Information Sciences Letters

Volume 12
Issue 9 *Sep. 2023*

Article 8

2023

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A. A. Abdellatif
Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt, dahy@azhar.edu.eg

E. Dahy
Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt, dahy@azhar.edu.eg

A. M. Zahran
Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt, dahy@azhar.edu.eg

Information Sciences Letters
An International Journal

http://dx.doi.org/10.18576/isl/120907

Mathematical analysis of HIV/HTLV-I co-infection model with saturated incidence rate

A. A. Abdellatif, E. Dahy and A. M. Zahran*

Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt

Received: 28 Feb. 2022, Revised: 16 Sep. 2022, Accepted: 20 Sep. 2022
Published online: 1 Sep. 2023

Abstract: Direct contact with specific contaminated body fluids is how both the human immunodeficiency virus (HIV) and the human T-lymphotropic virus type I (HTLV-I) are transmitted from one person to another. Therefore, the two viruses can co-infect same person. In the literature all the HIV/HTLV-I co-infection models assume that the infection rate is given by bilinear incidence. However, for high concentration of pathogens, the bilinear incidence is not suitable. Therefore, this study will focus on the dynamical behavior of an HIV/HTLV-I co-infection model with saturated incidence. The model includes the effect of Cytotoxic T lymphocytes (CTL) immune response. Through the non-negativity and boundedness of the solutions, we demonstrated that our proposed model is biologically acceptable. We calculate the threshold parameters which determine when the equibrium point exists and when it is globally asymptotically stable. Utilizing the Lyapunov function and Lyapunov-LaSalle asymptotic stability, we demonstrate the global asymptotic stability of all equilibrium. We performed numerical simulations to confirm the analytical solutions. The effect of saturation on The dynamics of HIV/HTLV-I co-infection are discussed.

Keywords: HIV/HTLV-I co-infection, global stability, CTLmediated immune response, Saturation Incidence, Lyapunov function.

1 Introduction

In recent years, many viruses affect the human health and cause fatal diseases. Examples of these viruses are human immunodeficiency virus (HIV), Type I human T-lymphotropic virus (HTLV-I), hepatitis C virus (HCV) and hepatitis B virus (HBV). A retrovirus called HIV infects and deactivates vulnerable $CD4^{+}$T cells. The advanced stage of HIV infection is known as acquired immunodeficiency syndrome (AIDS). Direct contact with specific bodily fluids (blood, semen (cum), pre-seminal fluid (pre-cum), vaginal fluids and rectal fluids) from an HIV-infected person can result in HIV infection. The antiviral medications that are currently available can dramatically reduce HIV replication, but they cannot completely eradicate HIV from the body. Globally, around 37.9 million individuals were living with HIV in 2018, and 770,000 people died from the disease, according to data from the WHO's Global Health Observatory (GHO 2018) [1]. The mathematical modeling of HIV infection within the host has advanced significantly during the past few decades . Various mathematical models of pathogen infection have been developed and tested by mathematical biologists over the last few decades [2]. These studies can aid in the understanding of pathogen dynamics within the host and the development of drug therapy strategies. Cytotoxic T lymphocytes (CTL) play a significant role in pathogen dynamics. CTLs attack the infected cells in order to eliminate or control the infection. The first mathematical model illustrating the impact of the CTL immune response on pathogen infection was reported by Nowak and Bangham [3]. Furthermore, stability analysis has emerged as one of the most important and useful methods for better understanding HIV dynamics within the host.

Mathematical modelling and analysis of HTLV-I infection has attracted the interest of many researchers, as shown in a variety of papers [4]-[5], It acts as a model for the kinetics of the CTL-mediated immune response in the host for HTLV-I [2]. HTLV-I is a retrovirus that infects the $CD4^{+}$T cells and can progress to serious diseases, adult T-cell leukemia (ATL) and HTLV-I-associated myelopathy/tropical spastic paraparesis (HAM/TSP).

Over the last decade, concurrent HIV and HTLV-I infection, as well as the aetiology of their disease outcomes and pathogenic, has become a global health concern [6]. This highlighted the significance of researching HTLV-I/HIV co-infection [7]. HIV and HTLV-I predominantly affect $CD4^{+}$ T cells, however these viruses exhibit distinct biological

* Corresponding author e-mail: E-mail: dahy@azhar.edu.eg

behaviours that have a variety of effects on ultimately and host immunity lead to a variety of clinical diseases [8]. Elaiw and AlShamrani [9] developed a model that describes the co-infection of the viruses HIV-I and HTLV-I:

$$\begin{aligned}
\dot{\mathscr{A}} &= \eth - \alpha\mathscr{A} - \eta_1\mathscr{A}\mathscr{K} - \eta_2\mathscr{A}\mathscr{U},\\
\dot{\mathscr{D}} &= \eta_1\mathscr{A}\mathscr{K} - a\mathscr{D} - \mu_1\mathscr{W}\mathscr{D},\\
\dot{\mathscr{U}} &= \varphi\eta_2\mathscr{A}\mathscr{U} - \delta\mathscr{U} - \mu_2\mathscr{Z}\mathscr{U},\\
\dot{\mathscr{K}} &= \xi\mathscr{D} - \varepsilon\mathscr{K},\\
\dot{\mathscr{W}} &= \sigma_1\mathscr{W}\mathscr{D} - \varsigma_1\mathscr{W},\\
\dot{\mathscr{Z}} &= \sigma_2\mathscr{Z}\mathscr{U} - \varsigma_2\mathscr{Z},
\end{aligned}$$

where $(\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z}) = (\mathscr{A}(t),\mathscr{D}(t),\mathscr{U}(t),\mathscr{K}(t),\mathscr{W}(t),\mathscr{Z}(t))$. The model describes the interaction of six compartments among time t; susceptible CD4$^+$T cells $\mathscr{A}$, active HIV-infected cells $\mathscr{D}$, Tax-expressing (active) HTLV-infected cells $\mathscr{U}$, free HIV particles $\mathscr{K}$, HIV-specific CTLs $\mathscr{W}$ and HTLV-specific CTLs $\mathscr{Z}$. The constant $\eth$ is the rate of creation of sensitive CD4$^+$T cells. Virus-to-cell transfer causes the HIV virions to replicate. The rate at which new infectious viruses emerge as a result of virus-to-cell interaction between free HIV particles and vulnerable CD4$^+$T cells is defined to as $\eta_1\mathscr{A}\mathscr{K}$. It is presumed that new infectious diseases emerge through cell-to-cell contact between susceptible CD4$^+$T cells and Tax-expressing HTLV-infected cells at a rate of $\eta_2\mathscr{A}\mathscr{U}$. A fraction $\varphi \in [0,1]$ of the newly HTLV-infected CD4$^+$T cells survive the antibody response.The death rates of Tax-expressing HTLV-infected cells and HIV-infected cells as a result of their unique immunity are denoted by the terms $\mu_1\mathscr{W}\mathscr{D}$ and $\mu_2\mathscr{Z}\mathscr{U}$. The proliferation rates of HIV-specific CTLs and HTLV-specific CTLs are given by $\sigma_1\mathscr{W}\mathscr{D}$ and $\sigma_2\mathscr{Z}\mathscr{U}$, respectively. The rate of free HIV virus generation is $\xi\mathscr{D}$. The death rates of the compartments $\mathscr{A}$, $\mathscr{D}$, $\mathscr{U}$, $\mathscr{K}$, $\mathscr{W}$ and $\mathscr{Z}$ are given by $\alpha\mathscr{A}$, $a\mathscr{D}$, $\delta\mathscr{U}$, $\varepsilon\mathscr{K}$, $\varsigma_1\mathscr{W}$ and $\varsigma_2\mathscr{Z}$, respectively. Table 1 provides a summary of all parameters and their meanings.

We observe that, in the presence of a high pathogen concentration, the bilinear incidence rate does not effectively represent the dynamics of the virus. As a result, it is reasonable to model the infection rate a saturated incidence [10]. As a result, the goal of this paper is to develop an HIV/HTLV-I co-infection with saturation. By demonstrating that the model's solutions are nonnegative and bounded, we prove that it is well-posed. We determine a set of threshold parameters that control the equilibria's existence and stability. By creating the Lyapunov function and using Lyapunov-LaSalle asymptotic stability, all equilibria are shown to be globally stable (L-LAS)[11]. To illustrate the theoretical results, we run a numerical simulations.

2 Model formulation

In this part, The dynamics of HIV/HTLV-I with saturation incidence are described by a model that we study as:

$$\dot{\mathscr{A}} = \eth - \alpha\mathscr{A} - \frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}} - \frac{\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}}, \tag{1}$$

$$\dot{\mathscr{D}} = \frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}} - a\mathscr{D} - \mu_1\mathscr{D}\mathscr{W}, \tag{2}$$

$$\dot{\mathscr{U}} = \frac{\phi\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}} - \delta\mathscr{U} - \mu_2\mathscr{U}\mathscr{Z}, \tag{3}$$

$$\dot{\mathscr{K}} = \xi\mathscr{D} - \varepsilon\mathscr{K}, \tag{4}$$

$$\dot{\mathscr{W}} = \sigma_1\mathscr{D}\mathscr{W} - \varsigma_1\mathscr{W}, \tag{5}$$

$$\dot{\mathscr{Z}} = \sigma_2\mathscr{U}\mathscr{Z} - \varsigma_2\mathscr{Z}. \tag{6}$$

where α_1 and α_2 are the saturated incidence rate constants.

3 Biologically realistic domain

Let $¥_j > 0,\ j = 1,...,5$ and define

$$\begin{aligned}
\mathfrak{I} = \{&(\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z}) \in \Re^6_{\geq 0} : 0 \leq \mathscr{A}(t),\mathscr{D}(t) \leq ¥_1, 0 \leq \mathscr{U}(t) \leq ¥_2,\ 0 \leq \mathscr{K}(t) \leq ¥_3,\\
&0 \leq \mathscr{W}(t) \leq ¥_4,\ 0 \leq \mathscr{Z}(t) \leq ¥_5\}
\end{aligned}$$

Proposition 1. The compact set $\mathfrak{I}$ is positively invariant for system (1)-(6).
Proof. We have

$$\dot{\mathscr{A}}\mid_{\mathscr{A}=0}=\eth>0,\quad \dot{\mathscr{D}}\mid_{\mathscr{D}=0}=\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}\geq 0\ \text{ for each }\ \mathscr{A},\mathscr{K}\geq 0,\quad \dot{\mathscr{U}}\mid_{\mathscr{U}=0}=0,$$
$$\dot{\mathscr{K}}\mid_{\mathscr{K}=0}=\xi\mathscr{D}\geq 0 \text{ for each } \mathscr{D}\geq 0,\quad \dot{\mathscr{W}}\mid_{\mathscr{W}=0}=0,\quad \dot{Z}\mid_{\mathscr{Z}=0}=0.$$

This insures that, $(\mathscr{A}(t),\mathscr{D}(t),\mathscr{U}(t),\mathscr{K}(t),\mathscr{W}(t),\mathscr{Z}(t))\in\mathfrak{R}^6_{\geq 0}$ for each $t\geq 0$ when $(\mathscr{A}(0),\mathscr{D}(0),\mathscr{U}(0),\mathscr{K}(0),\mathscr{W}(0),\mathscr{Z}(0))\in\mathfrak{R}^6_{\geq 0}$. To show the boundedness of all state variables, we define the function:

Table 1: Modeling parameters (1)-(6) and their interpretations .

