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Finding “Improvement Region” for the Inefficient Units in Data Envelopment Analysis

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Abstract: One of the most important issues in data envelopment analysis is sensitivity analysis of efficient and inefficient decision making units (DMUs). Sensitivity analysis of inefficient units has been more studied recently. We know that a specific inefficient DMUcan scarcely reach to the efficient frontier and achieving the score 1 in efficiency but it can easily obtain an efficiency score of α (α is a constant which is usually closed to 1 and defined by the decision maker). In this paper we are going to find a region which named Improvement Region (IR) for a specific inefficient DMU which can obtain at least an efficiency score of α. In this region the inefficient *DMU* which is under evaluation can satisfy the decision maker and also it can be improved itself to gain a new efficiency score and by these variations it is made more contented for decision maker. The procedure is illustrated by numerical examples.

Keywords: Data Envelopment Analysis (DEA), Sensitivity Analysis, Efficiency, Improvement Region (IR), Decision maker

1 Introduction

In 1978 data envelopment analysis is introduced by Charnes, Cooper, Rhodes [1] (CCR model) and extended by Banker, [2] (BCC model) .It is one of the best ways for assessing the relative efficiency of group of homogenous decision making units (DMUs) that use multiple inputs to produce multiple outputs. In recent years, one of the important issues in DEA is the sensitivity analysis included efficient and inefficient DMUswhich more researchers have great attention. In 1985, sensitivity analysis of CCR model for a specific efficient DMU with a single output was initiated by Charnes [3]. They built variations in data for DMU under consideration and led to alter the inverse matrix used to generate solutions in the usual simplex algorithm computer codes. In 1990 Charnes and Neralic considered additive model and they obtained sufficient conditions for remaining efficient [4]. Then in 1992, Charnes et al. obtained a specific stability region by using L_1 and L_∞ [5]. These researchers have studied the methods which simultaneous proportional change is assumed in inputs and outputs for a specific efficient DMU under evaluations. Then Zhu (1996) provides a modified DEA model to compute a stability region which DMU under evaluation remains efficient [6]. In 1998 Seiford and Zhu developed a procedure to determine an input stability region (ISR) and an output stability region (OSR) for efficient DMU [7]. They stated that an efficient DMU will remain efficient after the input increases or output decreases if and only if such changes occur within the ISR or OSR [7], and this subject are considering in recent years. Jahanshahloo et al. [8] extended the largest stability region for BCC model and Additive model by supporting hyperplanes for DMU under evaluation which all inputs and outputs of DMUs except DMU under evaluation are assumed fixed. The variations of inputs and outputs are included in four cases:

1.increase of outputs and increase of inputs,
2.decrease of outputs and the increase of inputs,
3.decrease of outputs and decrease of inputs,
4.increase of outputs and decrease of inputs.

By variation in case 4 the efficient unit preserves its efficiency because increase of outputs associated by decrease of inputs cannot worsen the efficiency of the DMU. They obtained this largest stability region by restricted their attention to the cases 1, 2 and 3 (see [8]). They consider the situation where data variations are only

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applied to the efficient DMU under evaluation and the data for the remaining DMUs are assumed fixed.In some works sensitivity analysis are based on the super efficiency DEA approach in which the efficient DMU under evaluation is not included in the reference set [9–12].
Sensitivity analysis of an inefficient DMU is studied less than sensitivity of an efficient DMU. In 1992, Charnes, et al. obtained an improvement for inefficient DMU by using Chebychev norm [5]. The model dealt with improvements in both inputs and outputs that could occur for an inefficient DMU before its statues would change to efficient. In the recent years data analysis of inefficient units has been more studied. In 2011 Jahanshahloo et al. supposed that DMU under evaluation is inefficient by the efficiency score of θ_o^* and $\theta_o^* < \alpha < 1$ which α is a fixed constant and defined by the manager.They obtained the new frontier $T_v'$with efficiency score of α. They proved that as the efficiency score of all points on the main frontier supposed to be 1, the efficiency score on the new frontier is α [13].
In this paper it is going to be found a region for those inefficient units whose efficiency score is less than a fixed constant α which is defined by the manager to obtain at least α. It means that a specific inefficient DMU with efficiency score $\theta_o^*$and $\theta_o^* < \alpha < 1$ can have an improvement in efficiency score for at least $\alpha - \theta_o^*$.This region which called "Improvement Region" (IR) is the region that the efficiency score of a specific inefficient DMU is become at least α. In this region the efficiency score of α is the least efficiency score which can be obtained by a specific inefficient *DMU*. Thus the inefficient *DMU*which is under evaluation can satisfy the decision maker and also it can be improved itself to gain a new efficiency score β_o^* and $\theta_o^* < \alpha \leq \beta_o^* \leq 1$. This new efficiency score β_o^* can be obtained by different ways such as decreasing inputs, increasing outputs or combination strategies. After defining "Improvement Region" for every inefficient unit with the usage of some theorems, it will be proved that the new efficiency score of each point of the Improvement Region for a specific inefficient DMU with efficiency score θ_o^* is β_o^* and $\alpha \leq \beta_o^* \leq 1$. ($\alpha$ is a constant which is defined by the manager). This paper proceeds as follows. The next section represents some basic DEA models. Section 3 develops a proposed method for finding"Improvement Region". Section 4 illustrates a numerical example. Section 5 presents method results using application in hospitals. and finally conclusions are given in section 6.

