{\Omega_1} + \left(\Lambda_3^{k,m}, v_1 - u_1^{\theta,k,m+1}\right)_{\Gamma_1}, \forall v_1 \in V_1 \tag{33}$$

and

$$c(u_3^{\theta,k,m+1}, v_3 - u_3^{\theta,k,m+1}) + \left(\alpha_3 u_3^{\theta,k,m+1}, v_3 - u_3^{\theta,k,m+1}\right)_{\Gamma_1} \geq \left(F^{\theta}(u_3^{\theta,k-1,m+1}), v_3 - u_3^{\theta,k,m+1}\right)_{\Omega 3} + \left(\Lambda_1^{k,m}, v_3 - u_3^{\theta,k,m+1}\right)_{\Gamma_1}, \forall v_3^i \in V_3. \tag{34}$$

On the other hand by taking

$$\theta_1^{k,m}=\frac{\partial \varepsilon_1^{\theta,k,m}}{\partial \eta_1}+\alpha_1\varepsilon_1^{\theta,k,m}, \tag{15}$$

we get

$$\begin{aligned}\Lambda_3^{\theta,k,m} &= \frac{\partial u_3^{\theta,k,m}}{\partial \eta_1}+\alpha_1 u_3^{\theta,k,m}+\frac{\partial (u_2^{\theta,k,m}-u_3^{\theta,k,m})}{\partial \eta_1}\\ &+\alpha_1(u_2^{\theta,k,m}-u_3^{\theta,k,m})\\ &=\frac{\partial u_3^{\theta,k,m}}{\partial \eta_1}+\alpha_1 u_3^{\theta,k,m}+\frac{\partial \varepsilon_1^{k,m}}{\partial \eta_1}+\alpha_1\varepsilon_1^{k,m} \qquad (16)\\ &=\frac{\partial u_3^{\theta,k,m}}{\partial \eta_1}+\alpha_1 u_3^{\theta,k,m}+\theta_1^{k,m}.\end{aligned}$$

Using (15) we have

$$\begin{aligned}&\Lambda_3^{k,m+1}=\frac{\partial u_3^{\theta,k,m}}{\partial \eta_1}+\alpha_1 u_3^{\theta,k,m}+\theta_1^{k,m+1}\\ &=-\frac{\partial u_3^{\theta,k,m}}{\partial \eta_3}+\alpha_1 u_3^{\theta,k,m}+\theta_1^{k,m+1}\\ &=\alpha_3 u_3^{\theta,k,m}-\frac{\partial u_1^{\theta,k,m}}{\partial \eta_3}-\alpha_3 u_1^{\theta,k,m}+\alpha_1 u_3^{\theta,k,m}+\theta_1^{k,m+1}\\ &=(\alpha_1+\alpha_3)u_3^{\theta,k,m}-\Lambda_1^{k,m}+\theta_1^{k,m+1}\end{aligned} \tag{17}$$

and the last equation in (17), we have

$$\begin{aligned}&\Lambda_1^{k,m+1}=-\frac{\partial u_1^{\theta,k,m}}{\partial \eta_1}+\alpha_3 u_1^{\theta,k,m}\\ &=\alpha_1 u_1^{\theta,k,m}-\frac{\partial u_2^{\theta,k,m}}{\partial \eta_1}-\alpha_1 u_2^{\theta,k,m}+\alpha_3 u_1^{\theta,k,m}+\alpha_3 u_1^{\theta,k,m}\\ &=(\alpha_1+\alpha_3)u_1^{\theta,k,m}-\Lambda_3^{k,m}+\theta_3^{k,m+1}.\end{aligned} \tag{18}$$

4.1 Semi discrete algorithm

The sequences $(u_1^{\theta,k,m},u_3^{\theta,k,m})_{m\in\mathbb{N}}$ solutions of (15) satisfy the following domain decomposition algorithm:

Step 1: $k=0,\ \theta=0.5$

Step 2: Let $\Lambda_s^{k,0}\in W_1^*$ be an initial value, $s=1,3$ ($W_1^*$is the dual of W_1).

Step 3; Given $\Lambda_t^{k,m}\in W^*$ solve for $s,t=1,3,s\neq t$: Find $u_s^{i,k,m+1}\in V_s$ solution of

$$c(u_s^{k,m},v_s^i-u_s^{k,m})+\Big(\alpha_s u_s^{k,m},v_s\Big)_{\Gamma s}\geq\Big(F^k(u_s^{k,m}),v_s\Big)_{\Omega s}+$$
$$+\Big(\Lambda_t^{k,m+1},v_s^i\Big)_{\Gamma s},\forall v_s\in V_s.$$

Step 4: Compute

$$\theta_1^{k,m+1}=\frac{\partial \varepsilon_1^{k,m+1}}{\partial \eta_1}+\alpha_1\varepsilon_1^{k,m+1}.$$

Step 5: Compute new data $\Lambda_t^{n+1,m}\in W^*$ solve for $s,t=1,3$, from

$$\Big(\Lambda_s^{k,m+1},\varphi\Big)_{\Gamma_i}=\Big((\alpha_s+\alpha_t)u_s^{k,m+1},v_s\Big)_{\Gamma s}-$$
$$\Big(\Lambda_t^{k,m+1},\varphi\Big)_{\Gamma_s}+\Big(\theta_t^{k,m+1},\varphi\Big)_{\Gamma t},\forall\varphi\in W_s,s\neq t.$$