Parameter	Description
$\eth$	Recruiting efficiency for the susceptible CD4$^+$T cells
α	Constant natural deaths for the sensitive CD4$^+$T cells
η_1	Constant incidence rate virus-cell between free HIV particles and susceptible CD4$^+$T cells
η_2	Constant incidence rate cell-cell between Tax-expressing HTLV-infected cells and susceptible CD4$^+$T cells
α_1	The parameter of the saturation infection rate between susceptible CD4$^+$T cells and unbound HIV particles
α_2	The constant of the saturation infection rate between Tax-expressing HTLV-infected cells and susceptible CD4$^+$T cells
a	Death rate constant of active HIV-infected cells
μ_1	Constant destruction of active HIV-infected cells by HIV-specific CTLs
μ_2	HTLV-specific CTLs destroy Tax-expressing HTLV-infected cells at a consistent rate
$\varphi\in(0,1)$	The potential for new HTLV infections through horizontal transmission could go dormant for a while
δ	Death rate constant of HTLV-infected cells that express Tax
ξ	Constant rate new HIV particle production
ε	Constant free HIV particle death rate
σ_1	Constant rate of HIV-specific CTL proliferation
σ_2	HTLV-specific CTLs proliferation rate constant
ς_1	HIV-specific CTLs' decay rate constant
ς_2	HTLV-specific CTL decay rate constant

$$\Psi(t)=\mathscr{A}(t)+\mathscr{D}(t)+\frac{1}{\varphi}\mathscr{U}(t)+\frac{a}{2\xi}\mathscr{K}(t)+\frac{\mu_1}{\sigma_1}\mathscr{W}(t)+\frac{\mu_2}{\varphi\sigma_2}\mathscr{Z}(t).$$

Then

$$\begin{aligned}\dot{\Psi}&=\eth-\alpha\mathscr{A}-\frac{a}{2}\mathscr{D}-\frac{\delta}{\varphi}\mathscr{U}-\frac{a\varepsilon}{2\xi}\mathscr{K}-\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}-\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}\\&\leq\eth-\phi\left(\mathscr{A}+\mathscr{D}+\frac{1}{\varphi}\mathscr{U}+\frac{a}{2\xi}\mathscr{K}+\frac{\mu_1}{\sigma_1}\mathscr{W}+\frac{\mu_2}{\varphi\sigma_2}\mathscr{Z}\right)=\eth-\phi\Psi,\end{aligned}$$

where ϕ is the minimum of the set $\{\alpha,\frac{a}{2},\delta,\varepsilon,\varsigma_1,\varsigma_2\}$. Then, $0\leq\Psi(t)\leq ¥_1$ if $\Psi(0)\leq ¥_1$ for $t\geq 0$, where $¥_1=\frac{\eth}{\phi}$. Since $\mathscr{A}$, $\mathscr{D}$, $\mathscr{U}$, $\mathscr{K}$, $\mathscr{W}$, and $\mathscr{Z}$ are all positive, and $0\leq\mathscr{A}(t),\mathscr{D}(t)\leq ¥_1$, $0\leq\mathscr{U}(t)\leq ¥_2$, $0\leq\mathscr{K}(t)\leq ¥_3$, $0\leq\mathscr{W}(t)\leq ¥_4$, $0\leq\mathscr{Z}(t)\leq ¥_5$ if $\mathscr{A}(0)+\mathscr{D}(0)+\frac{1}{\varphi}\mathscr{U}(0)+\frac{a}{2\xi}\mathscr{K}(0)+\frac{\mu_1}{\sigma_1}\mathscr{W}(0)+\frac{\mu_2}{\varphi\sigma_2}\mathscr{Z}(0)\leq ¥_1$, where $¥_2=\varphi ¥_1$, $¥_3=\dfrac{2\xi ¥_1}{a}$, $¥_4=\dfrac{\sigma_1 ¥_1}{\mu_1}$ and $¥_5=\dfrac{\varphi\sigma_2 ¥_1}{\mu_2}$. □

4 Equilibrium

Here, We concluded at We concluded at a collection of criteria for thresholds that guarantee presence of the equilibrium states in the model. Assume $(\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z})$ be any equilibrium point of the system (1)-(6) satisfying the following equations:

$$0 = \eth - \alpha\mathscr{A} - \frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}} - \frac{\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}}, \tag{7}$$

$$0 = \frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}} - a\mathscr{D} - \mu_1\mathscr{D}\mathscr{W}, \tag{8}$$

$$0 = \frac{\phi\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}} - \delta\mathscr{U} - \mu_2\mathscr{U}\mathscr{Z}, \tag{9}$$

$$0 = \xi\mathscr{D} - \varepsilon\mathscr{K}, \tag{10}$$

$$0 = \sigma_1\mathscr{D}\mathscr{W} - \varsigma_1\mathscr{W}, \tag{11}$$

$$0 = \sigma_2\mathscr{U}\mathscr{Z} - \varsigma_2\mathscr{Z}. \tag{12}$$

According to a simple calculation, system (1)-(6)admits nine equilibria.
(1) Infection-free equilibrium, $\mathfrak{U}_0 = (\mathscr{A}_0,0,0,0,0,0)$, where $\mathscr{A}_0 = \eth/\alpha$. In this instance, HTLV and HIV are both missing, illustrating the healthy state status.
(2) The equilibrium of HIV mono-infection without an active immune response, $\mathfrak{U}_1 = (\mathscr{A}_1,\mathscr{D}_1,0,\mathscr{K}_1,0,0)$ with

$$\mathscr{A}_1 = \frac{a\varepsilon + \eth\xi\alpha_1}{\xi\mathbb{Q}_1}, \quad \mathscr{D}_1 = \frac{\alpha\varepsilon}{\xi\mathbb{Q}_1}(\mathrm{R}_0-1), \mathscr{K}_1 = \frac{\alpha}{\mathbb{Q}_1}(\mathrm{R}_0-1).$$

where $\mathbb{Q}_1 = \eta_1 + \alpha\alpha_1$, Then $\mathfrak{U}_1$ exists when $\frac{\eth\xi\eta_1}{a\alpha\varepsilon} > 1$. The immune system is not stimulated at equilibrium $\mathfrak{U}_1$ where persistent HIV mono-infection continues. The fundamental System (1)-(6) HIV mono-infection reproductive ratio is provided by:

$$\mathrm{R}_0 = \frac{\eth\xi\eta_1}{a\alpha\varepsilon}.$$

The parameter R_0 dictates whether or not it is possible to establish a chronic HIV infection..
(3) Chronic mono-HTLV-I infection with a weakened immune system , $\mathfrak{U}_2 = (\mathscr{A}_2,0,\mathscr{U}_2,0,0,0)$, with

$$\mathscr{A}_2 = \frac{\delta + \eth\phi\alpha_2}{\phi\mathbb{Q}_2}, \quad \mathscr{U}_2 = \frac{\alpha}{A_2}(\mathrm{R}_1-1).$$

where $\mathbb{Q}_2 = \eta_2 + \alpha\alpha_2$, Then $\mathfrak{U}_2$ exists when $\frac{\eth\phi\eta_2}{\alpha\delta} > 1$. At the equilibrium $\mathfrak{U}_2$ The immune response is unstimulated where the chronic HTLV single-infection continues. The foundational reproduction ratio of the system (1)-(6) for HTLV single-infection is given as:

$$\mathrm{R}_1 = \frac{\eth\phi\eta_2}{\alpha\delta}.$$

The parameter R_1 determines the possibility of developing a chronic HTLV infection or not.
(4) Equilibrium of chronic HIV single-infection with just functional HIV-specific CTL, $\mathfrak{U}_3 = (\mathscr{A}_3,\mathscr{D}_3,0,\mathscr{K}_3,\mathscr{W}_3,0)$, where

$$\mathscr{A}_3 = \frac{\eth\varepsilon\sigma_1 + \eth\xi\varsigma_1\alpha_1}{\alpha\varepsilon\sigma_1 + \xi\varsigma_1\mathbb{Q}_1}, \quad \mathscr{D}_3 = \frac{\varsigma_1}{\sigma_1}, \quad \mathscr{K}_3 = \frac{\xi\varsigma_1}{\varepsilon\sigma_1}, \quad \mathscr{W}_3 = \frac{a}{\mu_1}(\mathrm{R}_2-1).$$

Then $\mathfrak{U}_3$ exists when $\frac{\eth\xi\eta_1\sigma_1}{a(\alpha\varepsilon\sigma_1+\xi\varsigma_1\mathbb{Q}_1)} > 1$. In the event of HIV mono-infection, the HIV-specific CTL-mediated immunity reproduction ratio is given as follows:

$$\mathrm{R}_2 = \frac{\eth\xi\eta_1\sigma_1}{a(\alpha\varepsilon\sigma_1+\xi\varsigma_1\mathbb{Q}_1)}.$$

The parameter R_2 determines whether the absence of HTLV infection stimulates the HIV-specific CTL-mediated immune response.
(5) Exclusively active HTLV-specific CTL equilibrium in chronic HTLV single-infection, $\mathfrak{U}_4 = (\mathscr{A}_4,0,\mathscr{U}_4,0,0,\mathscr{Z}_4)$, with

$$\mathscr{A}_4 = \frac{\eth\sigma_2 + \eth\varsigma_2\alpha_2}{\alpha\sigma_2 + \varsigma_2\mathbb{Q}_2}, \quad \mathscr{U}_4 = \frac{\varsigma_2}{\sigma_2}, \quad \mathscr{Z}_4 = \frac{\delta}{\mu_2}(\mathrm{R}_3-1).$$

It is clear that $\mathfrak{U}_4$ exists when $\frac{\phi\eth\eta_2\sigma_2}{\delta(\alpha\sigma_2+\varsigma_2\mathbb{Q}_2)} > 1$. In the event of HTLV single-infection, the HTLV-specific CTL-mediated immunity reproductive ratio is given as follows:

$$R_3 = \frac{\phi\eth\eta_2\sigma_2}{\delta(\alpha\sigma_2+\varsigma_2\mathbb{Q}_2)}.$$

The parameter R_3 gets to decide whether or not the absence of HIV infection stimulates the HTLV-specific CTL-mediated immune response.

(6) Having inactive HIV-specific CTL and HTLV-specific CTL, chronic HIV/HTLV co-infection equilibrium, $\mathfrak{U}_5 = (\mathscr{A}_5, \mathscr{D}_5, \mathscr{U}_5, \mathscr{K}_5, 0, 0)$, with

$$\mathscr{A}_5 = \frac{\eth\alpha_1\xi\alpha_2\phi + a\varepsilon\alpha_2\phi + \delta\alpha_1\xi}{\phi\xi(\alpha_2\mathbb{Q}_1+\eta_2\alpha_1)}, \quad \mathscr{K}_5 = \frac{\xi\eta_1\mathscr{A}_5 - a\varepsilon}{a\varepsilon\alpha_1} = \frac{\mathbb{Q}_2}{\alpha_2\mathbb{Q}_1+\eta_2\alpha_1}(R_4-1),$$
$$\mathscr{D}_5 = \frac{\varepsilon\mathbb{Q}_2}{\xi(\alpha_2\mathbb{Q}_1+\eta_2\alpha_1)}(R_4-1), \quad \mathscr{U}_5 = \frac{\phi\eta_2\mathscr{A}_5-\delta}{\delta\alpha_2} = \frac{\mathbb{Q}_1}{\alpha_2\mathbb{Q}_1+\eta_2\alpha_1}(R_5-1).$$

It is clear that $\mathfrak{U}_5$ exists when $\frac{\xi\eta_1(\eth\alpha_2\phi+\delta)}{a\varepsilon\phi\mathbb{Q}_2} > 1$, $\frac{\eta_2\phi(\eth\alpha_1\xi+a\varepsilon)}{\delta\xi\mathbb{Q}_1} > 1$. It is stated as if or not patients with HIV also have the HTLV-I virus:

$$R_4 = \frac{\xi\eta_1(\eth\alpha_2\phi+\delta)}{a\varepsilon\phi\mathbb{Q}_2}, \quad R_5 = \frac{\eta_2\phi(\eth\alpha_1\xi+a\varepsilon)}{\delta\xi\mathbb{Q}_1}.$$

The parameter $R_4 > 1$ identifies whether or not patients with HTLV-I infection may also have HIV. Thus, the parameter $R_5 > 1$ identifies whether or whether patients with HIV infection may also have HTLV-I.

(7) Equilibrium of persistent HIV/HTLV dual infection with active HIV-specific CTL only, $\mathfrak{U}_6 = (\mathscr{A}_6, \mathscr{D}_6, \mathscr{U}_6, \mathscr{K}_6, \mathscr{W}_6, 0)$, with

$$\mathscr{A}_6 = \frac{\delta R_6}{\eta_2\phi}, \quad \mathscr{D}_6 = \frac{\varsigma_1}{\sigma_1}, \quad \mathscr{K}_6 = \frac{\xi\varsigma_1}{\varepsilon\sigma_1}, \quad \mathscr{W}_6 = \frac{a}{\mu_1}(R_7-1),$$
$$\mathscr{U}_6 = \frac{1}{\alpha_2}\left(\frac{\phi\eta_2\mathscr{A}_6}{\delta}-1\right) = \frac{1}{\alpha_2}(R_6-1).$$

It is clear that $\mathfrak{U}_6$ exists when $\frac{a\phi\eta_2(\varepsilon\sigma_1+\xi\alpha_1\varsigma_1)}{\delta\xi\eta_1\sigma_1} > 1$ and $\frac{\xi\eta_1\sigma_1(\eth\phi\alpha_2+\delta)}{a\phi(\xi\varsigma_1(\alpha_2\mathbb{Q}_1+\alpha_1\eta_2)+\varepsilon\sigma_1\mathbb{Q}_2)} > 1$. In the presence of HIV infection, the HTLV infection reproductive ratio is given as:

$$R_6 = \frac{a\phi\eta_2(\varepsilon\sigma_1+\xi\alpha_1\varsigma_1)}{\delta\xi\eta_1\sigma_1}, \quad R_7 = \frac{\xi\eta_1\sigma_1(\eth\phi\alpha_2+\delta)}{a\phi(\xi\varsigma_1(\alpha_2\mathbb{Q}_1+\alpha_1\eta_2)+\varepsilon\sigma_1\mathbb{Q}_2)}.$$

The parameter R_6 determines whether or not HIV-infected persons may also have HTLV-I. The parameter R_7 determines whether the presence of HTLV-I infection enhances the HIV-specific CTL-mediated immune reaction.