2 Background

Data Envelopment Analysis is a nonparametric method for evaluating efficiency of systems with multiple inputs and multiple outputs. In this section we present some basic definitions, models and concepts that will be used in other sections in DEA. Consider DMU_j, $(j = 1,\ldots,n)$, where each DMU consumes m inputs to produce s outputs. Suppose that the observed input and output vectors of DMU_j are $\boldsymbol{x}_j = (\boldsymbol{x}_{1j},\ldots,\boldsymbol{x}_{mj})$ and $\boldsymbol{y}_j = (\boldsymbol{y}_{1j},\ldots,\boldsymbol{y}_{sj})$ respectively, and let $\boldsymbol{x}_j \geq 0$ and $\boldsymbol{x}_j \neq 0$ and $\boldsymbol{y}_j \geq 0$ and $\boldsymbol{y}_j \neq 0$ (This means that all data are assumed to be negative, but at least one component of every input and output vector is positive).
The production possibility set (PPS) T_c is defined by:

$$T_c = \left\{ (\boldsymbol{x},\boldsymbol{y})\,|\boldsymbol{x} \geq \sum_{j=1}^{n} \lambda_j \boldsymbol{x}_j, \boldsymbol{y} \leq \sum_{j=1}^{n} \lambda_j \boldsymbol{y}_j, \lambda_j \geq 0, j = 1,\ldots,n \right\}$$

The above definition implies that the CCR model is as follows:

$$\begin{array}{lll} \min\ \theta & & \\ \text{s.t} & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io}, & i = 1,\ldots,m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, & r = 1,\ldots,s \\ & \lambda_j \geq 0, & j = 1,\ldots,n. \end{array} \tag{1}$$

Where DMU_o is the DMU under evaluation.
In addition, the Production Possibility Set T_v is defined by:

$$T_v = \left\{ (\boldsymbol{x},\boldsymbol{y})\,|\boldsymbol{x} \geq \sum_{j=1}^{n} \lambda_j \boldsymbol{x}_j, \boldsymbol{y} \leq \sum_{j=1}^{n} \lambda_j \boldsymbol{y}_j, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1,\ldots,n \right\}$$

The above definition implies that the BCC model is as follows:

$$\begin{array}{lll} \min\ \theta & & \\ \text{s.t} & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io}, & i = 1,\ldots,m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, & r = 1,\ldots,s \\ & \sum_{j=1}^{n} \lambda_j = 1, & \\ & \lambda_j \geq 0 & j = 1,\ldots,n. \end{array} \tag{2}$$

Definition 1(Reference Set). *for a DMU_o , we define its reference set, $E_o = \left\{ j|\lambda_j^* > 0 \right\}$ in some optimal solution to 1 or 2 [14].*

Definition 2(Pareto-Koopmans Efficiency). *DMU_o $(o \in \{1,\ldots,n\})$ is a Pareto-Koopmans Efficiency if and only if it is not possible to improve any input or output without worsening some other input or output [14].*

Definition 3.*A DMU_o is extreme efficient, if and only if it satisfies the following two conditions: [13]*

(i)It is efficient (Pareto-Koopmans Efficient).
(ii)$|E_o| = 1$.

Definition 4.*A DMU_o is non-extreme efficient, if and only if it satisfies the following two conditions: [13]*

(i)It is efficient (Pareto-Koopmans Efficient).
(ii)$|E_o| > 1$ (that is the CCR or BCC envelopment model corresponding DMU_o has alternate optimal).

Definition 5. *A DMU_o with efficiency score of θ_o^* is inefficient if and only if $\theta_o^* < 1$. [14]*

Definition 6. *H is a hyperplane if $H = \{\mathbf{z}|\mathbf{p}^t\mathbf{z} + \alpha = 0\}$ where $\mathbf{z} = (\mathbf{x}_1,..,\mathbf{x}_m,\mathbf{y}_1,...,\mathbf{y}_s)$ and $\mathbf{p}$ is the gradient of the hyperplane and α is a scalar. Hyperplane $H = \{\mathbf{z}|\mathbf{p}^t\mathbf{z} + \alpha = 0\}$ is strong if none of components of p are zero. In PPS, based on inputs and outputs of the units, DEA forms efficient surfaces consist of strong and weak efficient surfaces. These surfaces are hyperplanes consist of strong and weak hyperplanes. In DEA these strong hyperplanes are defining too.*

For more details and the method of finding strong defining hyperplane of PPS, [15].
Afterwards, to find extreme efficient DMU in BCC model, the following linear programming is solved for each efficient DMU:

$$\begin{array}{ll} \max & \gamma_o = \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j \\ \text{s.t} & \sum_{j=1}^{n} \lambda_j x_j \leq x_o \\ & \sum_{j=1}^{n} \lambda_j y_j \geq y_o \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j \geq 0, \qquad j = 1,...,n \end{array} \tag{3}$$

DMU_o is an extreme efficient in BCC model if and only if the optimal value of (3) is equal to zero. [16, 17]
Let the set of extreme efficient DMUs in T_v be E and with determining the set of E, the set of E' is defined as follows [13]:

$$E' = \left\{ (\mathbf{x}'_j, \mathbf{y}'_j) \,|\, (\mathbf{x}'_j, \mathbf{y}'_j) = \left(\frac{1}{\alpha}\mathbf{x}_j, \mathbf{y}_j \right), j \in E \right\}$$