Step 6: Set $m=m+1$ go to **Step 3.**

Step 7: Set $k=k+1$ go to **Step 2.**

Lemma 1. *Let $u_s^k=u_{\Omega s}^k$, $e_s^{\theta,k,m+1}=u_s^{\theta,k,m+1}-u_s^k$ and $\eta_s^{k,m+1}=\Lambda_s^{k,m+1}-\Lambda_s^k$. Then for $s,t=1,3,s\neq t$, we have*

$$\begin{aligned}&c_s(e_s^{\theta,k,m+1},v_s-e_s^{\theta,k,m+1})+\Big(\alpha_s e_s^{\theta,k,m+1},v_s-e_s^{k,m+1}\Big)_{\Gamma s}\\ &=\Big(\eta_t^{k,m},v_s-e_s^{k,m+1}\Big)_{\Gamma_s},\forall v_s\in V_s\end{aligned} \tag{19}$$

and

$$\begin{aligned}&\Big(\eta_s^{k,m+1},\varphi\Big)_{\Gamma_s}=\Big((\alpha_s+\alpha_t)e_s^{k,m+1},v_s\Big)_{\Gamma_s}-\Big(\eta_t^{k,m},\varphi\Big)_{\Gamma_s}\\ &+\Big(\theta_t^{k,m+1},\varphi\Big)_{\Gamma s},\forall\varphi\in W_1.\end{aligned} \tag{20}$$

Proof. 1. We have

$$\begin{cases}c_s(u_s^{\theta,k,m+1},v_s-u_s^{\theta,k,m+1})+\Big(\alpha_s u_s^{\theta,k,m+1},v_s-u_s^{\theta,k,m+1}\Big)_{\Gamma_s},\\ \geq\Big(F^\theta(u_s^{\theta,k-1,m+1}),v_s-u_s^{\theta,k,m+1}\Big)_{\Omega_s},\\ +\Big(\Lambda_t^{k,m},v_s-u_s^{\theta,k,m+1}\Big)_{\Gamma s},\forall v_s\in V_s\end{cases}$$

and

$$\begin{cases}c_s(u_s^{\theta,k},v_s^i-u_s^{\theta,k})+\Big(\alpha_s u_s^{\theta,k},v_s-u_s^{\theta,k}\Big)_{\Gamma_s}\\ \geq\Big(F(u_s^{\theta,k-1}),v_s-u_s^{\theta,k}\Big)_{\Omega_s},\\ +\Big(\Lambda_t^k,v_s-u_s^{\theta,k}\Big)_{\Gamma_s},\forall v_s\in V_s.\end{cases}$$

Since $c(.,.)$ is a coercive bilinear form, it can be deduced

$$c(u_s^{\theta,k,m+1} - u_s^{\theta,k}, v_s) + \left(\alpha_s u_s^{\theta,k,m} - u_s^{\theta,k}, v_s\right)_{\Gamma_s} \geq \left(\Lambda_t^{k,m} - \Lambda_s^{k}, v_s\right)_{\Gamma_s}, \forall v_s \in V_s$$

and so

$$c_s^i(e_s^{\theta,k,m+1}, v_s^i - e_s^{\theta,k,m+1}) + \left(\alpha e_s^{\theta,k,m}, v_s - e_s^{\theta,k,m+1}\right)_{\Gamma_s} \geq \left(\eta_s^{k,m}, v_1 - e_s^{\theta,k,m+1}\right)_{\Gamma_s}, \forall v_s \in V_s.$$

2. We have $\lim_{m\to+\infty} \varepsilon_1^{\theta,k,m} = \lim_{m\to+\infty} \theta_1^{\theta,k,m} = 0$. Than

$$\Lambda_s^k = (\alpha_1+\alpha_3)\tilde{u}_s^{i,k} - \Lambda_t^{i,k}.$$

Therefore

$$\begin{aligned}\eta_s^{k,m+1} &= \Lambda_s^{k,m+1} - \Lambda_s^k \\ &= (\alpha_1+\alpha_3)u_s^{\theta,k,m} - \Lambda_t^{k,m} + \theta_t^{k,m+1} - (\alpha_1+\alpha_3)u_s^{\theta,k,m+1} \\ &\quad +\Lambda_t^k \\ &= (\alpha_1+\alpha_3)(u_s^{\theta,k,m+1} - u_s^{\theta,k}) - (\Lambda_t^{k,m} - \Lambda_t^k) + \theta_t^{k,m+1}.\end{aligned}$$

Lemma 2. *By letting C be a generic constant which has different values at different places, we get for $s,t = 1,3, s \neq t$*

$$\left(\eta_s^{k,m-1} - \alpha_s e_s^{k,m}, w\right)_{\Gamma_1} \leqslant C \left\|e_s^{k,m}\right\|_{1,\Omega s} \|w\|_{W_1} \tag{21}$$

and

$$\left(\alpha_s w_s + \theta_1^{k,m+1}, e_s^{k,m+1}\right)_{\Gamma_1} \leqslant C \left\|e_s^{k,m+1}\right\|_{1,\Omega_s} \|w\|_{W_1}. \tag{22}$$

Proof. Using Lemma 1 and the fact of the inverse of the trace mapping $Tr_i^{-1} : W_1 \longrightarrow V_s$ is continuous we have for $s,t = 1,3, s \neq t$

$$\begin{aligned}&\left(\eta_s^{k,m-1} - \alpha_s e_s^{k,m}, w^i\right)_{\Gamma s} = c(e_t^{k,m}, Tr^{-1}w) \\ &= \left(\nabla e_s^{k,m}, \nabla Tr^{-1}w\right)_{\Omega s} \\ &+ \left(\alpha e_s^{k,m}, Tr^{-1}w\right)_{\Omega_i} + \lambda\left(e_s^{k,m}, Tr^{-1}w\right)_{\Omega_s} \\ &\leqslant \left|e_s^{k,m}\right|_{1,\Omega_s} \left|Tr^{-1}w\right|_{1,\Omega_s} + \|\alpha\|_\infty \left\|e_s^{k,m}\right\|_{0,\Omega_s} \left\|Tr^{-1}w\right\|_{0,\Omega_s} \\ &+ |\lambda| \left\|e_s^{k,m}\right\|_{0,\Omega_s} \left\|Tr^{-1}w\right\|_{0,\Omega_s} \\ &\leqslant C \left\|e_s^{k,m}\right\|_{1,\Omega_i} \|w\|_{W_1}.\end{aligned}$$