(8) In having exclusively active HTLV-specific CTL, chronic HIV/HTLV co-infection equilibrium , $\mathfrak{U}_7 = (\mathscr{A}_7, \mathscr{D}_7, \mathscr{U}_7, \mathscr{K}_7, 0, \mathscr{Z}_7)$, with

$$\mathscr{A}_7 = \frac{\delta(\sigma_2+\alpha_2\varsigma_2)}{\phi\eta_2\sigma_2}, \quad \mathscr{U}_7 = \frac{\varsigma_2}{\sigma_2}, \quad \mathscr{D}_7 = \frac{\alpha\varepsilon\sigma_2+\varepsilon\varsigma_2\mathbb{Q}_2}{\xi(\mathbb{Q}_1(\alpha_2\varsigma_2+\sigma_2)+\eta_2\varsigma_2\alpha_1)}(R_8-1),$$
$$\mathscr{K}_7 = \frac{\alpha\sigma_2+\varsigma_2\mathbb{Q}_2}{\sigma_2\mathbb{Q}_1+\alpha_1\varsigma_2\mathbb{Q}_2+\eta_1\alpha_2\varsigma_2}(R_8-1), \quad \mathscr{Z}_7 = \frac{\delta}{\mu_2}(R_9-1).$$

It is clear that $\mathfrak{U}_7$ exists when $\frac{\xi\eth\eta_1(\sigma_2+\alpha_2\varsigma_2)}{a\varepsilon(\alpha\sigma_2+\varsigma_2\mathbb{Q}_2)} > 1$ and $\frac{\phi\eta_2\sigma_2(a\varepsilon+\alpha_1\xi\eth)}{\xi\delta(\mathbb{Q}_1(\sigma_2+\alpha_2\varsigma_2)+\eta_2\varsigma_2\alpha_1)} > 1$. When HTLV infection is present, the HIV infection reproductive ratio is given as:

$$R_8 = \frac{\xi\eth\eta_1(\sigma_2+\alpha_2\varsigma_2)}{a\varepsilon(\alpha\sigma_2+\varsigma_2\mathbb{Q}_2)}, \quad R_9 = \frac{\phi\eta_2\sigma_2(a\varepsilon+\alpha_1\xi\eth)}{\xi\delta(\mathbb{Q}_1(\sigma_2+\alpha_2\varsigma_2)+\eta_2\varsigma_2\alpha_1)}.$$

The parameter R_8 determines whether or whether patients with HTLV infection may also have HIV. The parameter R_9 determines whether the presence of HIV infection stimulates the HTLV-specific CTL-mediated immune response.

(9) Balance of active HIV-specific and HTLV-specific CTL during chronic HIV/HTLV co-infection, $\mathfrak{U}_8 = (\mathscr{A}_8, \mathscr{D}_8, \mathscr{U}_8, \mathscr{K}_8, \mathscr{W}_8, \mathscr{Z}_8)$, with

$$\mathscr{A}_8 = \frac{\delta\eth(\sigma_2+\alpha_2\varsigma_2)}{\eth\phi\eta_2\sigma_2},\ \mathscr{U}_8 = \frac{\varsigma_2}{\sigma_2},\ \mathscr{D}_8 = \frac{\varsigma_1}{\sigma_1},\ \mathscr{K}_8 = \frac{\xi\varsigma_1}{\varepsilon\sigma_1},$$
$$\mathscr{Z}_8 = \frac{\delta}{\mu_2}(\mathrm{R}_{10}-1),\ \mathscr{W}_8 = \frac{a}{\mu_1}(\mathrm{R}_{11}-1).$$

It is clear that $\mathfrak{U}_8$ exists when $\frac{a\phi\eta_2\sigma_2(\varepsilon\sigma_1+\alpha_1\xi\varsigma_1)}{\delta\xi\eta_1\sigma_1(\sigma_2+\alpha_2\varsigma_2)} > 1$ and $\frac{\xi\eta_1\sigma_1\eth(\sigma_2+\alpha_2\varsigma_2)}{a(\xi\varsigma_1\mathbb{Q}_1(\sigma_2+\varsigma_2\alpha_2)+\varepsilon\sigma_1(\alpha\sigma_2+\varsigma_2\mathbb{Q}_2)+\xi\alpha_1\eta_2\varsigma_1\varsigma_2)} > 1$. Now we define

$$\mathrm{R}_{10} = \frac{a\phi\eta_2\sigma_2(\varepsilon\sigma_1+\alpha_1\xi\varsigma_1)}{\delta\xi\eta_1\sigma_1(\sigma_2+\alpha_2\varsigma_2)},$$
$$\mathrm{R}_{11} = \frac{\xi\eta_1\sigma_1\eth(\sigma_2+\alpha_2\varsigma_2)}{a(\xi\varsigma_1\mathbb{Q}_1(\sigma_2+\varsigma_2\alpha_2)+\varepsilon\sigma_1(\alpha\sigma_2+\varsigma_2\mathbb{Q}_2)+\xi\alpha_1\eta_2\varsigma_1\varsigma_2)}.$$

It is clear that $\mathfrak{U}_8$ exists when $\mathrm{R}_{10} > 1$ and $\mathrm{R}_{11} > 1$. The parameter R_{10} refers to the ratio of HIV-specific CTL-mediated immunity to reproduction when HIV and HTLV-I are both present. Furthermore, the parameter R_{11} refers to the ratio of HTLV-specific CTL-mediated immunity to reproduction when HIV and HTLV-I are co-infected.

5 A study of global stability

In this part by creating the Lyapunov function and using Asymptotic Lyapunov-LaSalle stability, we demonstrate that all equilibria are globally asymptotically stable (L-LAS) [12]-[13]. The relation between the arithmetic and geometric mean inequality will be use here:

$$\frac{1}{n}\sum_{\Bbbk=1}^{n}\mathtt{C}_{\Bbbk} \geq \sqrt[n]{\prod_{\Bbbk=1}^{n}\mathtt{C}_{\Bbbk}},\quad \mathtt{C}_{\Bbbk}\geq 0,\ \Bbbk=1,2,... \tag{13}$$

We establish a function

$$\mathfrak{D}(\vartheta) = \vartheta - 1 - \ln\vartheta.$$

Define a function $F_i(\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z})$ and suppose Υ_i' where is the greatest invariant subset

$$\Upsilon_i = \left\{(\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z}) : \frac{dF_i}{dt} = 0\right\}, i = 0,1,...,8.$$

Theorem 1. If $\mathrm{R}_0 \leq 1$ and $\mathrm{R}_1 \leq 1$, then $\mathfrak{U}_0$ is globally asymptotically stable (G.A.S).
Proof. Define $F_0(\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z})$ as:

$$F_0 = \mathscr{A}_0\mathfrak{D}\left(\frac{\mathscr{A}}{\mathscr{A}_0}\right) + \mathscr{D} + \frac{1}{\varphi}\mathscr{U} + \frac{a}{\xi}\mathscr{K} + \frac{\mu_1}{\sigma_1}\mathscr{W} + \frac{\mu_2}{\varphi\sigma_2}\mathscr{Z},$$

Clearly, $F_0(\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z}) > 0$ for each $\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z} > 0$, and $F_0(\mathscr{A}_0,0,0,0,0,0) = 0$. Calculating $\frac{dF_0}{dt}$ existence the solutions of system (1)-(6) as:

$$\begin{aligned}\frac{dF_0}{dt} &= \left(1-\frac{\mathscr{A}_0}{\mathscr{A}}\right)\left(\eth - \alpha\mathscr{A} - \frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}} - \frac{\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}}\right) + \frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}} - a\mathscr{D} - \mu_1\mathscr{W}\mathscr{D}\\ &+ \frac{1}{\varphi}\left(\frac{\varphi\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}} - \delta\mathscr{U} - \mu_2\mathscr{Z}\mathscr{U}\right) + \frac{a}{\xi}(\xi\mathscr{D} - \varepsilon\mathscr{K}) + \frac{\mu_1}{\sigma_1}(\sigma_1\mathscr{W}\mathscr{D} - \varsigma_1\mathscr{W})\\ &+ \frac{\mu_2}{\varphi\sigma_2}(\sigma_2\mathscr{Z}\mathscr{U} - \varsigma_2\mathscr{Z})\\ &= \left(1-\frac{\mathscr{A}_0}{\mathscr{A}}\right)(\eth - \alpha\mathscr{A}) + \frac{\eta_1\mathscr{A}_0\mathscr{K}}{1+\alpha_1\mathscr{K}} + \frac{\eta_2\mathscr{A}_0\mathscr{U}}{1+\alpha_2\mathscr{U}} - \frac{\delta}{\varphi}\mathscr{U} - \frac{a\varepsilon}{\xi}\mathscr{K} - \frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W} - \frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}.\end{aligned}$$

Using $\mathscr{A}_0 = \eth/\alpha$, we obtain

$$\frac{dF_0}{dt} = -\alpha\frac{(\mathscr{A}-\mathscr{A}_0)^2}{\mathscr{A}} - \frac{a\varepsilon\alpha_1\mathrm{R}_0\mathscr{K}^2}{\xi(1+\alpha_1\mathscr{K})} - \frac{\alpha_2\delta\mathrm{R}_1\mathscr{U}^2}{\phi(1+\alpha_2\mathscr{U})} + \frac{a\varepsilon}{\xi}(\mathrm{R}_0-1)\mathscr{K} + \frac{\delta}{\varphi}(\mathrm{R}_1-1)\mathscr{U} - \frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W} - \frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}.$$

Therefore, $\frac{d\digamma_0}{dt} \leq 0$ for each $\mathscr{A},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z} > 0$, moreover, $\frac{d\digamma_0}{dt} = 0$ when $\mathscr{A}(t) = \mathscr{A}_0$ and $\mathscr{K}(t) = \mathscr{U}(t) = \mathscr{W}(t) = \mathscr{Z}(t) = 0$. The solutions of system (1)-(6) converge at Υ_0'. The items in the set Υ_0' have $\mathscr{K}(t) = 0$. Then, $\dot{\mathscr{K}} = 0$ and using system (1)-(6)fourth equation, we have

$$0 = \dot{\mathscr{K}} = b\,\mathscr{D}(t),$$

This yields, $\mathscr{D}(t) = 0$ for each t. Therefore, $\Upsilon_0' = \{\mathfrak{U}_0\}$ and applying L-LAS we get that $\mathfrak{U}_0$ is G.A.S .□

Theorem 2. Assume that $R_0 \geq 1$, $R_1 \leq 1$ and $R_2 \leq 1$, then $\mathfrak{U}_1$ is G.A.S.

Proof. Let us define the function $\digamma_1(\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z})$ as:

$$\digamma_1 = \mathscr{A}_1\mathfrak{D}\left(\frac{\mathscr{A}}{\mathscr{A}_1}\right) + \mathscr{D}_1\mathfrak{D}\left(\frac{\mathscr{D}}{\mathscr{D}_1}\right) + \frac{1}{\varphi}\mathscr{U} + \frac{a}{\xi}\mathscr{K}_1\mathfrak{D}\left(\frac{\mathscr{K}}{\mathscr{K}_1}\right) + \frac{\mu_1}{\sigma_1}\mathscr{W} + \frac{\mu_2}{\varphi\sigma_2}\mathscr{Z}.$$

Calculating $\frac{d\digamma_1}{dt}$ as:

$$\begin{aligned}
\frac{d\digamma_1}{dt} &= \left(1-\frac{\mathscr{A}_1}{\mathscr{A}}\right)\left(\eth - \alpha\mathscr{A} - \frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}} - \frac{\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}}\right) + \left(1-\frac{\mathscr{D}_1}{\mathscr{D}}\right)\left(\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}} - a\mathscr{D} - \mu_1\mathscr{W}\mathscr{D}\right)\\
&+ \frac{1}{\varphi}\left(\frac{\varphi\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}} - \delta\mathscr{U} - \mu_2\mathscr{Z}\mathscr{U}\right) + \frac{a}{\xi}\left(1-\frac{\mathscr{K}_1}{\mathscr{K}}\right)(\xi\mathscr{D} - \varepsilon\mathscr{K}) + \frac{\mu_1}{\sigma_1}(\sigma_1\mathscr{W}\mathscr{D} - \varsigma_1\mathscr{W})\\
&+ \frac{\mu_2}{\varphi\sigma_2}(\sigma_2\mathscr{Z}\mathscr{U} - \varsigma_2\mathscr{Z})\\
&= \left(1-\frac{\mathscr{A}_1}{\mathscr{A}}\right)(\eth - \alpha\mathscr{A}) + \frac{\eta_1\mathscr{A}_1\mathscr{K}}{1+\alpha_1\mathscr{K}} + \frac{\eta_2\mathscr{A}_1\mathscr{U}}{1+\alpha_2\mathscr{U}} - \frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}\frac{\mathscr{D}_1}{\mathscr{D}} + a\mathscr{D}_1 + \mu_1\mathscr{W}\mathscr{D}_1\\
&- \frac{\delta}{\varphi}\mathscr{U} - \frac{a\varepsilon}{\xi}\mathscr{K} - a\mathscr{D}\frac{\mathscr{K}_1}{\mathscr{K}} + \frac{a\varepsilon}{\xi}\mathscr{K}_1 - \frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W} - \frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}.
\end{aligned}$$