And the new production possibility set T'_v :

$$T'_v = \left\{ (\mathbf{x}', \mathbf{y}') \,|\mathbf{x}' \geq \frac{1}{\alpha} \sum_{j\in E} \lambda_j \mathbf{x}_j, \mathbf{y}' \leq \sum_{j\in E} \lambda_j \mathbf{y}_j, \sum_{j\in E} \lambda_j = 1 \right.$$
$$\left. , \lambda_j \geq 0, j \in E \right\}$$

The sensitivity analysis of an inefficient DMU is studied less than the sensitivity of an efficient units classification and it seems to be ignored but in the recent years this issue has been more studied. In 2011 Jahanshahloo et al. supposed that DMU under evaluation is inefficient by the efficiency score of θ_o^* and $\theta_o^* < \alpha < 1$ which α is a fixed constant and defined by the manager.They obtained the new frontier . They proved that as the efficiency score of all points on the main frontier supposed to be 1, the efficiency score on the new frontier is α. Then by using different ways such as decreasing inputs, increasing outputs or combination strategies, DMU_o with efficiency score of θ_o^* can obtain efficiency score of α and has an improvement for $\alpha - \theta_o^*$ in efficiency. [13]
To illustrate the subject, suppose an inefficient DMU_o with efficiency score of θ_o^* and $(\theta^* < \alpha < 1)$ is under evaluation.T_v frontier and T'_v frontier are depicted in figure 1. The efficiency score of all points on the T_v is supposed to be 1 and on the T'_v, α.

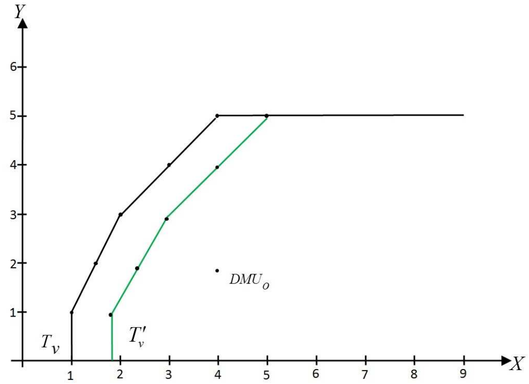

Fig. 1: T_v and T'_v frontier [13]
The efficiency score of each point on the T_v frontier is 1 (in T_v)
The efficiency score of each point on the T'_v frontier is α (in T_v)

Theorem 1. *The efficiency score of each point of E' in T_v is α.*

Proof. See [13]

Attention 1 *There is one- to- one correspondence between E and E'.*

Proof. See [13].

Attention 2 *There is one-to-one correspondence between T_v and T'_v frontier points.*

Proof. See [13]

Theorem 2. *The efficiency score of each point on the T'_v frontier is α in T_v.*

Proof. See [13]

The region for every inefficient unit whose efficiency score is smaller than α is called "Improvment Region". The efficiency score of DMU_o with efficiency score θ_o^* and $\theta_o^* < \alpha < 1$ has an improvement for at least $\alpha - \theta_o^*$. It will be looked more closely at the process in the next section.

3 Proposed method

In this method, it is supposed that DMU_o which is under evaluation is inefficient with efficiency score of θ_o^* *and* $\theta_o^* < \alpha < 1$ and α is a constant which is defined by the decision maker. The method to improve an inefficient DMU_o to obtain an exactly efficiency score of α has been developed by Jahanshahloo et al. [13] and it was extensively discussed in section 2. In the sequel, it is going to be defined a region which is called the "Improvement Region". In this region the efficiency score of α is the least efficiency score which can be obtained by

a specific inefficient DMU_o.Therefore the inefficient DMU_o can satisfy the decision maker and also it can be improved itself to gain a new efficiency score β_o^* and $\theta_o^* < \alpha \leq \beta_o^* \leq 1$. This new efficiency score β_o^* can be obtained by different ways such as decreasing inputs, increasing outputs or combination strategies.
To illustrate the subject, suppose that a specific inefficient DMU_o with efficiency score θ_o^* and $\theta_o^* < \alpha < 1$. Figure 2 shows Improvement Region (IR). It is the area consisting of the bold line ABF plus the vertical line FP and bold lines PQ, QR, horizontal line AR and all points between these line segments.

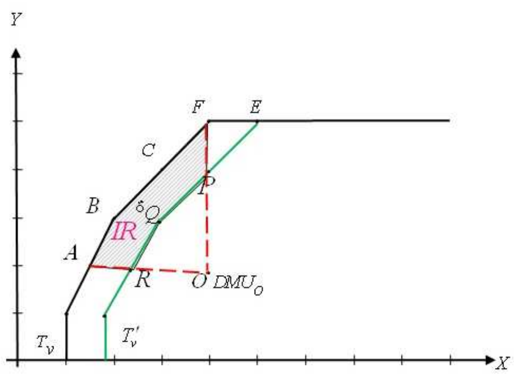

Fig. 2: Representing “mprovement Region” (IR) for DMU_o (inefficient unit)

