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A Controlled Contraction Principle in Partial S-Metric Spaces

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Abstract: In this paper, we introduce the notion of a partially $\alpha-$contractive self mapping and prove the existence and uniqueness of a fixed point for such mapping. Our results improve and generalize many results in S-metric spaces,

Keywords: Fixed point theory, Partial S-metric space, S-metric space

1 Introduction

The existence and uniqueness of fixed point for a self mapping was first introduced by Banach on a metric space. That was the starting point for many research work on this topic. Under different contraction principle and different types of metric space, such as partial metric space, and b-metric space,see [[3]-[19]]. In this article, we work in partial S-metric space.

The existence and uniqueness of a fixed point for a self mapping on different types of metric spaces were the main topic for many research papers [[2]-[18]]. The notion of S-metric space was introduced by Sedghi [4]. A generalization of S-metric space was given by Nabil in [1], where he introduced partial S-metric spaces. Moreover, he proved the existence of a fixed point for a self mapping in partial S-metric space. In this paper, we generalize the results in [1] by adding a control function to the contraction principle, which makes the results in [1] a direct consequences of our theorems.

Before proceeding to the main results, we set forth some definitions that will be used in the sequel.

Definition 1. [5] Let X be a nonempty set and $p : X \times X \longrightarrow [0,+\infty)$. We say that (X, p) is a *partial metric space* if for all $x,y,z \in X$ we have:

1.$x = y$ if and only if $p(x,y) = p(x,x) = p(y,y)$;
2.$p(x,x) \leq p(x,y)$;
3.$p(x,y) = p(y,x)$;
4.$p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

Definition 2. [4] Let X be a nonempty set. An *S-metric space* on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions, for all $x,y,z,a \in X$:

–$S(x;y;z) \geq 0$,

–$S(x;y;z) = 0$ if and only if $x = y = z$,

–$S(x;y;z) \leq S(x;x;a) + S(y;y;a) + S(z;z;a)$.

The pair $(X;S)$ is called an *S-metric space*.

Next, we give the definition of partial S-metric space.

Definition 3. [1] Let X be a nonempty set. A *partial S-metric space* on X is a function $S_p : X^3 \to [0,\infty)$ that satisfies the following conditions, for all $x,y,z,t \in X$:
(i) $x = y$ if and only if $S_p(x,x,x) = S_p(y,y,y) = S_p(x,x,y)$
(ii)$S_p(x,y,z) \leq S_p(x,x,t) + S_p(y,y,t) + S_p(z,z,t) - S_p(t,t,t)$
(iii)$S_p(x,x,x) \leq S_p(x,y,z)$
(iv)$S_p(x,x,y) = S_p(y,y,x)$.

The pair (X,S_p) is called a partial S-metric space.

Definition 4. A sequence $\{x_n\}_{n=0}^{\infty}$ of elements in (X,S_p) is called *p-Cauchy* if the limit $\lim_{n,m\to\infty} S_p(x_n,x_n,x_m)$ exists and finite. The partial S-metric space (X,S_p) is called *complete* if for each p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$ there exists $z \in X$ such that $S_p(z,z,z) = \lim_n S_p(z,z,x_n) = \lim_{n,m} S_p(x_n,x_n,x_m)$.

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Moreover, (X,S_p) is a complete partial S-metric space if and only if (X,S_p^s) is a complete S-metric space. A sequence $\{x_n\}_n$ in a partial S-metric space (X,S_p) is called 0-*Cauchy* if $\lim_{n,m\to\infty} S_p(x_n,x_n,x_m)=0$. We say that (X,S_p) is 0-*complete* if every 0-Cauchy in X converges to a point $x\in X$ such that $S_p(x,x,x)=0$.

One can easily construct an example of a partial S-metric space by using the ordinary partial metric space.

Example 1. [1] Let $X=[0,\infty)$ and p be the ordinary partial metric space on X. Define the mapping on X^3 to be $S_p(x,y,z)=p(x,z)+p(y,z)$. Then S_p defines a partial S-metric space.