For the second estimate, we have

$$\begin{aligned}&\left(\alpha_s w_s + \theta_1^{k,m+1}, e_s^{k,m+1}\right)_{\Gamma_s} = \left(\alpha_s w_s + \theta_1^{k,m+1}, e_s^{k,m+1}\right)_{\Gamma_s} \\ &\leqslant \left\|\alpha_s w_s + \theta_1^{k,m+1}\right\|_{0,\Gamma_1} \left\|e_s^{k,m+1}\right\|_{0,\Gamma_1} \\ &\leqslant \left(|\alpha_s| \|w_s\|_{0,\Gamma_1} + \left\|\theta_1^{k,m+1}\right\|_{0,\Gamma_1}\right) \left\|e_s^{k,m+1}\right\|_{0,\Gamma_1} \\ &\leq \max(|\alpha_s|, \left\|\theta_1^{k,m+1}\right\|_{0,\Gamma_1}) \|w_s\|_{0,\Gamma_1} \left\|e_s^{k,m+1}\right\|_{0,\Gamma_1} \\ &\leq C \left\|e_s^{k,m+1}\right\|_{0,\Gamma_1} \|w_s\|_{0,\Gamma_1} \leqslant C \left\|e_s^{k,m+1}\right\|_{0,\Gamma_1} \|w_s\|_{W_1}.\end{aligned}$$

Thus, it can be deduced

$$\begin{aligned}&|\alpha_s| \|w_s\|_{0,\Gamma_1} + \left\|\theta_1^{k,m+1}\right\|_{0,\Gamma_1} \\ &\leqslant \max\left(|\alpha_s|, \left\|\theta_1^{k,m+1}\right\|_{0,\Gamma_1}\right) \|w\|_{0,\Gamma_1}.\end{aligned}$$

Proposition 2. *For the sequences $(u_1^{\theta,k,m+1}, u_3^{\theta,k,m+1})_{m\in\mathbb{N}}$ solutions of (15) we have the following a posteriori error estimation*

$$\left\|u_1^{\theta,k,m+1} - u_1^k\right\|_{1,\Omega_1} + \left\|u_3^{\theta,k,m+1} - u_3^k\right\|_{3,\Omega_3} \leqslant C \left\|u_1^{\theta,k,m+1} - u_3^{k,m}\right\|_{W_1}.$$

Proof. From (19) and (20) and we take $v_s = v_1 - u^{k,m+1}$ in (33), then we have

$$\begin{aligned}&c(e_1^{k,m+1}, v_1) + c(e_3^{k,m}, v_3) \\ &= \left(\eta_3^{k,m} - \alpha_1 e_1^{k,m+1}, v_1^i\right)_{\Gamma_1} + \left(\eta_1^{k,m-1} - \alpha_3 e_3^{k,m}, v_3\right)_{\Gamma_1} \\ &= \left(\eta_3^{k+1,m} - \alpha_1 e_1^{k+1,m+1}, v_1\right)_{\Gamma_1} + \left(\eta_1^{k+1,m-1} - \alpha_3 e_3^{k+1,m}, v_3\right)_{\Gamma_1} \\ &+ \left(\eta_1^{k,m-1} - \alpha_3 e_3^{k+1,m}, v_1^i\right)_{\Gamma_1} - \left(\eta_1^{k+1,m-1} - \alpha_3 e_3^{k+1,m}, v_1\right)_{\Gamma_1}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&c(e_1^{k,m+1}, v_1) + c(e_3^{k,m}, v_3) = \\ &\left(\eta_3^{k,m} - \alpha_1 e_1^{k+1,m+1} + \eta_1^{k,m-1} - \alpha_3 e_3^{k,m}, v_1\right)_{\Gamma_1} \\ &+ \left(\eta_1^{k,m-1} - \alpha_3 e_3^{k,m}, v_3 - v_1\right)_{\Gamma_1} \\ &= \left((\alpha_1+\alpha_3) e_3^{k,m} + \theta_1^{k,m} - \alpha_1 e_1^{k,m+1} - \alpha_3 e_3^{k,m}, v_1\right)_{\Gamma_1} \\ &+ \left(\eta_1^{k,m-1} - \alpha_3 e_3^{k,m}, v_3 - v_1\right)_{\Gamma_1} \\ &= \left(\alpha_1(e_3^{k,m} - e_1^{k,m+1}) + \theta_1^{k,m}, v_1\right)_{\Gamma_1} + \left(\eta_1^{k,m-1} - \alpha_3 e_3^{k,m}, v_3 - v_1\right)_{\Gamma_1}.\end{aligned} \tag{43}$$