At first glance, it can be used model (1) or model (2) to evaluate$DMU_j\,(j=1,\ldots,n)$ and to be found all extreme efficient $DMUs$ for T_v and T_v' by using model (3). by attention 2 there is one-to-one correspondence between T_v and T_v' frontier points.
By definition 6 it can be found all strong supporting hyperplanes of product possibility set (PPS) [15]. Let be H_l The strong supporting hyperplanes of T_v frontier, with $l=1,\ldots,k$ given by:
$H_l = \{\boldsymbol{z}|\boldsymbol{p}^t\boldsymbol{z}_l + \alpha_l = 0,\, l=1,\ldots,k\}$ where $\boldsymbol{z} = (\boldsymbol{x}_1,..,\boldsymbol{x}_m,\boldsymbol{y}_1,\ldots,\boldsymbol{y}_s)$ and $\boldsymbol{p}$ is the gradient of the hyperplane and α is a scalar. Corresponding to the hyperplane H_l, the half spaces H_l^- *and* H_l^+ are defined as follow:

$$H_l^- = \left\{\boldsymbol{z}|\boldsymbol{p}^t\boldsymbol{z}_l + \alpha_l \leq 0, l=1,..,k\right\},$$
$$H_l^+ = \left\{\boldsymbol{z}|\boldsymbol{p}^t z_l + \alpha_l \geq 0, l=1,..,k\right\}$$

Similarly,the procedure will be repeated to find all of the supporting hyperplanes of T_v' frontier. They can be represented as $H_1',\ldots,H_k'$ which expressed as follows:

$$H_l^{'-} = \left\{\boldsymbol{z}'|\boldsymbol{p}'^t\boldsymbol{z}_l' + \alpha_l' = 0, l=1,..,k\right\}$$

where $\boldsymbol{z}' = (\boldsymbol{x}_1',..,\boldsymbol{x}_m',\boldsymbol{y}_1',\ldots,\boldsymbol{y}_s')$ and $\boldsymbol{p}'$ is the gradient of the hyperplane and α' is scalar.

Corresponding defining hyperplane H_l', the half space $H_l^{'-}$, *, and* , ,$H_l^{'+}$ are expressed as follows:

$$H_l^{'-} = \left\{\boldsymbol{z}'|\boldsymbol{p}'^t\boldsymbol{z}_l' + \alpha_l' \leq 0, l=1,..,k\right\},$$
$$H_l^{'+} = \left\{\boldsymbol{z}'|\boldsymbol{p}'^t z_l' + \alpha_l' \geq 0, l=1,..,k\right\}$$

Figure 2 shows the “Dominated Region” (D) where in red is the area that crosses the both frontier $T_v\, and\, T_v'$. This region is consisting of all points that are defined as follows:

$$D = \{(\boldsymbol{x},\boldsymbol{y})\,|\,(\boldsymbol{x},\boldsymbol{y}) \in T_v, (-\boldsymbol{x},\boldsymbol{y}) \geq (\boldsymbol{x}_o,\boldsymbol{y}_o)\}$$

Referring to defined half spaces, the set S is given by:

$$S_1 = \bigcap_{l=1}^{k} H_l^- \;, S_2 = \bigcup_{l=1}^{k} H_l^{'+} \;, S = \left(S_1 \bigcap S_2\right)$$

Finally “mprovement Region” (IR) is determined by:

$$IR = (S \cap D)$$

Theorem 3.*The efficiency score of each point of ”Improvement Region” (IR) is β^* that $\alpha \leq \beta^* \leq 1$.*

*Proof.*Let $\overline{M}$ with coordinates $(X_{\overline{M}}, Y_{\overline{M}})$ be an arbitrary point in IR as It is shown in Figure 3. There are three cases to discuss .First if $\overline{M}$ is a point on T_v frontier, the efficiency score $\beta^* = 1$. Second if $\overline{M}$ is a point on T_v' frontier by theorem 2 the efficiency score $\beta^* = \alpha$. Third supposed that $\overline{M}$ is a point of area between two frontiers. Respecting to point $\overline{M}$ there is a point like M' with coordinates $(X_{M'}, Y_{M'})$ on the T_v' frontier and there is a point like M with coordinates (X_M, Y_M) on the T_v frontier such that $(X_{\overline{M}}, Y_{\overline{M}}) = (X_{M'} - \varepsilon, Y_{M'})$ *where* $\varepsilon > 0$. Then, the point M' is evaluated by the BCC model in T_v frontier as follows:

$$\begin{array}{ll} \min & \theta_{M'} \\ \text{s.t} & \sum_{j\in E} \lambda_j \boldsymbol{x}_j \leq \theta_{M'} \boldsymbol{x}_{M'} \\ & \sum_{j\in E} \lambda_j \boldsymbol{y}_j' \geq \boldsymbol{y}_{M'} \\ & \sum_{j\in E} \lambda_j = 1 \\ & \lambda_j \geq 0, \quad j \in E. \end{array}$$

Theorem 2 asserts that there exists a feasible solution $\left(\theta_{M'}^* = \alpha,\, \lambda_M = 1,\, \lambda_j = 0,\, j \neq M\right)$ which is held in constraints. From the first constraint, it is concluded that $X_M = \alpha X_{M'}$ (1). Because $X_{\bar{M}} = X_{M'} - \varepsilon$, $\varepsilon \succ 0$, for having equation 1, α should increase. Morever, we know that

$$\begin{array}{l} (I)\; X_{\overline{M}} = X_{M'} - \varepsilon \\ (II) Y_{\overline{M}} = Y_{M'} \end{array}$$

If the point $\overline{M}$ is evaluated by the BCC model in T_v frontier, then by (I) and (II) $\beta^* > \alpha$ is obtained and it is complete the proof.