Definition 5. Let (X,S_p) be a partial S-metric space and $T:X\longrightarrow X$ be a given mapping. We say that T is *partially $\alpha-$contractive* if there exists a constant $k\in[0,1)$ and a function $\alpha: X\times X\longrightarrow[0,+\infty)$ such that for all $x,y\in X$ we have

$$\alpha(x,y)S_p(Tx,Tx,Ty)\le\max\{kS_p(x,x,y),S_p(x,x,x),S_p(y,y,y)\}. \tag{1}$$

Definition 6. Let (X,S_p) be a partial S-metric space and $T:X\longrightarrow X$ be a given mapping. We say that T is *$R_\alpha-$admissible* if $x,y\in X$, $\alpha(x,y)\ge 1$ implies that $\alpha(x,Ty)\ge 1$. Also, we say that T is *$\alpha-$admissible* if $x,y\in X$, $\alpha(x,y)\ge 1$ implies that $\alpha(Tx,Ty)\ge 1$.

Example 2. Let $X=[0,+\infty)$. Define $T:X\longrightarrow X$ by $Tx=\sqrt{x}$ and $\alpha: X\times X\longrightarrow X$ by

$$\alpha(x,y)=\begin{cases} e^{x-y} & \text{if } x\ge y\\ 0 & \text{if } x<y.\end{cases}$$

It is easy to see that T is $\alpha-$admissible and $R_\alpha-$admissible.

Now, set

$$\rho_{S_p}(\alpha):=\inf\{S_p(x,x,y)\mid x,y\in X:\alpha(x,y)\ge 1\}=\inf\{S_p(x,x,x)\mid x\in X:\alpha(x,x)\ge 1\},$$

$$X_{S_p}(\alpha)=\{x\in X\mid S_p(x,x,x)=\rho_{S_p}(\alpha)\},$$

$$Z_{S_p}(\alpha)=\{x\in X_{S_p}\mid \alpha(x,x)\ge 1\}.$$

2 Main Result

In this section, we prove the existence of a fixed point in partial S-metric space. We prove relevant corollary. This next theorem is considered to be our main result.

Theorem 1. *Let (X,S_p) be a complete partial S-metric space, T be a self mapping on X and assume that T is partially $\alpha-$contractive. If T is $\alpha-$admissible and $R_\alpha-$admissible and if $X_{S_p}(\alpha)$ is nonempty, then $Z_{S_p}(\alpha)$ is nonempty. Also, assume that there exists $x_0\in X$ such that $\alpha(x_0,x_0)\ge 1$ then:*

1.The set $Z_{S_p}(\alpha)$ is nonempty;
2.There exists $a\in Z_{S_p}(\alpha)$ such that $Ta=a$.

Moreover, if for all u,v in $Z_{S_p}(\alpha)$ with the property $Tu=u$ and $Tv=v$ we have $\alpha(u,v)\ge 1$, then T has a unique fixed point in $Z_{S_p}(\alpha)$.

Proof. Let $x_0\in X$ such that $\alpha(x_0,x_0)\ge 1$. Define a sequence $\{x_n\}$ for all $n\ge 0$ in X such that $x_1=Tx_0, x_2=Tx_1,\cdots,x_{n+1}=Tx_n,\cdots$. Since T is $R_\alpha-$admissible and $\alpha-$admissible, we have $\alpha(x_0,x_1)=\alpha(x_0,Tx_0)\ge 1$ and hence $\alpha(x_1,x_2)=\alpha(Tx_0,Tx_1)\ge 1$. So, by induction on n we get

$$\alpha(x_n,x_{n+1})\ge 1,$$

for all $n\ge 0$. Also, since T is $R_\alpha-$admissible; $\alpha(x_0,x_0)\ge 1$ implies $\alpha(x_0,x_1)=\alpha(x_0,Tx_0)\ge 1$. By induction on n, we also conclude that