Taking $v_1 = e_1^{k,m+1}$ and $v_3 = e_3^{k,m}$ in (22), then using $\frac{1}{2}(a+b) \leqslant a^2+b^2$ and the lemma 2, we get

$$\frac{1}{2}\left(\left\|u_1^{\theta,k,m+1}-\tilde{u}_1^{i,k+1}\right\|_{1,\Omega_1}+\left\|u_3^{\theta,k,m+1}-\tilde{u}_3^{i,k+1}\right\|_{3,\Omega_3}\right)^2$$
$$\leqslant \left\|u_1^{\theta,k,m+1}-u_1^{\theta,k}\right\|_{1,\Omega_1}^2+\left\|u_1^{\theta,k,m+1}-u_1^{\theta,k}\right\|_{3,\Omega_3}^2$$
$$\leq \left\|e_1^{k,m+1}\right\|_{1,\Omega_1}^2+\left\|e_3^{k,m}\right\|_{3,\Omega_3}^2$$
$$\leq \left(\nabla e_1^{k,m+1},\nabla e_1^{k,m+1}\right)_{\Omega_1}+\left(a_0 e_1^{k,m+1},e_1^{k,m+1}\right)_{\Omega_3}$$
$$+\left(\nabla e_3^{k,m},\nabla e_3^{n+1,m}\right)_{\Omega_1}+\left(a_0 e_3^{k,m},e_3^{k,m}\right)_{\Omega_3}$$
$$\leqslant \left(\nabla e_1^{k,m+1},\nabla e_1^{k,m+1}\right)_{\Omega_1}+\|a_0\|_\infty\left(e_1^{k,m+1},e_1^{k,m+1}\right)_{\Omega_1}$$
$$+\left(\nabla e_3^{k,m},\nabla e_3^{k,m}\right)_{\Omega_1}+\|a_0^i\|_\infty\left(e_3^{k,m},e_3^{k,m}\right)_{\Omega_3}.$$

Then

$$\frac{1}{2}\left(\left\|u_1^{\theta,k,m+1}-u_1^{\theta,k,}\right\|_{1,\Omega_1}+\left\|u_3^{\theta,k,m+1}-u_3^{\theta,k,m+1}\right\|_{3,\Omega_3}\right)^2$$
$$\leqslant \max(1,\|a_0\|_\infty)\left(c\left(e_1^{k,m+1},e_1^{k,m+1}\right)+c\left(e_3^{k,m},e_3^{k,m}\right)\right)$$
$$=\max(1,\|a_0\|_\infty)\left(\alpha_1(e_3^{k,m}-e_1^{k,m+1})+\theta_1^{k,m},e_1^{k,m+1}\right)_{\Gamma_1}$$
$$+\left(\eta_1^{k,m-1}-\alpha_3 e_3^{k,m},e_3^{k,m}-e_1^{k,m+1}\right)_{\Gamma_1}$$
$$\leqslant C_1\left\|e_1^{k,m+1}\right\|_{1,\Omega_1}\left\|e_3^{k,m}-e_1^{k,m+1}\right\|_{W_1}$$
$$+C_1\left\|e_3^{k,m}\right\|_{3,\Omega_3}\left\|e_3^{k,m}-e_1^{k,m+1}\right\|_{W_1}$$
$$\leqslant C_1\left[\left\|e_1^{k,m+1}\right\|_{1,\Omega_1}+\left\|e_3^{k,m}\right\|_{3,\Omega_3}\right]\left\|e_3^{k,m}-e_1^{k,m+1}\right\|_{W_1},$$

thus

$$\left\|e_1^{k,m+1}\right\|_{1,\Omega_1}+\left\|e_3^{k,m+1}\right\|_{3,\Omega_3}\leqslant\left\|e_1^{k,m+1}-e_3^{k,m+1}\right\|_{W_1}.$$

Therefore

$$\left\|u_1^{\theta,k,m+1}-u_1^{\theta,k}\right\|_{1,\Omega_1}+\left\|u_3^{\theta,k,m+1}-u_3^{\theta,k}\right\|_{3,\Omega_3}$$
$$\leqslant 2C_1\left\|u_1^{\theta,k,m+1}-u_3^{\theta,k,m+1}\right\|_{W_1}.$$

In the similar way, we define another nonoverlapping auxiliary problems over (Ω_2,Ω_4), we get the same result.

Proposition 3. *For the sequences $(u_2^{\theta,k,m+1},u_4^{\theta,k,m+1})_{m\in\mathbb{N}}$. We get the the similar following a posteriori error estimation*

$$\left\|u_2^{\theta,k,m+1}-u_2^{\theta,k}\right\|_{2,\Omega_2}+\left\|u_4^{\theta,k,m+1}-u_4^{\theta,k}\right\|_{4,\Omega_4}$$
$$\leqslant C\left\|u_2^{\theta,k,m+1}-u_4^{\theta,k,m+1}\right\|_{W_2}. \tag{23}$$

*Proof.*The proof is very similar to proof of Proposition 2.

Theorem 2. *Let $u_s^{\theta,k}=u_{\Omega_s}^{\theta,k}, s=1,2$. For the sequences $(u_1^{\theta,k,m+1},u_2^{\theta,k,m+1})_{m\in\mathbb{N}}$ solutions of problems (15), one have the following result*

$$\left\|u_1^{\theta,k,m+1}-u_1^{\theta,k}\right\|_{1,\Omega_1}+\left\|u_2^{\theta,k,m}-u_2^{\theta,k}\right\|_{2,\Omega_2}\leqslant$$
$$C\left(\left\|u_1^{\theta,k,m+1}-u_2^{\theta,k,m}\right\|_{W_1}+\left\|u_1^{\theta,k,m}-u_1^{\theta,k,m+1}\right\|_{W_2}+\right.$$
$$\left.+\left\|e_1^{i,k,m}\right\|_{W_1}+\left\|e_2^{k,m+1}\right\|_{W_2}\right).$$