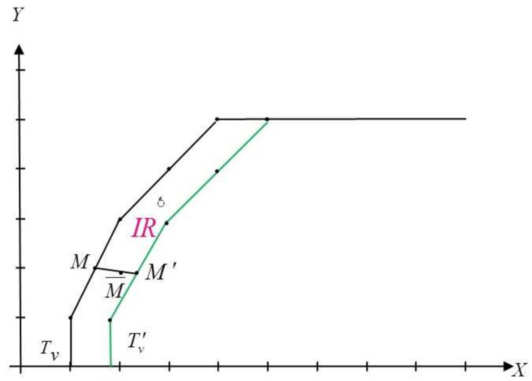

Fig. 3: $\overline{M}$ is an orbitrary point in (IR)

The procedure of finding "Improvement Region" can be expressed by an algorithm as follows:

Step1. Obtain all extreme points of T_v frontier by using model 2.3

Step2. Obtain all extreme points of T'_v frontier by using model 2.3

Step3. Calculate all supporting hyperplanes of T_v frontier which are named $H_1, H_2, \ldots, H_k$ and respectively for T'_v frontier which are called $H'_1, H'_2, \ldots, H'_k$ by using proposed method by Jahanshahloo et al [15].

Step4. Construct all half spaces H_l^- with $l = 1, \ldots, k$ given by:

$$H_l^- : \{\boldsymbol{z} | \boldsymbol{p}^t \boldsymbol{z}_l + \alpha_l \leq 0, l = 1, .., k\}.$$

Similarly according to define hyperplane H'_l, the half space $H_l^{'+}$ is given by:

$$H_l^{'+} : \{\boldsymbol{z}' | \boldsymbol{p}'^t \boldsymbol{z}'_l + \alpha'_l \geq 0, l = 1, .., k\}.$$

Step5. Determined the "Dominated Region" (D) as follows:

$$D = \{(\boldsymbol{x}, \boldsymbol{y}) \,|\, (\boldsymbol{x}, \boldsymbol{y}) \in T_v, (-\boldsymbol{x}, \boldsymbol{y}) \geq (\boldsymbol{x}_o, \boldsymbol{y}_o)\}$$

Step6. Formulate the region which is called "Improvement Region" (IR) as follows:

$$S_1 = \cap_{l=1}^k H_l^-, \;\; S_2 = \cup_{l=1}^k H_l^{'+},$$

$$S = (S_1 \cap S_2) = \left[\left(\cap_{l=1}^k H_l^- \right) \cap \left(\cup_{l=1}^k H_l^{'+} \right) \right]$$

$$IR = (S \cap D) = \left[\left(\cap_{l=1}^k H_l^- \right) \cap \left(\cup_{l=1}^k H_l^{'+} \right) \right] \cap \left\{ (\boldsymbol{x}, \boldsymbol{y}) \,|\, (\boldsymbol{x}, \boldsymbol{y}) \in T_v, (-\boldsymbol{x}, \boldsymbol{y}) \geq (\boldsymbol{x}_o, \boldsymbol{y}_o) \right\}$$

4 Numerical example.

In this section we are going to illustrate the proposed method by numerical example in CCR and BCC models.

4.1 Example (using BCC Model):

Consider a system of 6 DMUs with a single output and input as show in figure 4. Data is given in Table 1. Assume $\alpha = 0.800$

Table 1: Data of numerical example 4.1

DMUs	*A*	*B*	*C*	*G*	*E*	*F*
X	*1*	*2*	*4*	*3*	*5*	*3*
Y	*1*	*3*	*5*	*3*	*2*	*4*
Results	*1.000*	*1.000*	*1.000*	*0.670*	*0.300*	*1.000*

The extreme efficient DMUs are A, B and C. the set $E = \{A(1,1), B(2,3), C(4,5)\}$. F is non- extreme efficient DMU and E and G are inefficient DMUs.
The Strong hyperplanes by using[15] are:

$$\overline{AB}: \; H_1 = \{(\boldsymbol{x}, \boldsymbol{y}) \,|\, \boldsymbol{y} - 2\boldsymbol{x} = -1\},$$
$$\overline{BC}: \; H_2 = \{(\boldsymbol{x}, \boldsymbol{y}) \,|\, \boldsymbol{y} - \boldsymbol{x} = 1\}$$

DMU_E is inefficient with efficiency score $\theta_E^* = 0.3000 < \alpha = 0.800$. Now the set E' and Strong hyperplanes of T'_v frontier are defined as follows:

$$E' = \{A'(1.25, 1), B'(2.5, 3), C'(5, 5)\}$$
$$\overline{A'B'}: H'_1 = \{(\boldsymbol{x}, \boldsymbol{y}) \,|2\boldsymbol{x} - 1.25\boldsymbol{y} = 1.25\},$$
$$\overline{B'C'}: H'_2 = \{(\boldsymbol{x}, \boldsymbol{y}) \,|2\boldsymbol{x} - 2.5\boldsymbol{y} = -2.5\}$$

Figure 4.portrays the Improvement Region for inefficient DMU_E. The region represented by line segments $(\overline{HB})$, $(\overline{BC})$, $(\overline{CC'})$, $(\overline{C'B})$, $(\overline{B'E'})$ and $(\overline{E'H})$.