$$\alpha(x_0,x_n)\ge 1$$

for all $n\ge 0$. Also, given the fact that T is α-admissible and $\alpha(x_0,x_0)\ge 1$, it not difficult to see that $\alpha(x_n,x_n)\ge 1$ for all $n\ge 0$. Hence,

$$\begin{aligned} S_p(x_1,x_1,x_1)=&S_p(Tx_0,Tx_0,Tx_0)\\ \le&\alpha(x_0,x_0)S_p(Tx_0,Tx_0,Tx_0)\\ \le&\max\{kS_p(x_0,x_0,x_0),S_p(x_0,x_0,x_0)\}\\ =&S_p(x_0,x_0,x_0).\end{aligned}$$

By induction we obtain:

$$S_p(x_{n+1},x_{n+1},x_{n+1})\le S_p(x_n,x_n,x_n).$$

Therefore, $\{S_p(x_n,x_n,x_n)\}_{\{n\ge 0\}}$ is a nonincreasing sequence. Define

$$r_0:=\lim_n S_p(x_n,x_n,x_n)=\inf_n S_p(x_n,x_n,x_n)\ge 0$$

and

$$M_0:=\frac{2}{1-k}S_p(x_0,x_0,x_1)+S_p(x_0,x_0,x_0).$$

Next, we need to show that $S_p(x_0,x_0,x_n)\le M_0$, for any $n\ge 0$. If $n=0$; the case is trivial. For $n=1$ and using the fact that $k\in[0,1)$ we deduce that $S_p(x_0,x_0,x_1)\le\frac{2}{1-k}S_p(x_0,x_0,x_1)\le\frac{2}{1-k}S_p(x_0,x_0,x_1)+S_p(x_0,x_0,x_0)=M_0$. So, we may assume that is true for all $n\le n_0-1$ and prove it for $n=n_0\ge 2$.

$$\begin{aligned}&S_p(x_0,x_0,x_{n_0})\\ &\le S_p(x_0,x_0,x_1)+S_p(x_0,x_0,x_1)+S_p(x_{n_0},x_{n_0},x_1)-S_p(x_1,x_1,x_1)\\ &\le 2S_p(x_0,x_0,x_1)+S_p(x_1,x_1,x_{n_0})\\ &\le 2S_p(x_0,x_0,x_1)+\alpha(x_0,x_{n_0-1})S_p(Tx_0,Tx_0,Tx_{n_0-1})\\ &\le 2S_p(x_0,x_0,x_1)+\max\{kS_p(x_0,x_0,x_{n_0-1}),S_p(x_0,x_0,x_0),S_p(x_{n_0-1},x_{n_0-1},x_{n_0-1})\}\\ &\le 2S_p(x_0,x_0,x_1)+\max\{kS_p(x_0,x_0,x_{n_0-1}),S_p(x_0,x_0,x_0)\}.\end{aligned}$$

Also, by induction assumption, we have $S_p(x_0,x_0,x_{n_0-1}) \leq \frac{2}{1-k}S_p(x_0,x_0,x_1) + S_p(x_0,x_0,x_0)$. So, we have

$$\begin{aligned}S_p(x_0,x_0,x_{n_0}) &\leq 2S_p(x_0,x_0,x_1)+\\ &\max\{\frac{2k}{1-k}S_p(x_0,x_0,x_1)+kS_p(x_0,x_0,x_0),S_p(x_0,x_0,x_0)\}\\ &\leq 2S_p(x_0,x_0,x_1)+\frac{2k}{1-k}S_p(x_0,x_0,x_1)+S_p(x_0,x_0,x_0)\\ &= \frac{2}{1-k}S_p(x_0,x_0,x_1)+S_p(x_0,x_0,x_0)=M_0.\end{aligned}$$