Utilizing the $\mathfrak{U}_1$ equilibrium conditions:

$$\eth = \alpha\mathscr{A}_1 + \frac{\eta_1\mathscr{A}_1\mathscr{K}_1}{1+\alpha_1\mathscr{K}_1}, \quad \frac{\eta_1\mathscr{A}_1\mathscr{K}_1}{1+\alpha_1\mathscr{K}_1} = a\mathscr{D}_1 = \frac{a\varepsilon}{\xi}\mathscr{K}_1,$$

we obtain

$$\begin{aligned}
\frac{d\digamma_1}{dt} &= \left(1-\frac{\mathscr{A}_1}{\mathscr{A}}\right)(\alpha\mathscr{A}_1 - \alpha\mathscr{A}) + \frac{\eta_1\mathscr{A}_1\mathscr{K}_1}{1+\alpha_1\mathscr{K}_1}\left(1-\frac{\mathscr{A}_1}{\mathscr{A}}\right) + \frac{\eta_1\mathscr{A}_1\mathscr{K}}{1+\alpha_1\mathscr{K}} + \frac{\eta_2\mathscr{A}_1\mathscr{U}}{1+\alpha_2\mathscr{U}}\\
&- \frac{\eta_1\mathscr{A}_1\mathscr{K}_1}{1+\alpha_1\mathscr{K}_1}\frac{\mathscr{D}_1\mathscr{A}\mathscr{K}(1+\alpha_1\mathscr{K}_1)}{\mathscr{D}\mathscr{A}_1\mathscr{K}_1(1+\alpha_1\mathscr{K})} + \frac{\eta_1\mathscr{A}_1\mathscr{K}_1}{1+\alpha_1\mathscr{K}_1} + \mu_1\mathscr{W}\mathscr{D}_1 - \frac{\delta}{\varphi}\mathscr{U} - \frac{\eta_1\mathscr{A}_1\mathscr{K}_1}{1+\alpha_1\mathscr{K}_1}\frac{\mathscr{K}}{\mathscr{K}_1}\\
&- \frac{\eta_1\mathscr{A}_1\mathscr{K}_1}{1+\alpha_1\mathscr{K}_1}\frac{\mathscr{D}\mathscr{K}_1}{\mathscr{D}_1\mathscr{K}} + \frac{\eta_1\mathscr{A}_1\mathscr{K}_1}{1+\alpha_1\mathscr{K}_1} - \frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W} - \frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}\\
&= -\alpha\frac{(\mathscr{A}-\mathscr{A}_1)^2}{\mathscr{A}} + \frac{\eta_1\mathscr{A}_1\mathscr{K}_1}{1+\alpha_1\mathscr{K}_1}\left(\frac{\mathscr{K}(1+\alpha_1\mathscr{K}_1)}{\mathscr{K}_1(1+\alpha_1\mathscr{K})} - 1 - \frac{\mathscr{K}}{\mathscr{K}_1} + \frac{(1+\alpha_1\mathscr{K})}{(1+\alpha_1\mathscr{K}_1)}\right)\\
&+ \frac{\eta_1\mathscr{A}_1\mathscr{K}_1}{1+\alpha_1\mathscr{K}_1}\left(4 - \frac{\mathscr{A}_1}{\mathscr{A}} - \frac{\mathscr{D}_1\mathscr{A}\mathscr{K}(1+\alpha_1\mathscr{K}_1)}{\mathscr{D}\mathscr{A}_1\mathscr{K}_1(1+\alpha_1\mathscr{K})} - \frac{\mathscr{D}\mathscr{K}_1}{\mathscr{D}_1\mathscr{K}} - \frac{(1+\alpha_1\mathscr{K})}{(1+\alpha_1\mathscr{K}_1)}\right)\\
&+ \frac{\eta_2\mathscr{A}_1\mathscr{U}}{1+\alpha_2\mathscr{U}} - \frac{\delta}{\varphi}\mathscr{U} + \mu_1\left(\mathscr{D}_1 - \frac{\varsigma_1}{\sigma_1}\right)\mathscr{W} - \frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}.
\end{aligned}$$

We have

$$\frac{\mathscr{K}(1+\alpha_1\mathscr{K}_1)}{\mathscr{K}_1(1+\alpha_1\mathscr{K})} - 1 - \frac{\mathscr{K}}{\mathscr{K}_1} + \frac{(1+\alpha_1\mathscr{K})}{(1+\alpha_1\mathscr{K}_1)} = -\frac{\alpha_1(\mathscr{K}-\mathscr{K}_1)^2}{\mathscr{K}_1(1+\alpha_1\mathscr{K})(1+\alpha_1\mathscr{K}_1)},$$

$$\mathscr{D}_1 - \frac{\varsigma_1}{\sigma_1} = \frac{\alpha\varepsilon\sigma_1 + \xi\varsigma_1 A_1}{\xi\sigma_1 A_1}(R_2 - 1).$$

Moreover

$$\frac{\eta_2\mathscr{A}_1\mathscr{U}}{1+\alpha_2\mathscr{U}}-\frac{\delta}{\varphi}\mathscr{U}\leq\eta_2\mathscr{A}_1\mathscr{U}-\frac{\delta}{\varphi}\mathscr{U}=\frac{\delta}{\varphi}\left(\frac{\eta_2\mathscr{A}_1\varphi}{\delta}-1\right)\mathscr{U}=\frac{\delta}{\varphi}\left(\mathrm{R}_1\left(1+\frac{1-\mathrm{R}_0}{\frac{\xi\alpha_1\eth}{a\varepsilon}+\mathrm{R}_0}\right)-1\right)\mathscr{U}$$
$$=\frac{\delta}{\varphi}\left((\mathrm{R}_1-1)+\mathrm{R}_1\left(\frac{1-\mathrm{R}_0}{\frac{\xi\alpha_1\eth}{a\varepsilon}+\mathrm{R}_0}\right)\right).$$

If $\mathrm{R}_1\leq 1$, then $\mathrm{R}_1-1\leq 0$, and if $\mathrm{R}_0\geq 1$, then $1-\mathrm{R}_0\leq 0$. It follows that

$$\mathrm{R}_1\left(\frac{1-\mathrm{R}_0}{\frac{\xi\alpha_1\eth}{a\varepsilon}+\mathrm{R}_0}\right)\leq 0,$$

then

$$\frac{\eta_2\mathscr{A}_1\mathscr{U}}{1+\alpha_2\mathscr{U}}-\frac{\delta}{\varphi}\mathscr{U}\leq 0,$$

we obtain

$$\frac{d\digamma_1}{dt}=-\alpha\frac{(\mathscr{A}-\mathscr{A}_1)^2}{\mathscr{A}}-\frac{\eta_1\mathscr{A}_1\mathscr{K}_1}{1+\alpha_1\mathscr{K}_1}\frac{\alpha_1(\mathscr{K}-\mathscr{K}_1)^2}{\mathscr{K}_1(1+\alpha_1\mathscr{K})(1+\alpha_1\mathscr{K}_1)}$$
$$+\frac{\eta_1\mathscr{A}_1\mathscr{K}_1}{1+\alpha_1\mathscr{K}_1}\left(4-\frac{\mathscr{A}_1}{\mathscr{A}}-\frac{\mathscr{D}_1\mathscr{A}\mathscr{K}(1+\alpha_1\mathscr{K}_1)}{\mathscr{D}\mathscr{A}_1\mathscr{K}_1(1+\alpha_1\mathscr{K})}-\frac{\mathscr{D}\mathscr{K}_1}{\mathscr{D}_1\mathscr{K}}-\frac{(1+\alpha_1\mathscr{K})}{(1+\alpha_1\mathscr{K}_1)}\right)$$
$$+\frac{\delta}{\varphi}\left(\mathrm{R}_1\left(1+\frac{1-\mathrm{R}_0}{\frac{\xi\alpha_1\eth}{a\varepsilon}+\mathrm{R}_0}\right)-1\right)\mathscr{U}+\frac{\mu_1(\alpha\varepsilon\sigma_1+\xi\varsigma_1A_1)}{\xi\sigma_1A_1}(\mathrm{R}_2-1)\mathscr{W}-\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}.$$

Using inequality (13), we have $\frac{d\digamma_1}{dt}\leq 0$ for each $\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z}>0$, furthermore, $\frac{d\digamma_1}{dt}=0$ when $\mathscr{A}=\mathscr{A}_1,\mathscr{D}=\mathscr{D}_1,\mathscr{K}=\mathscr{K}_1$ and $\mathscr{U}=\mathscr{W}=\mathscr{Z}=0$. The system (1)-(6) solutions tend to Υ_1' it includes elements $\mathscr{A}(t)=\mathscr{A}_1$, $\mathscr{D}(t)=\mathscr{D}_1$, $\mathscr{K}(t)=\mathscr{K}_1$and $\mathscr{U}(t)=\mathscr{W}(t)=\mathscr{Z}(t)=0$. The result is,$\Upsilon_1'=\{\mathfrak{U}_1\}$. Then, $\mathfrak{U}_1$ is G.A.S. using L-LAS.□

Theorem 3. If $\mathrm{R}_0\leq 1$, $\mathrm{R}_1\geq 1$ and $\mathrm{R}_3\leq 1$, then $\mathfrak{U}_2$ is G.A.S.
Proof. a potential Lyapunov function is

$$\digamma_2(\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z})=\mathscr{A}_2\mathfrak{D}\left(\frac{\mathscr{A}}{\mathscr{A}_2}\right)+\mathscr{D}+\frac{1}{\varphi}\mathscr{U}_2\mathfrak{D}\left(\frac{\mathscr{U}}{\mathscr{U}_2}\right)+\frac{a}{\xi}\mathscr{K}+\frac{\mu_1}{\sigma_1}\mathscr{W}+\frac{\mu_2}{\varphi\sigma_2}\mathscr{Z}.$$

We calculate $\frac{d\digamma_2}{dt}$ as:

$$\frac{d\digamma_2}{dt}=\left(1-\frac{\mathscr{A}_2}{\mathscr{A}}\right)\left(\eth-\alpha\mathscr{A}-\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}-\frac{\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}}\right)+\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}-a\mathscr{D}-\mu_1\mathscr{W}\mathscr{D}$$
$$+\frac{1}{\varphi}\left(1-\frac{\mathscr{U}_2}{\mathscr{U}}\right)\left(\frac{\varphi\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}}-\delta\mathscr{U}-\mu_2\mathscr{Z}\mathscr{U}\right)+\frac{a}{\xi}(\xi\mathscr{D}-\varepsilon\mathscr{K})$$
$$+\frac{\mu_1}{\sigma_1}(\sigma_1\mathscr{W}\mathscr{D}-\varsigma_1\mathscr{W})+\frac{\mu_2}{\varphi\sigma_2}(\sigma_2\mathscr{Z}\mathscr{U}-\varsigma_2\mathscr{Z}). \tag{14}$$

Collecting terms of Eq. (14), we find

$$\frac{d\digamma_2}{dt}=\left(1-\frac{\mathscr{A}_2}{\mathscr{A}}\right)(\eth-\alpha\mathscr{A})+\frac{\eta_1\mathscr{A}_2\mathscr{K}}{1+\alpha_1\mathscr{K}}+\frac{\eta_2\mathscr{A}_2\mathscr{U}}{1+\alpha_2\mathscr{U}}-\frac{\eta_2\mathscr{A}\mathscr{U}_2}{1+\alpha_2\mathscr{U}}$$
$$-\frac{\delta}{\varphi}\mathscr{U}+\frac{\delta}{\varphi}\mathscr{U}_2+\frac{\mu_2}{\varphi}\mathscr{Z}\mathscr{U}_2-\frac{a\varepsilon}{\xi}\mathscr{K}-\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}-\frac{\varsigma_2\mu_2}{\varphi\sigma_2}\mathscr{Z}.$$