*Proof.*We use two nonoverlapping auxiliary problems over (Ω_1,Ω_3) and over (Ω_2,Ω_4) resp. From the previous two propositions, we have

$$\left\|u_1^{\theta,k,m+1}-u_1^{\theta,k}\right\|_{1,\Omega_1}+\left\|u_2^{\theta,k,m+1}-u_2^{\theta,k}\right\|_{2,\Omega_2}$$
$$\leqslant\left\|u_1^{\theta,k,m+1}-u_1^{\theta,k}\right\|_{1,\Omega_1}+\left\|u_3^{\theta,k,m+1}-u_3^{\theta,k}\right\|_{3,\Omega_3}$$
$$+\left\|u_2^{\theta,k,m+1}-u_2^{\theta,k}\right\|_{2,\Omega_2}+\left\|u_4^{\theta,k,m+1}-u_4^{\theta,k}\right\|_{4,\Omega_4}$$
$$\leqslant C\left\|u_1^{\theta,k,m+1}-u_3^{\theta,k,m}\right\|_{W_1}+C\left\|u_2^{\theta,k,m+1}-u_4^{\theta,k,m+1}\right\|_{W_2}$$
$$\leqslant C\left\|u_1^{\theta,k,m+1}-u_2^{\theta,k,m}+\varepsilon_1^{k,m}\right\|_{W_1}$$
$$+C\left\|u_2^{\theta,k,m}-u_1^{\theta,k,m+1}+\varepsilon_2^{k,m+}\right\|_{W_2},$$

then

$$\left\|u_1^{\theta,k,m+1}-u_1^{\theta,k}\right\|_{1,\Omega_1}+\left\|u_2^{\theta,k,m}-u_2^{\theta,k,m}\right\|_{2,\Omega_2}\leqslant$$
$$C\left(\left\|u_1^{\theta,k,m+1}-u_2^{\theta,k,m}+\varepsilon_1^{k,m}\right\|_{W_1}+\left\|u_2^{\theta,k,m}-u_1^{\theta,k,m+1}+\varepsilon_2^{k,m-1}\right\|_{W_2}\right.$$
$$\left.+\left\|\varepsilon_1^{k,m}\right\|_{W_1}+\left\|\varepsilon_2^{k,m+1}\right\|_{W_2}\right).$$

5 A Posteriori Error Estimate in the Discrete Case

5.1 The space discretization

Let Ω be decomposed into triangles and τ_h denote the set of all those elements $h>0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_i $i=\{1,\ldots,m(h)\}$ defined by $\varphi_i(M_j)=\delta_{ij}$, where M_j is a vertex of the considered triangulation.

We discretize in space, i.e., that we approach the space H_0^1 by a space discretization of finite dimensional $V^h\subset H_0^1$. In a second step, we discretize the problem with respect to time using the semi-implicit scheme. Therefore, we search a sequence of elements $u_h^{\theta,n}\in V^h$ which approaches $u_h(t_n,.)$, $t_n=n\Delta t$, $k=1,\ldots,n$, with initial data $u_h^0=u_{0h}$.

Let $u_h^{\theta,k,m+1} \in V^h$ be the solution of the discrete problem associated with (14), $u_{s,h}^{\theta,k,m+1} = u_{h,\Omega_s}^{\theta,k,m+1}$.

We construct the sequences $(u_{s,h}^{\theta,k,m+1})_{m\in\mathbb{N}}, u_{s,h}^{\theta,k,m+1} \in V_s^h, (s = 1,2)$ solutions of discrete problems associated with (33) and (34).

where r_h is the usual interpolation operator defined by

$$r_h v = \sum_{i=1}^{m(h)} v(M_j)\,\varphi_i(x). \tag{45}$$

In similar manner to that of the previous section, we introduce two auxiliary problems, we define for (Ω_1,Ω_3) the following full-discrete problems: find $u_{1,h}^{\theta,k,m+1} \in K_h$ solution of

$$\begin{cases} c(u_{1,h}^{\theta,k,m+1}, \tilde{v}_{1,h} - u_{1,h}^{\theta,k,m+1}) + \left(\alpha_{1,h} u_{1,h}^{\theta,k,m+1}, \tilde{v}_{1,h} - u_{1,h}^{\theta,k,m+1}\right)_{\Gamma_1} \\ \geq \left(F^{\theta}(u_{1,h}^{\theta,k-1,m+1}), \tilde{v}_{1,h} - \tilde{u}_{1,h}^{i,k,m+1}\right)_{\Omega_1}, \quad \tilde{v}_{1,h} \in V^h, \\ u_{1,h}^{\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \\ \dfrac{\partial u_{1,h}^{\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_{1,h}^{\theta,k,m+1} = \dfrac{\partial u_{2,h}^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_{2,h}^{\theta,k,m}, \text{ on } \Gamma_1 \end{cases} \tag{24}$$

by taking the trial function $\tilde{v}_{1,h} = v_{1,h} - u_{1,h}^{\theta,k,m+1}$ in (24), we get

$$\begin{cases} c(u_{1,h}^{\theta,k,m+1}, v_{1,h}) + \left(\alpha_{1,h} u_{1,h}^{\theta,k,m+1}, v_{1,h}\right)_{\Gamma_1} \\ \leq \left(F(u_{1,h}^{\theta,k-1,m+1}), v_{1,h}\right)_{\Omega_1}, \quad v_{1,h} \in V^h, \\ u_{1,h}^{\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \\ \dfrac{\partial u_{1,h}^{\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_{1,h}^{\theta,k,m+1} = \dfrac{\partial u_{2,h}^{\theta,k,m}}{\partial \eta_1} + \alpha_1 u_{2,h}^{\theta,k,m}, \text{ on } \Gamma_1. \end{cases} \tag{47}$$