Hence, by induction we conclude that $S_p(x_0,x_0,x_n) \leq M_0$. Next, we need to show that

$$\lim_{n,m} S_p(x_n,x_n,x_m) = r_0.$$

For all n,m we have $S_p(x_n,x_n,x_m) \geq S_p(x_n,x_n,x_n) \geq r_0$. Let $\varepsilon > 0$ find a natural number n_0 such that $S_p(x_{n_0},x_{n_0},x_{n_0}) < r_0+\varepsilon$ and $2M_0k^{n_0} < r_0+\varepsilon$. Now for any $n,m \geq 2n_0$, since T is $R_\alpha-$admissible and using the fact that $\alpha(x_n,x_{n+1}) \geq 1$ we deduce that $\alpha(x_n,x_m) \geq 1$. Hence,

$$\begin{aligned}&S_p(x_n,x_n,x_m)\\ &\leq \alpha(x_n,x_m)S_p(x_n,x_n,x_m)\\ &\leq \max\{kS_p(x_{n-1},x_{n-1},x_{m-1}),S_p(x_{n-1},x_{n-1},x_{n-1}),S_p(x_{m-1},x_{m-1},x_{m-1})\}\\ &\leq \max\{k^2S_p(x_{n-2},x_{n-2},x_{m-2}),S_p(x_{n-2},x_{n-2},x_{n-2}),S_p(x_{m-2},x_{m-2},x_{m-2})\}\\ &\leq \cdots \leq \max\{k^{n_0}S_p(x_{n-n_0},x_{n-n_0},x_{m-n_0}),S_p(x_{n-n_0},x_{n-n_0},x_{n-n_0}),\\ &S_p(x_{m-n_0},x_{m-n_0},x_{m-n_0})\}\\ &\leq r_0+\varepsilon.\end{aligned}$$

Hence,

$$\lim_{n,m} S_p(x_n,x_n,x_m) = r_0.$$

Since (X,p) is a complete partial S-metric space; there exists $\widetilde{x} \in X$ such that

$$r_0 = S_p(\widetilde{x},\widetilde{x},\widetilde{x}) = \lim_n S_p(\widetilde{x},\widetilde{x},x_n) = \lim_{n,m} S_p(x_n,x_n,x_m).$$

Next, we show that $S_p(\widetilde{x},\widetilde{x},\widetilde{x}) = S_p(\widetilde{x},\widetilde{x},T\widetilde{x})$. For each natural number n we have

$$S_p(\widetilde{x},\widetilde{x},T\widetilde{x}) \leq 2S_p(\widetilde{x},\widetilde{x},x_n) - S_p(x_n,x_n,x_n) + S_p(T\widetilde{x},T\widetilde{x},x_n).$$

Using the property that T is $\alpha-$contractive we deduce that there exists a subsequence of natural numbers $\{n_l\}$ such that

$$\begin{aligned}&S_p(T\widetilde{x},T\widetilde{x},x_{n_l})\\ &\leq \alpha(\widetilde{x},x_{n_l-1})S_p(T\widetilde{x},T\widetilde{x},x_{n_l})\\ &\leq \max\{kS_p(\widetilde{x},\widetilde{x},x_{n_l-1}),S_p(\widetilde{x},\widetilde{x},\widetilde{x}),S_p(x_{n_l-1},x_{n_l-1},x_{n_l-1})\}.\end{aligned}$$

So, for $l \geq 1$, we have either $S_p(T\widetilde{x},T\widetilde{x},x_{n_l}) \leq kS_p(\widetilde{x},\widetilde{x},x_{n_l-1})$ or less than or equal $S_p(\widetilde{x},\widetilde{x},\widetilde{x})$ or less than or equal $S_p(x_{n_l-1},x_{n_l-1},x_{n_l-1})$.
In all of these three cases, if we take the limit as l goes toward ∞ we get $S_p(\widetilde{x},\widetilde{x},T\widetilde{x}) \leq S_p(\widetilde{x},\widetilde{x},\widetilde{x})$. But, we know by the property (ii) of the partial S-metric space definition that $S_p(\widetilde{x},\widetilde{x},\widetilde{x}) \leq S_p(\widetilde{x},\widetilde{x},T\widetilde{x})$. Therefore, $S_p(\widetilde{x},\widetilde{x},\widetilde{x}) = S_p(\widetilde{x},\widetilde{x},T\widetilde{x})$.