Utilizing the $\mathfrak{U}_2$ equilibrium conditions:

$$\eth=\alpha\mathscr{A}_2+\frac{\eta_2\mathscr{A}_2\mathscr{U}_2}{1+\alpha_2\mathscr{U}_2},\quad\frac{\eta_2\mathscr{A}_2\mathscr{U}_2}{1+\alpha_2\mathscr{U}_2}=\frac{\delta}{\phi}\mathscr{U}_2,$$

we obtain

$$\begin{aligned}\frac{dF_2}{dt} &= \left(1-\frac{\mathscr{A}_2}{\mathscr{A}}\right)(\alpha\mathscr{A}_2-\alpha\mathscr{A}) + \frac{\eta_2\mathscr{A}_2\mathscr{U}_2}{1+\alpha_2\mathscr{U}_2}\left(1-\frac{\mathscr{A}_2}{\mathscr{A}}\right) + \frac{\eta_1\mathscr{A}_2\mathscr{K}}{1+\alpha_1\mathscr{K}} + \frac{\eta_2\mathscr{A}_2\mathscr{U}_2}{1+\alpha_2\mathscr{U}_2}\frac{\mathscr{U}(1+\alpha_2\mathscr{U}_2)}{\mathscr{U}_2(1+\alpha_2\mathscr{U})}\\
&-\frac{\eta_2\mathscr{A}_2\mathscr{U}_2}{1+\alpha_2\mathscr{U}_2}\frac{\mathscr{A}(1+\alpha_2\mathscr{U}_2)}{\mathscr{A}_2(1+\alpha_2\mathscr{U})} - \frac{\eta_2\mathscr{A}_2\mathscr{U}_2}{1+\alpha_2\mathscr{U}_2}\frac{\mathscr{U}}{\mathscr{U}_2} + \frac{\eta_2\mathscr{A}_2\mathscr{U}_2}{1+\alpha_2\mathscr{U}_2} + \frac{\mu_2}{\varphi}\mathscr{U}_2\mathscr{Z} - \frac{a\varepsilon}{\xi}\mathscr{K} - \frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W} - \frac{\varsigma_2\mu_2}{\varphi\sigma_2}\mathscr{Z}\\
&= -\alpha\frac{(\mathscr{A}-\mathscr{A}_2)^2}{\mathscr{A}} + \frac{\eta_2\mathscr{A}_2\mathscr{U}_2}{1+\alpha_2\mathscr{U}_2}\left(\frac{\mathscr{U}(1+\alpha_2\mathscr{U}_2)}{\mathscr{U}_2(1+\alpha_2\mathscr{U})} - \frac{\mathscr{U}}{\mathscr{U}_2} - 1 + \frac{1+\alpha_2\mathscr{U}}{1+\alpha_2\mathscr{U}_2}\right)\\
&+ \frac{\eta_2\mathscr{A}_2\mathscr{U}_2}{1+\alpha_2\mathscr{U}_2}\left(3-\frac{\mathscr{A}_2}{\mathscr{A}} - \frac{\mathscr{A}(1+\alpha_2\mathscr{U}_2)}{\mathscr{A}_2(1+\alpha_2\mathscr{U})} - \frac{1+\alpha_2\mathscr{U}}{1+\alpha_2\mathscr{U}_2}\right)\\
&+ \frac{\eta_1\mathscr{A}_2\mathscr{K}}{1+\alpha_1\mathscr{K}} - \frac{a\varepsilon}{\xi}\mathscr{K} + \frac{\mu_2}{\varphi}\left(\mathscr{U}_2 - \frac{\varsigma_2}{\sigma_2}\right)\mathscr{Z} - \frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}.\end{aligned}$$

But also

$$\frac{\mathscr{U}(1+\alpha_2\mathscr{U}_2)}{\mathscr{U}_2(1+\alpha_2\mathscr{U})} - \frac{\mathscr{U}}{\mathscr{U}_2} - 1 + \frac{1+\alpha_2\mathscr{U}}{1+\alpha_2\mathscr{U}_2} = -\frac{\alpha_2(\mathscr{U}-\mathscr{U}_2)^2}{\mathscr{U}_2(1+\alpha_2\mathscr{U})(1+\alpha_2\mathscr{U}_2)},$$

$$\mathscr{U}_2 - \frac{\varsigma_2}{\sigma_2} = \frac{\alpha\sigma_2+\varsigma_2 A_2}{\sigma_2 A_2}\left(\frac{\eth\phi\eta_2\sigma_2}{\delta(\alpha\sigma_2+\varsigma_2 A_2)} - 1\right) = \frac{\alpha\sigma_2+\varsigma_2 A_2}{\sigma_2 A_2}(\mathrm{R}_3-1),$$

and

$$\begin{aligned}\left(\frac{\eta_1\mathscr{A}_2}{1+\alpha_1\mathscr{K}} - \frac{a\varepsilon}{\xi}\right)\mathscr{K} &\le \left(\eta_1\mathscr{A}_2 - \frac{a\varepsilon}{\xi}\right)\mathscr{K} = \frac{a\varepsilon}{\xi}\left(\frac{\xi\eta_1\mathscr{A}_2}{a\varepsilon} - 1\right)\mathscr{K} = \frac{a\varepsilon}{\xi}\left(\mathrm{R}_0\left(1+\frac{1-\mathrm{R}_1}{\frac{\varphi\alpha_2\eth}{\delta}+\mathrm{R}_1}\right)-1\right)\mathscr{K}\\
&= \frac{a\varepsilon}{\xi}\left(\mathrm{R}_0 - 1 + \mathrm{R}_0\left(\frac{1-\mathrm{R}_1}{\frac{\varphi\alpha_2\eth}{\delta}+\mathrm{R}_1}\right)\right)\mathscr{K}.\end{aligned}$$

If $\mathrm{R}_0 \le 1$ then $\mathrm{R}_0 - 1 \le 0$. and if $\mathrm{R}_1 \ge 1$ then $1-\mathrm{R}_1 \le 0$, and its follows that

$$\mathrm{R}_0\left(\frac{1-\mathrm{R}_1}{\frac{\varphi\alpha_2\eth}{\delta}+\mathrm{R}_0}\right) \le 0.$$

Then

$$\left(\frac{\eta_1\mathscr{A}_2}{1+\alpha_1\mathscr{K}} - \frac{a\varepsilon}{\xi}\right)\mathscr{K} \le 0,$$

and we obtain

$$\begin{aligned}\frac{dF_2}{dt} &= \alpha\frac{(\mathscr{A}-\mathscr{A}_2)^2}{\mathscr{A}} - \frac{\eta_2\mathscr{A}_2\mathscr{U}_2}{1+\alpha_2\mathscr{U}_2}\left(\frac{\alpha_2(\mathscr{U}-\mathscr{U}_2)^2}{\mathscr{U}_2(1+\alpha_2\mathscr{U})(1+\alpha_2\mathscr{U}_2)}\right)\\
&+ \frac{\eta_2\mathscr{A}_2\mathscr{U}_2}{1+\alpha_2\mathscr{U}_2}\left(3-\frac{\mathscr{A}_2}{\mathscr{A}} - \frac{\mathscr{A}(1+\alpha_2\mathscr{U}_2)}{\mathscr{A}_2(1+\alpha_2\mathscr{U})} - \frac{1+\alpha_2\mathscr{U}}{1+\alpha_2\mathscr{U}_2}\right)\\
&+ \frac{a\varepsilon}{\xi}\left(\mathrm{R}_0\left(1+\frac{1-\mathrm{R}_1}{\frac{\varphi\alpha_2\eth}{\delta}+\mathrm{R}_1}\right)-1\right)\mathscr{K} + \frac{\mu_2(\alpha\sigma_2+\varsigma_2 A_2)}{\varphi\sigma_2 A_2}(\mathrm{R}_3-1)\mathscr{Z} - \frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}.\end{aligned}$$

Using inequality (13) and $\mathrm{R}_1 \ge 1$, $\mathrm{R}_0 \le 1$ and $\mathrm{R}_3 \le 1$, we have $\frac{dF_2}{dt} \le 0$ for each $\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z} > 0$, furthermore, $\frac{dF_2}{dt} = 0$ when $\mathscr{A}(t) = \mathscr{A}_2, \mathscr{U}(t) = \mathscr{U}_2$ and $\mathscr{K}(t) = \mathscr{W}(t) = \mathscr{Z}(t) = 0$. The system (1)-(6) solutions tend to Υ_2' it includes elements $\mathscr{A}(t) = \mathscr{A}_2, \mathscr{U}(t) = \mathscr{U}_2$ and $\mathscr{K}(t) = \mathscr{W}(t) = \mathscr{Z}(t) = 0$. From Eqs. (4), we get

$$0 = \dot{\mathscr{K}} = \zeta\mathscr{D}(t) - \varepsilon\mathscr{K}(t),$$

which leads to $\mathscr{D}(t) = 0$. The result is, $\Upsilon_2' = \{\mathfrak{U}_2\}$. Then, $\mathfrak{U}_2$ is G.A.S. using L-LAS.□

Theorem 4. Suppose that $\mathrm{R}_1 \leq 1$ and $\mathrm{R}_2 \geq 1$, then $\mathfrak{U}_3$ is G.A.S.
Proof. a potential Lyapunov function is $\mathcal{F}_3(\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z})$ as:

$$\mathcal{F}_3 = \mathscr{A}_3 \mathfrak{D}\left(\frac{\mathscr{A}}{\mathscr{A}_3}\right) + \mathscr{D}_3 \mathfrak{D}\left(\frac{\mathscr{D}}{\mathscr{D}_3}\right) + \frac{1}{\varphi}\mathscr{U} + \frac{\eta_1 \mathscr{A}_3}{\varepsilon(1+\alpha_1 \mathscr{K}_3)} \mathscr{K}_3 \mathfrak{D}\left(\frac{\mathscr{K}}{\mathscr{K}_3}\right) + \frac{\mu_1}{\sigma_1} \mathscr{W}_3 \mathfrak{D}\left(\frac{\mathscr{W}}{\mathscr{W}_3}\right) + \frac{\mu_2}{\varphi\sigma_2}\mathscr{Z}.$$

We calculate $\frac{d\mathcal{F}_3}{dt}$ as:

$$\begin{aligned}\frac{d\mathcal{F}_3}{dt} &= \left(1-\frac{\mathscr{A}_3}{\mathscr{A}}\right)\left(\eth - \alpha\mathscr{A} - \frac{\eta_1 \mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}} - \frac{\eta_2 \mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}}\right) + \left(1-\frac{\mathscr{D}_3}{\mathscr{D}}\right)\left(\frac{\eta_1 \mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}} - a\mathscr{D} - \mu_1 \mathscr{W}\mathscr{D}\right)\\ &+ \frac{1}{\varphi}\left(\frac{\varphi\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}} - \delta\mathscr{U} - \mu_2\mathscr{Z}\mathscr{U}\right) + \frac{\eta_1\mathscr{A}_3}{\varepsilon(1+\alpha_1\mathscr{K}_3)}\left(1-\frac{\mathscr{K}_3}{\mathscr{K}}\right)(\xi\mathscr{D} - \varepsilon\mathscr{K})\\ &+ \frac{\mu_1}{\sigma_1}\left(1-\frac{\mathscr{W}_3}{\mathscr{W}}\right)(\sigma_1\mathscr{W}\mathscr{D} - \varsigma_1\mathscr{W}) + \frac{\mu_2}{\varphi\sigma_2}(\sigma_2\mathscr{Z}\mathscr{U} - \varsigma_2\mathscr{Z}).\end{aligned} \tag{15}$$

We collect the terms of Eq. (15) as:

$$\begin{aligned}\frac{d\mathcal{F}_3}{dt} &= \left(1-\frac{\mathscr{A}_3}{\mathscr{A}}\right)(\eth - \alpha\mathscr{A}) + \frac{\eta_1\mathscr{A}_3\mathscr{K}}{1+\alpha_1\mathscr{K}} + \frac{\eta_2\mathscr{A}_3\mathscr{U}}{1+\alpha_2\mathscr{U}} - a\mathscr{D} - \frac{\mathscr{D}_3}{\mathscr{D}}\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}} + a\mathscr{D}_3\\ &+ \mu_1\mathscr{W}\mathscr{D}_3 - \frac{\delta}{\varphi}\mathscr{U} + \frac{\eta_1\mathscr{A}_3}{\varepsilon(1+\alpha_1\mathscr{K}_3)}\xi\mathscr{D} - \frac{\eta_1\mathscr{A}_3\mathscr{K}}{1+\alpha_1\mathscr{K}_3} - \frac{\eta_1\mathscr{A}_3\mathscr{K}_3}{\varepsilon(1+\alpha_1\mathscr{K}_3)\mathscr{K}}\xi\mathscr{D}\\ &+ \frac{\eta_1\mathscr{A}_3\mathscr{K}_3}{1+\alpha_1\mathscr{K}_3} - \frac{\varsigma_1\mu_1}{\sigma_1}\mathscr{W} - \mu_1\mathscr{W}_3\mathscr{D} + \frac{\varsigma_1\mu_1}{\sigma_1}\mathscr{W}_3 - \frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}.\end{aligned}$$