Similarly, we get

$$\begin{cases} c(u_{3,h}^{\theta,k,m+1}, v_{1,h}) + \left(\alpha_{3,h} u_{3,h}^{\theta,k,m+1}, v_{1,h}\right)_{\Gamma_1} \\ \leq \left(F^{\theta}(u_{3,h}^{\theta,k-1,m+1}), v_{1,h}\right)_{\Omega_3}, \\ u_{3,h}^{\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_3 \cap \partial\Omega, \\ \dfrac{\partial u_{3,h}^{\theta,k,m+1}}{\partial \eta_3} + \alpha_3 u_{3,h}^{\theta,k,m+1} = \dfrac{\partial u_1^{\theta,k,m}}{\partial \eta_3} + \alpha_3 u_1^{\theta,k,m}, \text{ on } \Gamma_1. \end{cases} \tag{48}$$

For (Ω_2,Ω_4), we have

$$\begin{cases} c(u_{2,h}^{\theta,k,m+1}, v_{2,h}) + \left(\alpha_{2,h} u_{2,h}^{\theta,k,m+1}, v_{2,h}\right)_{\Gamma_1} \\ \leq \left(F^{\theta}(u_{2,h}^{\theta,k-1,m+1}), v_{2,h}\right)_{\Omega_2}, \\ u_{2,h}^{\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_2 \cap \partial\Omega, \\ \dfrac{\partial u_{2,h}^{\theta,k,m+1}}{\partial \eta_2} + \alpha_2 u_{2,h}^{\theta,k,m+1} = \dfrac{\partial u_1^{\theta,k,m}}{\partial \eta_{2,h}} + \alpha_2 u_1^{\theta,k,m}, \text{ on } \Gamma_2 \end{cases} \tag{49}$$

and

$$\begin{cases} c(u_{4,h}^{\theta,k,m+1}, v_{4,h}) + \left(\alpha_{4,h} u_4^{\theta,k,m+1}, v_{4,h}\right)_{\Gamma_1} \\ \leq \left(F^{\theta}(u_4^{\theta,k-1,m+1}), v_{4,h}\right)_{\Omega_4}, \\ u_{4,h}^{\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \\ \dfrac{\partial u_{4,h}^{\theta,k,m+1}}{\partial \eta_4} + \alpha_4 u u_{4,h}^{\theta,k,m+1} = \dfrac{\partial u_{2,h}^{\theta,k,m+1}}{\partial \eta_4} + \alpha_4 u_{2,h}^{\theta,k,m+1}, \text{ on } \Gamma_2. \end{cases} \tag{50}$$

Theorem 3.[8]*The solution of the system of QVI (47) and (48), and 49 is the maximum element the set of discrete subsolutions.*

We can obtain the discrete counterparts of propositions 1 and 2 by doing almost the same analysis as in section above (i.e., passing from continuous spaces to discrete subspaces and from continuous sequences to discrete ones). Therefore,

$$\left\|u_{1,h}^{\theta,k,m+1} - u_{1,h}^{\theta,k}\right\|_{1,\Omega_1} + \left\|u_{3,h}^{\theta,k,m+1} - u_{3,h}^{\theta,k}\right\|_{1,\Omega_3} \leqslant C\left\|u_{1,h}^{\theta,k,m+1} - u_{3,h}^{\theta,k,m}\right\|_{W_1} \tag{51}$$

and

$$\left\|u_{2,h}^{\theta,k,m+1} - u_{2,h}^{\theta,k}\right\|_{1,\Omega_2} + \left\|u_{4,h}^{\theta,k,m+1} - u_{4,h}^{\theta,k}\right\|_{1,\Omega_4} \leqslant C\left\|u_{2,h}^{\theta,k,m+1} - u_{4,h}^{\theta,k,m}\right\|_{W_2}. \tag{52}$$

Similar to that in the proof of Theorem 2 we get the following discrete estimates

$$\begin{aligned} &\left\|u_{1,h}^{\theta,k,m+1} - u_{1,h}^{\theta,k}\right\|_{1,\Omega_1} + \left\|u_{2,h}^{\theta,k,m} - u_{2,h}^{\theta,k}\right\|_{1,\Omega_2} \leqslant \\ &C\ (\left\|u_{1,h}^{\theta,k,m+1} - u_{2,h}^{\theta,k,m}\right\|_{W_1} + \left\|u_{2,h}^{\theta,k,m} - u_{1,h}^{\theta,k,m}\right\|_{W_2} \\ &+ \left\|e_{1,h}^{k+1,m}\right\|_{W_1} + \left\|e_{2,h}^{k+1,m}\right\|_{W_2}). \end{aligned}$$

Next we will obtain an error estimate between the approximated solution $u_{s,h}^{\theta,k,m+1}$ and the semi discrete solution in time $u^{\theta,k}$. We introduce some necessary notations. We denote by

$$\varepsilon_h = \{E \in T : T \in \tau_h \text{ and } E \notin \partial\Omega\}$$

and for every $T \in \tau_h$ and $E \in \varepsilon_h$, we define

$$\omega_T = \{T' \in \tau_h : T' \cap T \neq \varnothing\}, \quad \omega_E = \{T' \in \tau_h : T' \cap E \neq \varnothing\}.$$

The right hand side f is not necessarily continuous function across two neighboring elements of τ_h having E as a common side, $[f]$ denotes the jump of f across E and η_E the normal vector of E.

We have the following theorem which gives an a posteriori error estimate for the discrete GODDM.