Now, we show that $X_{S_p}(\alpha)$ is nonempty. For each natural number l pick $x_l \in X$ with $\alpha(x_l,x_l) \geq 1$ and $S_p(x_l,x_l,x_l) < \rho_{S_p}(\alpha) + \frac{1}{l}$ and show that

$$\lim_{n,m} S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_m) = \rho_{S_p}(\alpha).$$

Let $\varepsilon > 0$ put $n_0 := (\frac{3}{\varepsilon(1-k)})+1$ if $l \geq n_0$ then we have: $\rho_{S_p}(\alpha) \leq S_p(\widetilde{x}_l,\widetilde{x}_l,T\widetilde{x}_l) \leq S_p(\widetilde{x}_l,\widetilde{x}_l,T\widetilde{x}_l) \leq r_{x_l} \leq S_p(\widetilde{x}_l,\widetilde{x}_l,T\widetilde{x}_l) < \rho_{S_p}(\alpha) + \frac{1}{l} \leq \rho_{S_p}(\alpha) + \frac{1}{n_0} < \rho_{S_p}(\alpha) + \frac{\varepsilon(1-k)}{3}$. Hence, we deduce that:

$$U_l := S_p(\widetilde{x}_l,\widetilde{x}_l,\widetilde{x}_l) - S_p(T\widetilde{x}_l,T\widetilde{x}_l,T\widetilde{x}_l) < \frac{\varepsilon(1-k)}{3},$$

for $i \geq n_0$. Also, if $l \geq n_0$, then $S_p(\widetilde{x}_l,\widetilde{x}_l,\widetilde{x}_l) = r_{x_l} \leq S_p(x_l,x_l,x_l) < \rho_{S_p}(\alpha) + \frac{1}{n_0}$. Which implies $S_p(\widetilde{x}_l,\widetilde{x}_l,\widetilde{x}_l) \leq \rho_{S_p}(\alpha) + \frac{\varepsilon(1-k)}{3}$ for all $l \geq n_0$. Now, if $n,m \geq n_0$, then $S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_m) \leq 2S_p(\widetilde{x}_n,\widetilde{x}_n,T\widetilde{x}_n) + S_p(T\widetilde{x}_n,T\widetilde{x}_n,T\widetilde{x}_m) + 2S_p(T\widetilde{x}_m,T\widetilde{x}_m,\widetilde{x}_m) - S_p(T\widetilde{x}_n,T\widetilde{x}_n,T\widetilde{x}_n) - S_p(T\widetilde{x}_m,T\widetilde{x}_m,T\widetilde{x}_m)$.
We know that $S_p(\widetilde{x},\widetilde{x},\widetilde{x}) = S_p(\widetilde{x},\widetilde{x},T\widetilde{x})$ which implies:

$$\begin{aligned}&S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_m)\\ &\leq U_n+U_m+\alpha(\widetilde{x}_n,\widetilde{x}_m)S_p(T\widetilde{x}_n,T\widetilde{x}_n,T\widetilde{x}_m)\\ &\leq S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_m) \leq U_n+U_m+S_p(T\widetilde{x}_n,T\widetilde{x}_n,T\widetilde{x}_m)\\ &< U_n+U_m+\max\{kS_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_m),S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_n),S_p(\widetilde{x}_m,\widetilde{x}_m,\widetilde{x}_m)\}.\end{aligned}$$