Utilizing the $\mathfrak{U}_3$ equilibrium conditions:

$$\eth = \alpha\mathscr{A}_3 + \frac{\eta_1\mathscr{A}_3\mathscr{K}_3}{1+\alpha_1\mathscr{K}_3}, \quad \frac{\eta_1\mathscr{A}_3\mathscr{K}_3}{1+\alpha_1\mathscr{K}_3} = a\mathscr{D}_3 + \mu_1\mathscr{C}_3^{\mathscr{D}}\mathscr{D}_3, \quad \mathscr{D}_3 = \frac{\varsigma_1}{\sigma_1}, \quad \mathscr{K}_3 = \frac{\xi}{\varepsilon}\mathscr{D}_3 = \frac{\xi\varsigma_1}{\varepsilon\sigma_1}.$$

we obtain

$$\begin{aligned}\frac{d\mathcal{F}_3}{dt} &= \left(1-\frac{\mathscr{A}_3}{\mathscr{A}}\right)(\alpha\mathscr{A}_3 - \alpha\mathscr{A}) + \frac{\eta_1\mathscr{A}_3\mathscr{K}_3}{1+\alpha_1\mathscr{K}_3}\left(1-\frac{\mathscr{A}_3}{\mathscr{A}}\right) + \frac{\eta_1\mathscr{A}_3\mathscr{K}}{1+\alpha_1\mathscr{K}} + \frac{\eta_2\mathscr{A}_3\mathscr{U}}{1+\alpha_2\mathscr{U}}\\ &- \frac{\eta_1\mathscr{A}_3\mathscr{K}_3}{1+\alpha_1\mathscr{K}_3}\frac{\mathscr{D}}{\mathscr{D}_3} + \mu_1\mathscr{W}_3\mathscr{D} - \frac{\eta_1\mathscr{A}_3\mathscr{K}_3}{1+\alpha_1\mathscr{K}_3}\frac{\mathscr{D}_3\mathscr{A}\mathscr{K}(1+\alpha_1\mathscr{K}_3)}{\mathscr{D}\mathscr{A}_3\mathscr{K}_3(1+\alpha_1\mathscr{K})} + \frac{\eta_1\mathscr{A}_3\mathscr{K}_3}{1+\alpha_1\mathscr{K}_3} - \frac{\delta}{\varphi}\mathscr{U}\\ &+ \frac{\eta_1\mathscr{A}_3\mathscr{K}_3}{1+\alpha_1\mathscr{K}_3}\frac{\mathscr{D}}{\mathscr{D}_3} - \frac{\eta_1\mathscr{A}_3\mathscr{K}_3}{1+\alpha_1\mathscr{K}_3}\frac{\mathscr{K}}{\mathscr{K}_3} - \frac{\eta_1\mathscr{A}_3\mathscr{K}_3}{1+\alpha_1\mathscr{K}_3}\frac{\mathscr{K}_3\mathscr{D}}{\mathscr{K}\mathscr{D}_3} + \frac{\eta_1\mathscr{A}_3\mathscr{K}_3}{1+\alpha_1\mathscr{K}_3}\\ &- \frac{\varsigma_1\mu_1}{\sigma_1\mathscr{W}_3} - \mu_1\mathscr{W}_3\mathscr{D} + \frac{\varsigma_1\mu_1}{\sigma_1}\mathscr{W}_3 - \frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}\\ &\leq -\alpha\frac{(\mathscr{A}-\mathscr{A}_3)^2}{\mathscr{A}} + \frac{\eta_1\mathscr{A}_3\mathscr{K}_3}{1+\alpha_1\mathscr{K}_3}\left(-1 - \frac{\mathscr{K}}{\mathscr{K}_3} + \frac{\mathscr{K}(1+\alpha_1\mathscr{K}_3)}{\mathscr{K}_3(1+\alpha_1\mathscr{K})} + \frac{1+\alpha_1\mathscr{K}}{1+\alpha_1\mathscr{K}_3}\right)\\ &+ \frac{\eta_1\mathscr{A}_3\mathscr{K}_3}{1+\alpha_1\mathscr{K}_3}\left(4 - \frac{\mathscr{A}_3}{\mathscr{A}} - \frac{\mathscr{K}_3\mathscr{D}}{\mathscr{K}\mathscr{D}_3} - \frac{\mathscr{D}_3\mathscr{A}\mathscr{K}(1+\alpha_1\mathscr{K}_3)}{\mathscr{D}\mathscr{A}_3\mathscr{K}_3(1+\alpha_1\mathscr{K})} - \frac{1+\alpha_1\mathscr{K}}{1+\alpha_1\mathscr{K}_3}\right)\\ &+ \frac{\eta_2\mathscr{A}_3\mathscr{U}}{1+\alpha_2\mathscr{U}} - \frac{\delta}{\varphi}\mathscr{U} - \frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}.\end{aligned} \tag{16}$$

Since we have

$$\begin{aligned}\frac{\eta_2\mathscr{A}_3\mathscr{U}}{1+\alpha_2\mathscr{U}} - \frac{\delta}{\varphi}\mathscr{U} &\leq \frac{\delta}{\varphi}\left(\frac{\varphi\eta_2\mathscr{A}_3}{\delta} - 1\right)\mathscr{U} = \frac{\delta}{\varphi}\left(\frac{\varphi\eta_2}{\delta}\left(\frac{\eth\varepsilon\sigma_1 + \eth\xi\varsigma_1\alpha_1}{\xi\varsigma_1\mathbb{Q}_1 + \alpha\varepsilon\sigma_1}\right) - 1\right)\mathscr{U}\\ &= \frac{\delta}{\varphi}\left(\mathrm{R}_1\left(\frac{\varepsilon\alpha\sigma_1 + \xi\varsigma_1\alpha\alpha_1 + \xi\varsigma_1\eta_1 - \xi\varsigma_1\eta_1}{b\varsigma_1\mathbb{Q}_1 + \alpha\varepsilon\sigma_1}\right) - 1\right)\mathscr{U} = \frac{\delta}{\varphi}\left(\mathrm{R}_1\left(1 - \frac{a}{\eth\sigma_1}\mathrm{R}_2\right) - 1\right)\mathscr{U},\end{aligned}$$

Hence, if $R_1 \leq 1$ and $R_2 \geq 1$ then $\frac{\delta}{\varphi}\left(R_1\left(1-\frac{a\varsigma_1}{\eth\sigma_1}R_2\right)-1\right)\mathscr{U} \leq 0$ Then, Eq. (16) becomes

$$\begin{aligned}\frac{dF_3}{dt} &= -\alpha\frac{(\mathscr{A}-\mathscr{A}_3)^2}{\mathscr{A}} - \frac{\eta_1\mathscr{A}_3\mathscr{K}_3}{1+\alpha_1\mathscr{K}_3}\left(\frac{\alpha_1(\mathscr{K}-\mathscr{K}_3)^2}{\mathscr{K}_3(1+\alpha_1\mathscr{K})(1+\alpha_1\mathscr{K}_3)}\right)\\ &+\frac{\eta_1\mathscr{A}_3\mathscr{K}_3}{1+\alpha_1\mathscr{K}_3}\left(4-\frac{\mathscr{A}_3}{\mathscr{A}}-\frac{\mathscr{K}_3\mathscr{D}}{\mathscr{K}\mathscr{D}_3}-\frac{\mathscr{D}_3\mathscr{A}\mathscr{K}(1+\alpha_1\mathscr{K}_3)}{\mathscr{D}\mathscr{A}_3\mathscr{K}_3(1+\alpha_1\mathscr{K})}-\frac{1+\alpha_1\mathscr{K}}{1+\alpha_1\mathscr{K}_3}\right)\\ &+\frac{\delta}{\varphi}\left(R_1\left(1-\frac{a}{\eth\sigma_1}R_2\right)-1\right)\mathscr{U}-\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}.\end{aligned}$$

Therefore $\frac{dF_3}{dt} \leq 0$ for each $\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z} > 0$, furthermore, $\frac{dF_3}{dt} = 0$ when $\mathscr{A}(t) = \mathscr{A}_3$, $\mathscr{D}(t) = \mathscr{D}_3, \mathscr{K}(t) = \mathscr{K}_3, \mathscr{W}(t) = \mathscr{W}_3$ and $\mathscr{U}(t) = \mathscr{Z}(t) = 0$. The system (1)-(6) solutions tend to Υ_3' which contains elements with $\mathscr{A}(t) = \mathscr{A}_3, \mathscr{D}(t) = \mathscr{D}_3, \mathscr{K}(t) = \mathscr{K}_3$, $\mathscr{W}(t) = \mathscr{W}_3$ and $\mathscr{U}(t) = \mathscr{Z}(t) = 0$. The result is, $\Upsilon_3' = \{\mathfrak{U}_3\}$. Then, $\mathfrak{U}_3$ is G.A.S. using L-LAS.□

Theorem 5. Let $R_0 \leq 1$and $R_3 \geq 1$, then $\mathfrak{U}_4$ is G.A.S.
Proof. Define $F_4(\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z})$ as:

$$F_4 = \mathscr{A}_4\mathfrak{D}\left(\frac{\mathscr{A}}{\mathscr{A}_4}\right)+\mathscr{D}+\frac{1}{\varphi}\mathscr{U}_4\mathfrak{D}\left(\frac{\mathscr{U}}{\mathscr{U}_4}\right)+\frac{a}{\xi}\mathscr{K}+\frac{\mu_1}{\sigma_1}\mathscr{W}+\frac{\mu_2}{\varphi\sigma_2}\mathscr{Z}_4\mathfrak{D}\left(\frac{\mathscr{Z}}{\mathscr{Z}_4}\right).$$

Calculating $\frac{dF_4}{dt}$ as:

$$\begin{aligned}\frac{dF_4}{dt} &= \left(1-\frac{\mathscr{A}_4}{\mathscr{A}}\right)\left(\eth-\alpha\mathscr{A}-\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}-\frac{\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}}\right)+\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}-a\mathscr{D}-\mu_1\mathscr{W}\mathscr{D}\\ &+\frac{1}{\varphi}\left(1-\frac{\mathscr{U}_4}{\mathscr{U}}\right)\left(\frac{\varphi\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}}-\delta\mathscr{U}-\mu_2\mathscr{Z}\mathscr{U}\right)+\frac{a}{\xi}(\xi\mathscr{D}-\varepsilon\mathscr{K})\\ &+\frac{\mu_1}{\sigma_1}(\sigma_1\mathscr{W}\mathscr{D}-\varsigma_1\mathscr{W})+\frac{\mu_2}{\varphi\sigma_2}\left(1-\frac{\mathscr{Z}_4}{\mathscr{Z}}\right)(\sigma_2\mathscr{Z}\mathscr{U}-\varsigma_2\mathscr{Z}).\end{aligned} \tag{17}$$

Collecting terms of Eq. (17), we obtain

$$\begin{aligned}\frac{dF_4}{dt} &= \left(1-\frac{\mathscr{A}_4}{\mathscr{A}}\right)(\eth-\alpha\mathscr{A})+\frac{\eta_1\mathscr{A}_4\mathscr{K}}{1+\alpha_1\mathscr{K}}+\frac{\eta_2\mathscr{A}_4\mathscr{U}}{1+\alpha_2\mathscr{U}}-\frac{\eta_2\mathscr{A}\mathscr{U}_4}{1+\alpha_2\mathscr{U}}-\frac{\delta}{\varphi}\mathscr{U}+\frac{\delta}{\varphi}\mathscr{U}_4\\ &+\frac{\mu_2}{\varphi}\mathscr{Z}\mathscr{U}_4-\frac{\varepsilon a}{\xi}\mathscr{K}-\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}-\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}-\frac{\mu_2}{\varphi}\mathscr{Z}_4\mathscr{U}+\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}_4.\end{aligned}$$

Utilizing the $\mathfrak{U}_4$ equilibrium conditions:

$$\eth = \alpha\mathscr{A}_4+\frac{\eta_2\mathscr{A}_4\mathscr{U}_4}{1+\alpha_2\mathscr{U}_4}, \quad \frac{\eta_2\mathscr{A}_4\mathscr{U}_4}{1+\alpha_2\mathscr{U}_4} = \frac{\delta}{\varphi}\mathscr{U}_4+\frac{\mu_2}{\varphi}\mathscr{Z}_4\mathscr{U}_4, \quad \mathscr{U}_4 = \frac{\varsigma_2}{\sigma_2},$$

we obtain

$$\begin{aligned}\frac{dF_4}{dt} &= \left(1-\frac{\mathscr{A}_4}{\mathscr{A}}\right)(\alpha\mathscr{A}_4-\alpha\mathscr{A})+\frac{\eta_2\mathscr{A}_4\mathscr{U}_4}{1+\alpha_2\mathscr{U}_4}\left(1-\frac{\mathscr{A}_4}{\mathscr{A}}\right)+\frac{\eta_1\mathscr{A}_4\mathscr{K}}{1+\alpha_1\mathscr{K}}+\frac{\eta_2\mathscr{A}_4\mathscr{U}}{1+\alpha_2\mathscr{U}}\\ &-\frac{\eta_2\mathscr{A}_4\mathscr{U}_4}{1+\alpha_2\mathscr{U}_4}\frac{\mathscr{A}(1+\alpha_2\mathscr{U}_4)}{\mathscr{A}_4(1+\alpha_2\mathscr{U})}-\frac{\eta_2\mathscr{A}_4\mathscr{U}}{1+\alpha_2\mathscr{U}_4}+\frac{\mu_2}{\varphi}\mathscr{Z}_4\mathscr{U}+\frac{\eta_2\mathscr{A}_4\mathscr{U}_4}{1+\alpha_2\mathscr{U}_4}-\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}_4\\ &+\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}-\frac{\varepsilon a}{\xi}\mathscr{K}-\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}-\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}-\frac{\mu_2}{\varphi}\mathscr{Z}_4\mathscr{U}+\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}_4\\ &\leq -\alpha\frac{(\mathscr{A}-\mathscr{A}_4)^2}{\mathscr{A}}+\frac{\eta_2\mathscr{A}_4\mathscr{U}_4}{1+\alpha_2\mathscr{U}_4}\left(1-\frac{\mathscr{A}_4}{\mathscr{A}}\right)+\frac{\eta_2\mathscr{A}_4\mathscr{U}}{1+\alpha_2\mathscr{U}}\\ &-\frac{\eta_2\mathscr{A}_4\mathscr{U}_4}{1+\alpha_2\mathscr{U}_4}\frac{\mathscr{A}(1+\alpha_2\mathscr{U}_4)}{\mathscr{A}_4(1+\alpha_2\mathscr{U})}-\frac{\eta_2\mathscr{A}_4\mathscr{U}}{1+\alpha_2\mathscr{U}_4}+\frac{\eta_2\mathscr{A}_4\mathscr{U}_4}{1+\alpha_2\mathscr{U}_4}\\ &+\frac{\eta_1\mathscr{A}_4\mathscr{K}}{1+\alpha_1\mathscr{K}}-\frac{\varepsilon a}{\xi}\mathscr{K}-\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}.\end{aligned} \tag{18}$$