Theorem 4.*Let $u_s^{\theta,k} = u^{\theta,k}|_{\Omega_s}$ where u is the solution of problem (1), the sequences $\left(u_{1,h}^{\theta,k,m+1}, u_{2,h}^{\theta,k,m}\right)_{m\in\mathbb{N}}$ are solutions of the discrete problems (33) and (34). Then there exists a constant C independent of h such that*

$$\left\|u_{1,h}^{\theta,k,m+1} - u_1^{\theta,k}\right\|_{1,\Omega_1} + \left\|u_{2,h}^{\theta,k,m} - u_2^{\theta,k}\right\|_{1,\Omega_2} \leqslant C\left\{\sum_{i=1}^{2}\sum_{T\in\tau_h}\left(\eta_i^T\right) + \eta_{\Gamma_s}\right\},$$

where

$$\eta_{\Gamma_s} = \left\|u_{h,s}^{\theta,k,*} - u_{h,t}^{\theta,k,*-1}\right\|_{W_{h,s}} + \left\|\varepsilon_{i,h}^{\theta,k,*}\right\|_{W_{h,s}}$$

and

$$\eta_s^T = h_T\left\|F\left(u_{h,s}^{\theta,k-1,*}\right) + u_{h,s}^{\theta,k-1} + \Delta\, u_{h,s}^{\theta,k,*} - \left(1+\lambda a_{h0}^k\right)u_{h,s}^{\theta,k}\right\|_{0,T} + \sum_{E\in\varepsilon_h} h_E^{\frac{1}{2}}\left\|\left[\frac{\partial u_{h,s}^{\theta,k,*}}{\partial\eta_E}\right]\right\|_{0,E}, \tag{25}$$

where c is Lipschitz constant of the right hand side and the symbol $*$ is corresponds to $m+1$ when $s=1$ and to m when $s=2$.

*Proof.*The proof is based on the technique of the residual a posteriori estimation see [24] and Theorem 3. We give the main steps by the triangle inequality we have

$$\sum_{s=1}^{2}\left\|u_s^{\theta,k} - u_{h,s}^{\theta,k,*}\right\|_{1,\Omega_s} \leqslant \sum_{s=1}^{2}\left\|u_s^{\theta,k} - u_{h,s}^{\theta,k}\right\|_{1,\Omega_s} + \sum_{s=1}^{2}\left\|u_{h,s}^{\theta,k} - u_{s,h}^{*}\right\|_{1,\Omega_s}. \tag{26}$$

The second term on the right hand side of (26) is bounded by

$$\sum_{s=1}^{2}\sum_{i=1}^{2}\left\|u_{h,s}^{\theta,k} - u_{s,h}^{*}\right\|_{1,\Omega_s} \leqslant C\sum_{s=1}^{2}\eta_{\Gamma_s}.$$

To bound the first term on the right hand side of (26) we use the residual equation and apply the technique of the residual a posteriori error estimation [24] , to get for $v_h \in V^h$

$$\begin{cases} c(u_s^{\theta,k} - u_{h,s}^{\theta,k}, v_s) = c(u_s^{\theta,k} - u_{h,s}^{\theta,k}, v_s - v_{h,s}) \\ \leq \sum\limits_{T\subset\Omega_s}\int_T\left(F^{i,\theta}\left(u_{h,s}^{\theta,k-1}\right) + u_{h,s}^{\theta,k-1} + \mu\Delta\, u_{h,s}^{\theta,k}\right. \\ \left. -\left(1+\mu a_{h0}^k\right)u_{h,s}^{\theta k}\right)\left(v_s - v_{h,s}\right)ds \\ -\sum\limits_{E\subset\Omega_s}\int_E\left[\frac{\partial u_{h,s}^{\theta k}}{\partial\eta_E}\right]\left(v_s - v_{h,s}\right)ds - \sum\limits_{E\subset\Gamma_s}\int_E\frac{\partial u_{h,s}^{\theta k}}{\partial\eta_E}\left(v_s - v_{h,s}\right)ds' \\ +\sum\limits_{E\subset\Omega_s}\int_T\left(F^{\theta}\left(u_s^{\theta,k}\right) - F^{\theta}\left(u_{h,s}^{\theta k}\right)\right)\left(v_s - v_{h,s}\right)d\sigma \\ +\left(\frac{\partial u_{h,s}^{\theta k}}{\partial\eta_s}, v_s - v_{h,s}\right)_{\Gamma_s}, \end{cases}$$

where $F^{\theta}\left(u_{h,s}^{\theta,k}\right)$ is any approximation of $F^{\theta}\left(u_s^{\theta,k}\right)$. Therefore

$$\begin{aligned} &\sum_{s=1}^{2}c(u_s^{\theta,k} - u_{h,s}^{\theta,k}, v_s) \\ &\leq \sum_{s=1}^{2}\sum_{T\subset\Omega_s}\left\|F^{\theta}\left(u_{h,s}^{\theta,k}\right) + u_{h,s}^{\theta,k-1} + \mu\Delta u_{h,s}^{\theta,k}\right. \\ &\left.-\left(1+\mu a_{h0}^k\right)u_{h,s}^{\theta,k}\right\|_{0,T}\left\|v_s - v_{h,s}\right\|_{0,T} \\ &+\sum_{s=1}^{2}\sum_{E\subset\Omega_s}\left\|\left[\frac{\partial u_{h,s}^{\theta,k}}{\partial\eta_E}\right]\right\|_{0,E}\left\|v_s - v_{h,s}\right\|_{0,E} + \\ &\sum_{s=1}^{2}\sum_{E\subset\Gamma_s}\left\|\frac{\partial u_{h,s}^{\theta,k}}{\partial\eta_E}\right\|_{0,E}\left\|v_s - v_{h,s}\right\|_{0,E} \\ &+\sum_{s=1}^{2}\sum_{T\subset\Omega_s}c\left\|u_s^{\theta,k} - u_{h,s}^{\theta,k}\right\|_{0,T}\left\|v_s - v_{h,s}\right\|_{0,T} \\ &+\sum_{s=1}^{2}\sum_{T\subset\Omega_s}\left\|\frac{\partial u_{h,s}^{\theta,k}}{\partial\eta_s}\right\|_{0,T}\left\|v_s - v_{h,s}\right\|_{0,T}, \end{aligned}$$