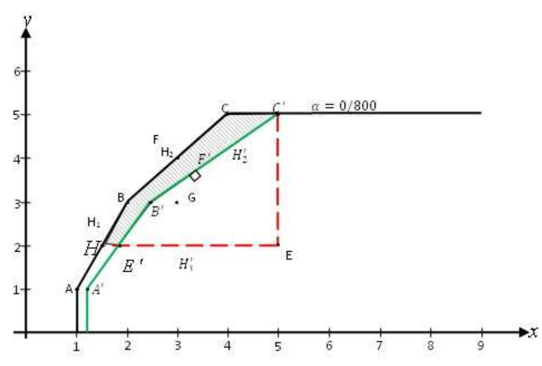

Fig. 4: Improvement Region for inefficient DMU_E

$$S_1 = H_1^- \bigcap H_2^- = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{y} - 2\boldsymbol{x} \leq -1, \boldsymbol{y} - \boldsymbol{x} \leq 1\},$$
$$S_2 = H_1^{'+} \bigcup H_2^{'+} = \{(\boldsymbol{x}, \boldsymbol{y}) | 2\boldsymbol{x} - 1.25\boldsymbol{y} \geq 1.25, 2\boldsymbol{x} - 2.5\boldsymbol{y} \geq -2.5\}$$
$$S = S_1 \bigcap S_2 = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{y} - 2\boldsymbol{x} \leq -1, \boldsymbol{y} - \boldsymbol{x} \leq 1\} \bigcap \{(\boldsymbol{x}, \boldsymbol{y}) | 2\boldsymbol{x} - 1.25\boldsymbol{y} \geq 1.25, \; 2\boldsymbol{x} - 2.5\boldsymbol{y} \geq -2.5\}$$

Dominated region D is defined as follows which is restricted to:

$$D = \{(\boldsymbol{x},\boldsymbol{y}) \,|\, (\boldsymbol{x},\boldsymbol{y}) \in T_v,\ (-\boldsymbol{x},\boldsymbol{y}) \geq (\boldsymbol{x}_E, \boldsymbol{y}_E)\}$$

Finally the “Improvement Region” is defined as follows:

$$\begin{aligned} IR &= (S \cap D) \\ &= \{(\boldsymbol{x},\boldsymbol{y}) | \boldsymbol{y} - 2\boldsymbol{x} \leq -1, \boldsymbol{y} - \boldsymbol{x} \leq 1\} \\ &\bigcap \{(\boldsymbol{x},\boldsymbol{y}) | 2\boldsymbol{x} - 1.25\boldsymbol{y} \geq 1.25,\ 2\boldsymbol{x} - 2.5\boldsymbol{y} \geq -2.5\} \\ &\bigcap \{(\boldsymbol{x},\boldsymbol{y}) \,|\, (\boldsymbol{x},\boldsymbol{y}) \in T_v,\ (-\boldsymbol{x},\boldsymbol{y}) \geq (\boldsymbol{x}_E, \boldsymbol{y}_E)\} \end{aligned}$$

One of The points in the frontier $T_v'$with efficiency score of $\alpha = 0.8$ is $E' = (1.875, 2)$. The other points such as $H = (1.75, 2.1)$, $K = (1.80, 2.05)$, $L = (1.75, 2.5)$, $M = (4, 4.5)$ and $N = (3, 3.9)$ are in the ”Improvment Region” with efficiency score 0.89, 0.89, 0.85, 0.93, 0.88 and 0.9 respectively. All of these points satisfy in the ”Improvment Region” and half spaces.Thus these points are points that their input is less than input of DMU_E and their output is more than output of DMU_E. Therefore efficiency scores of these points are better than efficiency score DMU_E.

4.2 Example (using CCR Model):

Consider a system of 4 DMUs with a single output and 2 inputs as show in figure 5. Data is given in Table 2. Assume $\alpha = 0.800$

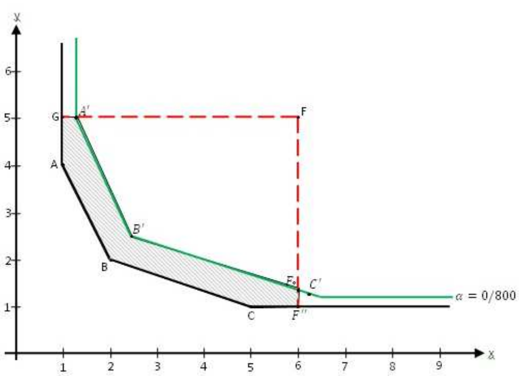

Fig. 5: Data set in T_v

The extreme efficient DMUs are A, B and C. the set $E = \{A(1,4,1), B(2,2,1), C(5,1,1)\}$. The inefficient DMU_F is under evaluation with efficiency score $\theta^* = 0.380 < \alpha = 0.800$