Hence,

$$\rho_{S_p}(\alpha) \leq S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_m)$$

$$\leq max\{\frac{2}{3}\varepsilon, \frac{2}{3}\varepsilon(1-k)+S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_n), \frac{2}{3}\varepsilon(1-k)+S_p(\widetilde{x}_m,\widetilde{x}_m,\widetilde{x}_m)\}$$

$$\leq \max\{\frac{2}{3}\varepsilon, \rho_{S_p}(\alpha)+\varepsilon(1-k)\} < \rho_{S_p}(\alpha)+\varepsilon.$$

Thus,

$$\lim_{n,m} S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_m) = \rho_{S_p}(\alpha).$$

Since (X,S_p) is complete there exists $a \in X$ such that,

$$S_p(a,a,a) = \lim_n S_p(a,a,\widetilde{x}_n) = \lim_{n,m} S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_m) = \rho_{S_p}(\alpha).$$

Therefore, we conclude that $a \in X_{S_p}(\alpha)$ and thus $X_{S_p}(\alpha)$ is nonempty. Therefore, $Z_{S_p}(\alpha)$ is nonempty.

Now, let $x_0 \in Z_{S_p}(\alpha)$ be arbitrary. Then by the above argument we have

$$\rho_{S_p}(\alpha) \leq S_p(T\widetilde{x},T\widetilde{x},T\widetilde{x}) \leq S_p(\widetilde{x},\widetilde{x},T\widetilde{x}) = S_p(\widetilde{x},\widetilde{x},\widetilde{x}) = r_0 = \rho_{S_p}(\alpha).$$

Thus, $T\widetilde{x} = \widetilde{x}$, Now, assume that T has two fixed points in $Z_{S_p}(\alpha)$ say u and v. By our hypothesis, we know that $\alpha(u,v) \geq 1$. Thus,

$$S_p(u,u,v) = S_p(Tu,Tu,Tv) \leq \alpha(u,v)S_p(Tu,Tu,Tv)$$

$$\leq \max\{kS_p(u,u,v),S_p(u,u,u),S_p(v,v,v)\}.$$

Now, if $S_p(u,u,v) \leq kS_p(u,u,v)$ we deduce that $S_p(u,u,v) = 0$ and in this case $u = v$, or $S_p(u,u,v) \leq S_p(u,u,u) = S_p(v,v,v)$ and in this case by condition (ii) of the definition of the partial S-metric space we obtain $S_p(u,u,v) = S_p(u,u,u) = S_p(v,v,v)$ and hence by condition (i) of the same definition we conclude that $u = v$. Therefore, we obtain the uniqueness as desired.

As a consequence of the above result, the following corollary follows easily.

Corollary 1.*Let (X,S_p) be a 0-complete partial S-metric space, $k \in [0,1)$ and consider the map $T : X \longrightarrow X$ to be α-admissible and R_α-admissible,and there exists $x_0 \in X$ such that $\alpha(x_0,x_0) \geq 1$, also for every $x,y \in X$ we have $\alpha(x,y)S_p(Tx,Tx,Ty) \leq kS_p(x,x,y)$. Then there exists $\widetilde{x} \in X$ such that $T\widetilde{x} = \widetilde{x}$.*

*Proof.*Using the same technique and notation in the proof of Theorem 1, we deduce that $S_p(x_n,x_n,x_n) \leq \alpha(x_n,x_n)S_p(x_n,x_n,x_n) \leq k^n S_p(x_0,x_0,x_0)$. Thus,

$$r_0 = S_p(\widetilde{x},\widetilde{x},\widetilde{x}) = lim_n S_p(\widetilde{x},\widetilde{x},x_n) = lim_{n,m} S_p(x_n,x_n,x_m) = 0.$$

This implies that $S_p(\widetilde{x},\widetilde{x},\widetilde{x}) = 0$. Since $S_p(\widetilde{x},\widetilde{x},\widetilde{x}) = S_p(\widetilde{x},\widetilde{x},T\widetilde{x}) = 0$, we have $\widetilde{x} = T\widetilde{x}$ as required.

In closing, we change the contraction principle in Theorem 1, to show that there exist a unique fixed point in the whole space X.