We have

$$\frac{\eta_1\mathscr{A}_4\mathscr{K}}{1+\alpha_1\mathscr{K}}-\frac{\varepsilon a}{\xi}\mathscr{K}\leq\frac{\varepsilon a}{\xi}\left(\frac{\xi\eta_1\mathscr{A}_4}{\varepsilon a}-1\right)\mathscr{K}=\frac{\varepsilon a}{\xi}\left(\frac{\xi\eta_1}{\varepsilon a}\left(\frac{\eth\sigma_2+\eth\varsigma_2\alpha_2}{\alpha\sigma_2+\varsigma_2\mathbb{Q}_2}\right)-1\right)\mathscr{K}$$
$$=\frac{\varepsilon a}{\xi}\left(\mathrm{R}_0\left(\frac{\alpha\sigma_2+\varsigma_2\alpha\alpha_2+\varsigma_2\eta_2-\varsigma_2\eta_2}{\alpha\sigma_2+\varsigma_2\mathbb{Q}_2}\right)-1\right)\mathscr{K}=\frac{\varepsilon a}{\xi}\left(\mathrm{R}_0\left(1-\frac{\delta\varsigma_2}{\varphi\eth\sigma_2}\mathrm{R}_3\right)-1\right)\mathscr{K},$$

Hence, if $\mathrm{R}_0\leq 1$ and $\mathrm{R}_3\geq 1$ then $\frac{\varepsilon a}{\xi}\left(\mathrm{R}_0\left(1-\frac{\delta\varsigma_2}{\varphi\eth\sigma_2}\mathrm{R}_3\right)-1\right)\mathscr{K}\leq 0$. Furthermore, Eq. (18) will be in the form:

$$\frac{dF_4}{dt}=-\alpha\frac{(\mathscr{A}-\mathscr{A}_4)^2}{\mathscr{A}}-\frac{\eta_2\mathscr{A}_4\mathscr{U}_4}{1+\alpha_2\mathscr{U}_4}\left(\frac{\alpha_2(\mathscr{U}-\mathscr{U}_4)^2}{\mathscr{U}_4(1+\alpha_2\mathscr{U})(1+\alpha_2\mathscr{U}_4)}\right)$$
$$+\frac{\eta_2\mathscr{A}_4\mathscr{U}_4}{1+\alpha_2\mathscr{U}_4}\left(3-\frac{\mathscr{A}_4}{\mathscr{A}}-\frac{\mathscr{A}(1+\alpha_2\mathscr{U}_4)}{\mathscr{A}_4(1+\alpha_2\mathscr{U})}-\frac{1+\alpha_2\mathscr{U}}{1+\alpha_2\mathscr{U}_4}\right)$$
$$+\frac{\varepsilon a}{\xi}\left(\mathrm{R}_0\left(1-\frac{\delta\varsigma_2}{\varphi\eth\sigma_2}\mathrm{R}_3\right)-1\right)\mathscr{K}-\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}.$$

Therefore $\frac{dF_4}{dt}\leq 0$ for each $\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z}>0$, Furthermore, $\frac{dF_4}{dt}=0$ when $\mathscr{A}(t)=\mathscr{A}_4,\mathscr{U}(t)=\mathscr{U}_4,\mathscr{Z}(t)=\mathscr{Z}_4$ and $\mathscr{D}(t)=\mathscr{K}(t)=\mathscr{W}(t)=0$. The system (1)-(6) solutions tend to Υ_4' which includes elements with $\mathscr{A}(t)=\mathscr{A}_4,\mathscr{U}(t)=\mathscr{U}_4,\mathscr{Z}(t)=\mathscr{Z}_4$ and $\mathscr{D}(t)=\mathscr{K}(t)=\mathscr{W}(t)=0$. The result is, $\Upsilon_4'=\{\mathfrak{U}_4\}$.Then, $\mathfrak{U}_4$ is G.A.S. using L-LAS.□

Theorem 6. Let $\mathrm{R}_4>1$, $\mathrm{R}_5>1$, $\mathrm{R}_7\leq 1$ and $\mathrm{R}_9\leq 1$, then $\mathfrak{U}_5$ is G.A.S.
Proof. Define $F_5(\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z})$ as:

$$F_5=\mathscr{A}_5\mathfrak{D}\left(\frac{\mathscr{A}}{\mathscr{A}_5}\right)+\mathscr{D}_5\mathfrak{D}\left(\frac{\mathscr{D}}{\mathscr{D}_5}\right)+\frac{1}{\varphi}\mathscr{U}_5\mathfrak{D}\left(\frac{\mathscr{U}}{\mathscr{U}_5}\right)+\frac{\eta_1\mathscr{A}_5}{\varepsilon(1+\alpha_1\mathscr{K}_5)}\mathscr{K}_5\mathfrak{D}\left(\frac{\mathscr{K}}{\mathscr{K}_5}\right)+\frac{\mu_1}{\sigma_1}\mathscr{W}_5+\frac{\mu_2}{\varphi\sigma_2}\mathscr{Z}_5.$$

Calculating $\frac{dF_5}{dt}$ as:

$$\frac{dF_5}{dt}=\left(1-\frac{\mathscr{A}_5}{\mathscr{A}}\right)\left(\eth-\alpha\mathscr{A}-\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}-\frac{\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}}\right)+\left(1-\frac{\mathscr{D}_5}{\mathscr{D}}\right)\left(\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}-a\mathscr{D}-\mu_1\mathscr{W}\mathscr{D}\right)$$
$$+\frac{1}{\varphi}\left(1-\frac{\mathscr{U}_5}{\mathscr{U}}\right)\left(\frac{\varphi\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}}-\delta\mathscr{U}-\mu_2\mathscr{Z}\mathscr{U}\right)+\frac{\eta_1\mathscr{A}_5}{\varepsilon(1+\alpha_1\mathscr{K}_5)}\left(1-\frac{\mathscr{K}_5}{\mathscr{K}}\right)(\xi\mathscr{D}-\varepsilon\mathscr{K})$$
$$+\frac{\mu_1}{\sigma_1}(\sigma_1\mathscr{W}\mathscr{D}-\varsigma_1\mathscr{W})+\frac{\mu_2}{\varphi\sigma_2}(\sigma_2\mathscr{Z}\mathscr{U}-\varsigma_2\mathscr{Z})$$
$$=\left(1-\frac{\mathscr{A}_5}{\mathscr{A}}\right)(\eth-\alpha\mathscr{A})+\frac{\eta_1\mathscr{A}_5\mathscr{K}}{1+\alpha_1\mathscr{K}}+\frac{\eta_2\mathscr{A}_5\mathscr{U}}{1+\alpha_2\mathscr{U}}-a\mathscr{D}-\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}\frac{\mathscr{D}_5}{\mathscr{D}}+a\mathscr{D}_5$$
$$+\mu_1\mathscr{D}_5\mathscr{W}-\frac{\delta}{\varphi}\mathscr{U}-\frac{\eta_2\mathscr{A}\mathscr{U}_5}{1+\alpha_2\mathscr{U}}+\frac{\delta}{\varphi}\mathscr{U}_5+\frac{\mu_2}{\varphi}\mathscr{U}_5\mathscr{Z}+\frac{\eta_1\mathscr{A}_5}{1+\alpha_1\mathscr{K}_5}\frac{\xi\mathscr{D}}{\varepsilon}$$
$$-\frac{\eta_1\mathscr{A}_5\mathscr{K}}{1+\alpha_1\mathscr{K}_5}-\frac{\eta_1\mathscr{A}_5\mathscr{K}_5}{1+\alpha_1\mathscr{K}_5}\frac{\xi\mathscr{D}}{\varepsilon\mathscr{K}}+\frac{\eta_1\mathscr{A}_5\mathscr{K}_5}{1+\alpha_1\mathscr{K}_5}-\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}-\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}.$$

Utilizing the $\mathfrak{U}_5$ equilibrium conditions:

$$\eth=\alpha\mathscr{A}_5+\frac{\eta_1\mathscr{A}_5\mathscr{K}_5}{1+\alpha_1\mathscr{K}_5}+\frac{\eta_2\mathscr{A}_5\mathscr{U}_5}{1+\alpha_2\mathscr{U}_5},\quad\frac{\phi\eta_2\mathscr{A}_5\mathscr{U}_5}{1+\alpha_2\mathscr{U}_5}=\delta\mathscr{U}_5,$$
$$\mathscr{K}_5=\frac{\xi}{\varepsilon}\mathscr{D}_5,\ \frac{\eta_1\mathscr{A}_5\mathscr{K}_5}{1+\alpha_1\mathscr{K}_5}=a\mathscr{D}_5,$$

we obtain

$$\begin{aligned}\frac{dF_5}{dt} &= \left(1-\frac{\mathscr{A}_5}{\mathscr{A}}\right)(\alpha\mathscr{A}_5-\alpha\mathscr{A})+\left(1-\frac{\mathscr{A}_5}{\mathscr{A}}\right)\left(\frac{\eta_1\mathscr{A}_5\mathscr{K}_5}{1+\alpha_1\mathscr{K}_5}+\frac{\eta_2\mathscr{A}_5\mathscr{U}_5}{1+\alpha_2\mathscr{U}_5}\right)+\frac{\eta_1\mathscr{A}_5\mathscr{K}}{1+\alpha_1\mathscr{K}}+\frac{\eta_2\mathscr{A}_5\mathscr{U}}{1+\alpha_2\mathscr{U}}\\
&-\frac{\eta_1\mathscr{A}_5\mathscr{K}_5}{1+\alpha_1\mathscr{K}_5}\frac{\mathscr{D}}{\mathscr{D}_5}-\frac{\eta_1\mathscr{A}_5\mathscr{K}_5}{1+\alpha_1\mathscr{K}_5}\frac{\mathscr{A}\mathscr{K}\mathscr{D}_5(1+\alpha_1\mathscr{K}_5)}{\mathscr{A}_5\mathscr{K}_5\mathscr{D}(1+\alpha_1\mathscr{K})}+\frac{\eta_1\mathscr{A}_5\mathscr{K}_5}{1+\alpha_1\mathscr{K}_5}+\mu_1\mathscr{D}_5\mathscr{W}-\frac{\eta_2\mathscr{A}_5\mathscr{U}_5}{1+\alpha_2\mathscr{U}_5}\frac{\mathscr{U}}{\mathscr{U}_5}\\
&-\frac{\eta_2\mathscr{A}_5\mathscr{U}_5}{1+\alpha_2\mathscr{U}_5}\frac{\mathscr{A}(1+\alpha_2\mathscr{U}_5)}{\mathscr{A}_5(1+\alpha_2\mathscr{U})}+\frac{\eta_2\mathscr{A}_5\mathscr{U}_5}{1+\alpha_2\mathscr{U}_5}+\frac{\mu_2}{\varphi}\mathscr{U}_5\mathscr{Z}+\frac{\eta_1\mathscr{A}_5\mathscr{K}_5}{(1+\alpha_1\mathscr{K}_5)}\frac{\mathscr{D}}{\mathscr{D}_5}-\frac{\eta_1\mathscr{A}_5\mathscr{K}_5}{(1+\alpha_1\mathscr{K}_5)}\frac{\mathscr{K}}{\mathscr{K}_5}\\
&-\frac{\eta_1\mathscr{A}_5\mathscr{K}_5}{1+\alpha_1\mathscr{K}_5}\frac{\mathscr{K}_5\mathscr{D}}{\mathscr{D}_5\mathscr{K}}+\frac{\eta_1\mathscr{A}_5\mathscr{K}_5}{1+\alpha_1\mathscr{K}_5}-\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}-\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}\\
&=-\alpha\frac{(\mathscr{A}-\mathscr{A}_5)^2}{\mathscr{A}}-\frac{\alpha_2(\mathscr{U}-\mathscr{U}_5)^2}{\mathscr{U}_5(1+\alpha_2\mathscr{U})(1+\alpha_2\mathscr{U}_5)}-\frac{\alpha_1(\mathscr{K}-\mathscr{K}_5)^2}{\mathscr{K}_5(1+\alpha_1\mathscr{K})(1+\alpha_1\mathscr{K}_5)}\\
&+\frac{\eta_1\mathscr{A}_5\mathscr{K}_5}{1+\alpha_1\mathscr{K}_5}\left(4-\frac{\mathscr{A}_5}{\mathscr{A}}+\frac{\mathscr{A}\mathscr{K}\mathscr{D}_5(1+\alpha_1\mathscr{K}_5)}{\mathscr{A}_5\mathscr{K}_5\mathscr{D}(1+\alpha_1\mathscr{K})}-\frac{\mathscr{K}_5\mathscr{D}}{\mathscr{K}\mathscr{D}_5}-\frac{1+\alpha_1\mathscr{K}}{1+\alpha_1\mathscr{K}_5}\right)\\
&+\frac{\eta_2\mathscr{A}_5\mathscr{U}_5}{1+\alpha_2\mathscr{U}_5}\left(3-\frac{\mathscr{A}_5}{\mathscr{A}}-\frac{\mathscr{A}(1+\alpha_2\mathscr{U}_5)}{\mathscr{A}_5(1+\alpha_2\mathscr{U})}-\frac{1+\alpha_2\mathscr{U}}{1+\alpha_2\mathscr{U}_5}\right)\\
&+\mu_1\left(\mathscr{D}_5-\frac{\varsigma_1}{\sigma_1}\right)\mathscr{W}+\frac{\mu_2}{\varphi}\left(\mathscr{U}_5-\frac{\varsigma_2}{\sigma_2}\right)\mathscr{Z}.\end{aligned}$$