The Strong Supporting hyperplanes by using [15] are:

$$\overline{AB} : H_1 = \{(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}) \,| 6\boldsymbol{y} - 2\boldsymbol{x}_1 - \boldsymbol{x}_2 = 0\}$$
$$\overline{BC} : H_2 = \{(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}) \,| 8\boldsymbol{y} - \boldsymbol{x}_1 - 3\boldsymbol{x}_2 = 0\}$$

Table 2: Data of numerical example 4.1

DMUs	*A*	*B*	*C*	*F*
x_1	*1*	*2*	*5*	*6*
x_2	*4*	*2*	*1*	*5*
y_1	*1*	*1*	*1*	*1*
Results	*1.000*	*1.000*	*1.000*	*0.380*

Now the set E' and Strong hyperplanes of T_v' are defined as follows:

$$\begin{aligned} &E' = \{A'(1.25, 5, 1), B'(2.5, 2.5, 1), C'(6.25, 1.25, 1)\} \\ &\overline{A'B'} : H_1' = \{(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}) \,| -2.5\boldsymbol{x}_1 - 1.25\boldsymbol{x}_2 + 9.375\boldsymbol{y} = 0\} \\ &\overline{B'C'} : H_2' = \{(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}) \,| -1.25\boldsymbol{x}_1 - 3.75\boldsymbol{x}_2 + 12.5\boldsymbol{y} = 0\} \end{aligned}$$

$$\begin{aligned} S_1 &= H_1^- \bigcap H_2^- \\ &= \{(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}) \,| 6\boldsymbol{y} - 2\boldsymbol{x}_1 - \boldsymbol{x}_2 \leq 0, 8\boldsymbol{y} - \boldsymbol{x}_1 - 3\boldsymbol{x}_2 \leq 0\} \\ S_2 &= H_1'^+ \bigcup H_2'^+ \\ &= \{(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}) \,| -2.5\boldsymbol{x}_1 - 1.25\boldsymbol{x}_2 + 9.375\boldsymbol{y} \geq 0 \\ &\quad , -1.25\boldsymbol{x}_1 - 3.75\boldsymbol{x}_2 + 12.5\boldsymbol{y} \geq 0\} \\ S &= S_1 \bigcap S_2 \end{aligned}$$

$$D = \{(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}) \,|\, (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}) \in T_v, (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}) \geq (\boldsymbol{x}_F, \boldsymbol{x}_F, \boldsymbol{y}_F)\}$$

Finally the Improvement Region (IR) is defined as follows:

$$IR = (S \cap D)$$

The points, such as $D = (2, 2.5, 1)$, $E = (3, 2, 1)$, $K = (2, 3.5, 1)$ and $L = (1.5, 3, 2, 1)$ are in the ”Improvment Region” with efficiency score 0.92, 0.89, 0.80 and 0.97 respectively. All of these points satisfy in the ”Improvment Region” and half spaces.Thus these points are points that their inputs are less than inputs of DMU_F. Therefore efficiency scores of these points are better than efficiency score DMU_F. It is clear that $M = (4, 3, 1)$ doesn’t satisfy in the ”Improvement Region” and half spaces, however its inputs are less than inputs DMU_F.

5 Application in hospitals

The examples used in previous section have been very limited in the number of inputs and outputs used. This made it possible to use simple graphic displays to clarify ”Improvment Region” but, of course, this was at the expanse of the realism needed to deal with the multiple inputs and multiple outputs. Hence, we illustrate our approach in finding ”Improvment Region” for data set 12 hospitals. A list of hospitals used is provided in Table 3. In this report, there are number of doctors, number of nurses, number of outpatients and inpatients which number of doctors and nurses are considered as inputs and number of outpatients and inpatients are as outputs. Assume $\alpha = 0.70$. The example is received from [14] and is about evaluation the relative efficiency of 12 hospitals.

model 2 (CCR) is used for efficiency evaluation. The extreme efficient hospitals are A, B and D. The set $E = \{A(20,151,100,90)), B(19,131,150,50), D(27,104,180,72)\}$. The other hospitals are inefficient. $hospital_E$ is inefficient with efficiency score $\theta_E^* = 0.21 < \alpha = 0.70$. Now set E' is defined as follows:

$$E' = \left\{A'(28.4, 214.42, 100, 90), B'(26.98, 186.2, 150, 50), \right.$$
$$\left. D'(38.34, 147.68, 180, 72)\right\}$$

Table 3: Data of Application in hospitals

Hospitals	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
Doctors	*20*	*19*	*25*	*27*	*55*	*55*	*33*	*31*	*30*	*50*	*53*	*38*
Nurses	*151*	*131*	*160*	*104*	*285*	*255*	*235*	*206*	*244*	*268*	*306*	*284*
Outpatient	*100*	*150*	*160*	*180*	*45*	*230*	*220*	*152*	*190*	*250*	*260*	*250*
Inpatient	*90*	*50*	*55*	*72*	*39*	*90*	*88*	*80*	*100*	*100*	*147*	*120*
Results	*1.0*	*1.0*	*0.83*	*1.0*	*0.21*	*0.61*	*0.90*	*0.76*	*0.96*	*0.71*	*0.81*	*0.96*

$hospital_E$ is inefficient and it can scarcely reach to the efficient frontier and achieving the score 1 in efficiency but it can obtain an efficiency score closed to 1 namely $\alpha = 0.70$ and defined by the manager of hospital. Thus this hospital can satisfy the manager of hospital and it can be improved itself to gain a new efficiency score more than 0.7. In order to it should decrease inputs or increase outputs or combination them. This developing places in the "Improvement Region". One of The points in the frontier $T_v'$with efficiency score of $\alpha = 0.70$ is $K = (26.5, 149, 125, 56)$. Some of the suggestions are designated which $0.7 < \theta_{new}^* < 1$ as following:
$L = (20.5, 150, 160, 55)$, $M = (24.3, 140, 135, 59)$, $N = (27.2, 171, 160, 75)$, $P = (35.2, 140.81, 190, 81)$ with efficiency score 0.75, 0.81, 0.87 and 0.84, respectively.