Theorem 2.*Let (X,S_p) be a complete partial S-metric space, $k \in [0,1)$ and assume the there exists $x_0 \in X$ such that $\alpha(x_0,x_0) \geq 1$. Consider the map $T : X \longrightarrow X$ to be $\alpha-$admissible and $R_\alpha-$admissible. Assume that for every $x,y \in X$ we have*

$$\alpha(x,y)S_p(Tx,Tx,Ty) \leq \max\{kS_p(x,x,y), \frac{S_p(x,x,x)+S_p(y,y,y)}{2}\}, \quad (2)$$

then there exists a unique $u \in X$ such that $Tu = u$.

Proof. Note that, for every $x,y \in X$ we have:

$$\alpha(x,y)S_p(Tx,Tx,Ty) \leq \max\{kS_p(x,x,y), \frac{S_p(x,x,x)+S_p(y,y,y)}{2}\}$$
$$\leq \max\{kS_p(x,x,y), S_p(x,x,x), S_p(y,y,y)\}.$$

Thus, all the conditions of Theorem 1 are satisfied. Hence, there exists $u \in X$ such that $Tu = u$. Assume that there exists two fixed points $u,v \in X$ for T such that $\alpha(u,v) \geq 1$. Hence,

$$S_p(u,u,v) = S_p(Tu,Tu,Tv) \leq \alpha(u,v)S_p(Tu,Tu,Tv)$$
$$\leq \max\{kS_p(u,u,v), \frac{S_p(u,u,u)+S_p(v,v,v)}{2}\}.$$

Thus, we either have $S_p(u,u,v) \leq kS_p(u,u,v)$ which implies that $S_p(u,u,v) = 0$ and hence $u = v$, or $0 = 2S_p(u,u,v) - S_p(u,u,u) - S_p(v,v,v)$ which also implies that $u = v$ as desired.

Example 3. Let (X,S_p) be a partial S-metric space, where $X = [0,1] \cup [2,3]$ and the partial S-metric space $S_p : X^3 \longrightarrow [0,+\infty)$ is defined by

$$S_p(x,y,z) = \begin{cases} \|\max\{x,y\} - z\| & \text{if } \{x,y,z\} \cap [2,3] \neq \emptyset \\ |x-y-z| & \text{if } \{x,y,z\} \subset [0,1]. \end{cases}$$

Define the functions $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\infty)$ as follows $Tx = \frac{x+1}{2}$ if $0 \leq x \leq 1$, $T2 = 1$, and $Tx = \frac{x+2}{2}$ if $2 < x \leq 3$,

$$\alpha(x,y) = \begin{cases} e^{x-y} & \text{if } x \geq y \\ 0 & \text{if } x < y. \end{cases}$$

It is easy to see that T is $\alpha-$admissible and $R_\alpha-$admissible. Note that, we can always pick our x, y and z such that $\max\{x,y\} > z$. Also T is an increasing function. So, for every $x \geq y \in X$ we have:

$$S_p(Tx,Tx,Ty) \leq \alpha(x,y)S_p(Tx,Tx,Ty) \leq \frac{1}{2}S_p(x,x,y), \text{ if } x,y \in [0,1],$$

and

$$S_p(Tx,Tx,Ty) \leq \alpha(x,y)S_p(Tx,Tx,Ty)$$
$$\leq \frac{S_p(x,x,x)+S_p(y,y,y)}{2}, \quad \{x,y\} \cap [2,3] \neq \emptyset.$$

One can verify that the function T in this example satisfies the conditions of Theorem 2 and the unique fixed point will be 1.

3 Conclusion

In closing, the author would like to bring to the reader's attention the possibility of obtaining the same result of Theorem 2.1 by changing the hypothesis where T is partially α-contractive with the following contraction principle $\alpha(x,y)S_p(Tx,Tx,Ty) \leq \psi(S_p(x,x,y))$, where ψ is a self-function on $(0,\infty)$.

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