Using the following fact

$$\left\|u_s^{\theta,k} - u_{h,s}^{\theta,k}\right\|_{1,\Omega_s} \leqslant \sup_{v_s^i\in K}\frac{c(u_s^{\theta,k} - u_{h,s}^{\theta,k}, v_s + ch_s^T)}{\|v_s^i + ch_s^T\|_{1,\Omega_i}},$$

we get

$$\sum_{s=1}^{2}c(u_s^{\theta,k} - u_{h,s}^{\theta,k}, v_s + ch_s^T) \leq \sum_{s=1}^{2}\left(\sum_{T\subset\Omega_s}\eta_s^T\right)\sum_{s=1}^{2}\|v_s\|_{1,\Omega_s}. \tag{55}$$

Finally, by combining (52), (26) and (27) the required result follows.

6 Numerical example

In this section we give a simple numerical example. Consider the following evolutionary HJB equation which can approximated by PQVIs [7] and [10] which investigated the stationary case.

$$\begin{cases} \max\limits_{1\leq i\leq 2}\left(\dfrac{\partial u^i}{\partial t}+A^iu^i-f^i\right)=0, \text{ in } \Omega\times[0,T], \\ u(0,t) \text{ in } \Omega=0, \end{cases} \tag{56}$$

where $\Omega=]0.1[,\ u(0,x)=0,\ T=1$ and

$$A^1u=\frac{\partial^2 u}{\partial x^2},\ A^2u=\frac{\partial^2 u}{\partial x^2}+u,\ \ f^1=f^2=x+t$$

The exact solution of the problem is

$$u(x,t)=\left(x^4-x^5\right)\sin(10x)\cos(20\pi t).$$

For the finite element approximation, we take uniform partition and linear conforming element. For the domain decomposition, we use the following decompositions $\Omega_1=]0,\ 0.55[,\ \Omega_2=]0.45,\ 1[$.

We compute the bilinear semi implicit scheme combined with Galerkin solution in Ω and and we apply the generalized overlapping domain decomposition method to compute the bilinear sequences $u_{h,s}^{i,k,m+1}$, $(s=1,2)$ to be able to look at the behavior of the constant C, where the space steps $h=\frac{1}{10},\ \frac{1}{100},\ \frac{1}{1000}$ and the time steps of discetization $\Delta t=\frac{1}{10},\ \frac{1}{50},\frac{1}{100}$.

We denote by $E_s=\left\|u_s^{i,k}-u_{h,s}^{i,k,m}\right\|_{1,\Omega_s}$, $T_1=\left\|u_{h,1}^{i,k,m+1}-u_{h,2}^{i,k,m}\right\|_{W_h^1}$ and $T_2=\left\|u_{h,2}^{i,k,m}-u_{h,1}^{i,k,m-1}\right\|_{W_h^2}$.

The generalized overlapping domain decomposition method, with $\alpha_1=\alpha_2=0.55$, converges. The iterations have been stopped when the relative error between two subsequent iterates is less than 10^{-6}, we get the following results

$\Delta t=\frac{1}{10},\ \theta=\frac{1}{2}$

h	$1/10$	$1/100$	$1/1000$
E_s	$0.5081043(-4)$	$0.264825(-6)$	$0.4725905(-6)$
E_s	$0.6265874(-4)$	$03852017(-6)$	$0.3837247(-6)$
T_1	$0.9650827(-4)$	$0.573981(-6)$	$0.1286211(-6)$
T_2	$0.892843(-4)$	$0.6418371(-6)$	$0.9430526(-6)$
Iterations	8	14	20

$\Delta t=\frac{1}{20},\theta=\frac{1}{2}$

h	$1/10$	$1/100$	$1/1000$
E_s	$0.4759595(-3)$	$0.8496273(-4)$	$0.9482601(-4)$
E_s	$0.5083649(-3)$	$0.7892758(-4)$	$0.8542894(-4)$
T_1	$0.7592478(-3)$	$0.927307(-4)$	$0.9785809(-4)$
T_2	$0.8584208(-3)$	$0.855012(-4)$	$0.9438526(-4)$
Iterations	8	14	20

$\Delta t=1/40$

h	$1/10$	$1/100$	$1/1000$
E_s	$0.9276183(-2)$	$0.2937842(-3)$	$0.8297682(-4)$
E_s	$0.8524725(-2)$	$0.2572064(-3)$	$0.87085497(-4)$
T_1	$0.9793482(-2)$	$0.6079027(-3)$	$0.5433127(-4)$
T_2	$0.7582921(-2)$	$0.51975802(-3)$	$0.517528(-4)$
Iterations	8	14	20

7 Conclusion

In this paper, a posteriori error estimates for PQVI with linear source terms are derived using the theta time scheme combined with a finite element spatial approximation. Also the techniques of the residual a posteriori error analysis are used. Furthermore the results of some numerical experiments are presented to support the theory. In the future work, the a posteriori error analysis for similar results will be obtained in the general case of more than two subdomains.

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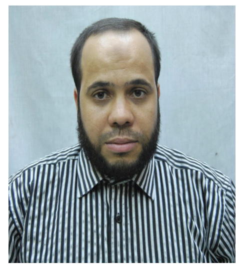

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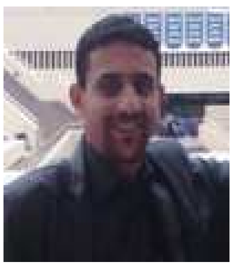

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