6 Conclusion

In this paper, we developed a new approach for the sensitivity analysis of an inefficient unit whose efficiency score is less than α. The presented method in this paper specifies an"Improvement Region" for an inefficient DMU. This region for the inefficient DMU has an improvement for at least $\alpha - \theta^*$. By choosing different strategies the specific inefficient DMU can improve itself to the level α That is defined by the manager and also to the level that is greater than α. It means that an inefficient DMU can obtain more contentment and satisfaction of the manager.
There are many places such as schools, universities, hospitals, banks, companies and etc. whose staffs should have at least a defined efficiency score so those people with efficiency score less than the least should come up with the level by themselves. By the proposed method, the efficiency score of a specific inefficient DMU changes to at least a defined efficiency score. Sometimes a change in strategy in input (input decreasing) or a change in output (output increasing) or simultaneous changes in input and output is impossible but the Improvement Region is available for each inefficient DMU and the manager can examine different strategies and decided more explicitly for the future.

References

[1] A. Charnes,W. W. Cooper, E. Rhodes, *Measuring the efficiency of decision making units European*, Journal of operational Research ,2,429-444 (1987)

[2] R.D. Banker,A. Charnes W. W. Coopper,*Some models for estimating technicaland scale inefficiencies in data envelopment analysis*, Management Science,30 (9) , 1078-1092 (1984)

[3] A. Charnes, W.W. Cooper, A.Y. Lewin, R.C. Morey, J. Rousseau, *Sensitivity and Stability analysis in DEA*, Annals of Operation Research, 2 139-156(1985)

[4] A. Charnes, L. Neralic, *Sensitivity analysis of the additive model in data envelopment analysis*, European Journal of Operational Research ,48, 332-341 (1990)

[5] A. Charnes, S.Haag, P. Jaska, J. Semple, *Sensitivity of efficiency classification in the additive model of data envelopment analysis*, International Journal of system Science, 23,789-798 (1992)

[6] J. Zhu, *Robustness of the efficient DMUs in data envelopment analysis*, European Journal of Operation Research, 90 ,451-460. (1996)

[7] L.M. Seiford, J. Zhu, *Stability regions for maintaining efficiency in data envelopment analysis*, European Journal of Operational Research ,108 ,127-139 (1998)

[8] G.R.Jahanshahloo, F.HosseinzadehLotfi,,M. Sanei, N.Shoja, G. Tohidi, *Sensitivity and stability analysis in data envelopment analysis*, Applied Mathematics and Computation ,169, 887-896 (2005)

[9] ValterBolljunicic, *Sensitivity analysis of an efficient DMU in DEA model with variable returns to scale (VRS)*, Journal of Productivity Analysis ,25 ,173-192 (2006)

[10] A. Charnes, S. Haag, P. Jaska, J. Semple, *Sensitivity of efficiency classifications in the additive model of data envelopment analysis*, International Journal of Systems Science ,23 ,789-798 (1992): Operation Research 2, 429-444.

[11] A. Charnes, J. Rousseau, J. Semple, *Sensitivity and stabilityof efficiency classifications in data envelopment analysis*, Journal of Productivity Analysis, 7 ,5-18 (1996)

[12] Franklin Dexter, Liam O Neill, Lei Xin, Johannes Ledolter, *Sensitivity of Super-efficient data envelopment analysis results to individual decision-making units: an example of surgical workload by specialty*, Health Care Management Science ,11,307-318 (2008)

[13] G.R.Jahanshahloo, F. HosseinzadehLotfi, N.Shoja, A. GholamAbri M. FallahJelodar, *Sensitivity analysis of inefficient units in data envelopment analysis*, Mathematical and Computer Modeling, 53 ,587-596 (2011)

[14] W.Cooper, L. Seiford, K. Tone, *Introduction to Data Envelopment Analysis and its uses with DEA-Solver Software and References* (2005).

[15] G.R. Jahanshahloo, F. HosseinzadehLotfi,,H. ZhianiReza,F. RezaiBalf, *Finding strong defining hyperplanes of Production Possibility Set*, European Journal of Operational Research, 177, 42-54 (2007)
[16] Chames, A., W. W. Cooper and R. M. Thrall, *Classifying and Characterizing Efficiencies and Inefficiencies in Data Envelopment Analysis*, Oper. Res. Letters, 5, 3 , 105-110.(1986)
[17] Chames, A., W. W. Cooper and R. M. Thrall, *A Structure for Classifying and Characterizing Efficiency and Inefficiency in Data Envelopment Analysis*,J. Productivity Analysis, 2 , 197-237(1991)

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