We have

$$\begin{aligned}\mathscr{D}_5-\frac{\varsigma_1}{\sigma_1}&=\frac{\xi\varsigma_1(\alpha_2\mathbb{Q}_1+\alpha_1\eta_2)+\varepsilon\sigma_1\mathbb{Q}_2}{\xi\sigma_1(\alpha_2\mathbb{Q}_1+\alpha_1\eta_2)}\left(\frac{\xi\eta_1\sigma_1(\eth\phi\alpha_2+\delta)}{a\phi(\xi\varsigma_1(\alpha_2\mathbb{Q}_1+\alpha_1\eta_2)+\varepsilon\sigma_1\mathbb{Q}_2)}-1\right),\\
\mathscr{U}_5-\frac{\varsigma_2}{\sigma_2}&=\frac{\mathbb{Q}_1(\sigma_2+\alpha_2\varsigma_2)+\eta_2\varsigma_2\alpha_1}{\sigma_2(\alpha_2\mathbb{Q}_1+\alpha_1\eta_2)}\left(\frac{\phi\eta_2\sigma_2(a\varepsilon+\alpha_1\xi\eth)}{\xi\delta(\mathbb{Q}_1(\sigma_2+\alpha_2\varsigma_2)+\eta_2\varsigma_2\alpha_1)}-1\right).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\frac{dF_5}{dt} &=-\alpha\frac{(\mathscr{A}-\mathscr{A}_5)^2}{\mathscr{A}}-\frac{\alpha_2(\mathscr{U}-\mathscr{U}_5)^2}{\mathscr{U}_5(1+\alpha_2\mathscr{U})(1+\alpha_2\mathscr{U}_5)}-\frac{\alpha_1(\mathscr{K}-\mathscr{K}_5)^2}{\mathscr{K}_5(1+\alpha_1\mathscr{K})(1+\alpha_1\mathscr{K}_5)}\\
&+\frac{\eta_1\mathscr{A}_5\mathscr{K}_5}{1+\alpha_1\mathscr{K}_5}\left(4-\frac{\mathscr{A}_5}{\mathscr{A}}+\frac{\mathscr{A}\mathscr{K}\mathscr{D}_5(1+\alpha_1\mathscr{K}_5)}{\mathscr{A}_5\mathscr{K}_5\mathscr{D}(1+\alpha_1\mathscr{K})}-\frac{\mathscr{K}_5\mathscr{D}}{\mathscr{K}\mathscr{D}_5}-\frac{1+\alpha_1\mathscr{K}}{1+\alpha_1\mathscr{K}_5}\right)\\
&+\frac{\eta_2\mathscr{A}_5\mathscr{U}_5}{1+\alpha_2\mathscr{U}_5}\left(3-\frac{\mathscr{A}_5}{\mathscr{A}}-\frac{\mathscr{A}(1+\alpha_2\mathscr{U}_5)}{\mathscr{A}_5(1+\alpha_2\mathscr{U})}-\frac{1+\alpha_2\mathscr{U}}{1+\alpha_2\mathscr{U}_5}\right).\\
&+\frac{\mu_1(\xi\varsigma_1(\alpha_2\mathbb{Q}_1+\alpha_1\eta_2)+\varepsilon\sigma_1\mathbb{Q}_2)}{\xi\sigma_1(\alpha_2\mathbb{Q}_1+\alpha_1\eta_2)}(\mathrm{R}_7-1)\mathscr{W}+\frac{\mu_2(\mathbb{Q}_1(\sigma_2+\alpha_2\varsigma_2)+\eta_2\varsigma_2\alpha_1)}{\varphi\sigma_2(\alpha_2\mathbb{Q}_1+\alpha_1\eta_2)}(\mathrm{R}_9-1)\mathscr{Z}.\end{aligned}$$

Therefore $\frac{dF_5}{dt}\leq 0$ for each $\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z}>0$, Furthermore, $\frac{dF_5}{dt}=0$ when $\mathscr{A}(t)=\mathscr{A}_5,\mathscr{D}(t)=\mathscr{D}_5,\mathscr{U}(t)=\mathscr{U}_5,\mathscr{K}(t)=\mathscr{K}_5$ and $\mathscr{Z}(t)=\mathscr{W}(t)=0$. The system (1)-(6) solutions tend to Υ_5' which has elements with $\mathscr{A}(t)=\mathscr{A}_5$, $\mathscr{D}(t)=\mathscr{D}_5$, $\mathscr{U}(t)=\mathscr{U}_5$, $\mathscr{K}(t)=\mathscr{K}_5$ and $\mathscr{Z}(t)=\mathscr{W}(t)=0$. The result is, $\Upsilon_5'=\{\mathfrak{U}_5\}$. Then, $\mathfrak{U}_5$ is G.A.S. using L-LAS.□

Theorem 7. If $\mathrm{R}_6>1$, $\mathrm{R}_7>1$,and $\mathrm{R}_{10}\leq 1$, then $\mathfrak{U}_6$ is G.A.S.
Proof. Define $F_6(\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z})$ as:

$$F_6=\mathscr{A}_6\mathfrak{D}\left(\frac{\mathscr{A}}{\mathscr{A}_6}\right)+\mathscr{D}_6\mathfrak{D}\left(\frac{\mathscr{D}}{\mathscr{D}_6}\right)+\frac{1}{\varphi}\mathscr{U}_6\mathfrak{D}\left(\frac{\mathscr{U}}{\mathscr{U}_6}\right)+\frac{\eta_1\mathscr{A}_6}{\varepsilon(1+\alpha_1\mathscr{K}_6)}\mathscr{K}_6\mathfrak{D}\left(\frac{\mathscr{K}}{\mathscr{K}_6}\right)+\frac{\mu_1}{\sigma_1}\mathscr{W}_6\mathfrak{D}\left(\frac{\mathscr{W}}{\mathscr{W}_6}\right)+\frac{\mu_2}{\varphi\sigma_2}\mathscr{Z}.$$

Calculating $\frac{dF_6}{dt}$ as:

$$\begin{aligned}\frac{dF_6}{dt} &= \left(1-\frac{\mathscr{A}_6}{\mathscr{A}}\right)\left(\eth-\alpha\mathscr{A}-\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}-\frac{\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}}\right)+\left(1-\frac{\mathscr{D}_6}{\mathscr{D}}\right)\left(\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}-a\mathscr{D}-\mu_1\mathscr{W}\mathscr{D}\right)\\
&+\frac{1}{\varphi}\left(1-\frac{\mathscr{U}_6}{\mathscr{U}}\right)\left(\frac{\varphi\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}}-\delta\mathscr{U}-\mu_2\mathscr{Z}\mathscr{U}\right)+\frac{\eta_1\mathscr{A}_6}{\varepsilon(1+\alpha_1\mathscr{K}_6)}\left(1-\frac{\mathscr{K}_6}{\mathscr{K}}\right)(\xi\mathscr{D}-\varepsilon\mathscr{K})\\
&+\frac{\mu_1}{\sigma_1}\left(1-\frac{\mathscr{W}_6}{\mathscr{W}}\right)(\sigma_1\mathscr{W}\mathscr{D}-\varsigma_1\mathscr{W})+\frac{\mu_2}{\varphi\sigma_2}(\sigma_2\mathscr{Z}\mathscr{U}-\varsigma_2\mathscr{Z})\\
&=\left(1-\frac{\mathscr{A}_6}{\mathscr{A}}\right)(\eth-\alpha\mathscr{A})+\frac{\eta_2\mathscr{A}_6\mathscr{U}}{1+\alpha_2\mathscr{U}}+\frac{\eta_1\mathscr{A}_6\mathscr{K}}{1+\alpha_1\mathscr{K}}-a\mathscr{D}-\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}\frac{\mathscr{D}_6}{\mathscr{D}}+a\mathscr{D}_6+\mu_1\mathscr{W}\mathscr{D}_6-\frac{\delta}{\varphi}\mathscr{U}\\
&-\frac{\eta_2\mathscr{A}\mathscr{U}_6}{1+\alpha_2\mathscr{U}}+\frac{\delta}{\varphi}\mathscr{U}_6+\frac{\mu_2}{\varphi}\mathscr{Z}\mathscr{U}_6+\frac{\eta_1\mathscr{A}_6}{(1+\alpha_1\mathscr{K}_6)}\frac{\xi\mathscr{D}}{\varepsilon}-\frac{\eta_1\mathscr{A}_6\mathscr{K}_6}{1+\alpha_1\mathscr{K}_6}\frac{b\mathscr{D}}{\varepsilon\mathscr{K}}+\frac{\eta_1\mathscr{A}_6\mathscr{K}_6}{1+\alpha_1\mathscr{K}_6}-\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}\\
&-\frac{\eta_1\mathscr{A}_6\mathscr{K}}{(1+\alpha_1\mathscr{K}_6)}-\mu_1\mathscr{W}_6\mathscr{D}+\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}_6-\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}.\end{aligned}$$

Utilizing the $\mathfrak{U}_6$ equilibrium conditions:

$$\eth=\alpha\mathscr{A}_6+\frac{\eta_1\mathscr{A}_6\mathscr{K}_6}{1+\alpha_1\mathscr{K}_6}+\frac{\eta_2\mathscr{A}_6\mathscr{U}_6}{1+\alpha_2\mathscr{U}_6},\ \frac{\eta_1\mathscr{A}_6\mathscr{K}_6}{1+\alpha_1\mathscr{K}_6}=a\mathscr{D}_6+\mu_1\mathscr{W}_6\mathscr{D}_6,\ \frac{\eta_2\mathscr{A}_6\mathscr{U}_6}{1+\alpha_2\mathscr{U}_6}=\frac{\delta}{\varphi}\mathscr{U}_6,\ \mathscr{D}_6=\frac{\varsigma_1}{\sigma_1},\ \mathscr{K}_6=\frac{\xi\mathscr{D}_6}{\varepsilon},$$

we obtain

$$\begin{aligned}\frac{dF_6}{dt}&=\left(1-\frac{\mathscr{A}_6}{\mathscr{A}}\right)(\alpha\mathscr{A}_6-\alpha\mathscr{A})+\left(1-\frac{\mathscr{A}_6}{\mathscr{A}}\right)\left(\frac{\eta_1\mathscr{A}_6\mathscr{K}_6}{1+\alpha_1\mathscr{K}_6}+\frac{\eta_2\mathscr{A}_6\mathscr{U}_6}{1+\alpha_2\mathscr{U}_6}\right)+\frac{\eta_2\mathscr{A}_6\mathscr{U}}{1+\alpha_2\mathscr{U}}+\frac{\eta_1\mathscr{A}_6\mathscr{K}}{1+\alpha_1\mathscr{K}}\\
&-\frac{\eta_1\mathscr{A}_6\mathscr{K}_6}{1+\alpha_1\mathscr{K}_6}\frac{\mathscr{D}}{\mathscr{D}_6}+\mu_1\mathscr{C}_6^{\mathscr{D}}\mathscr{D}-\frac{\eta_1\mathscr{A}_6\mathscr{K}_6}{1+\alpha_1\mathscr{K}_6}\frac{\mathscr{A}\mathscr{K}\mathscr{D}_6(1+\alpha_1\mathscr{K}_6)}{\mathscr{A}_6\mathscr{K}_6\mathscr{D}(1+\alpha_1\mathscr{K})}+\frac{\eta_1\mathscr{A}_6\mathscr{K}_6}{1+\alpha_1\mathscr{K}_6}-\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}_6\\
&-\frac{\eta_2\mathscr{A}_6\mathscr{U}_6}{1+\alpha_2\mathscr{U}_6}\frac{\mathscr{U}}{\mathscr{U}_6}-\frac{\eta_2\mathscr{A}_6\mathscr{U}_6}{1+\alpha_2\mathscr{U}_6}\frac{\mathscr{A}(1+\alpha_2\mathscr{U}_6)}{\mathscr{A}_6(1+\alpha_2\mathscr{U})}+\frac{\eta_2\mathscr{A}_6\mathscr{U}_6}{1+\alpha_2\mathscr{U}_6}+\frac{\mu_2}{\varphi}\mathscr{Z}\mathscr{U}_6+\frac{\eta_1\mathscr{A}_6\mathscr{K}_6}{1+\alpha_1\mathscr{K}_6}\frac{\mathscr{D}}{\mathscr{D}_6}\\
&-\frac{\eta_1\mathscr{A}_6\mathscr{K}_6}{1+\alpha_1\mathscr{K}_6}\frac{\mathscr{K}_6\mathscr{D}}{\mathscr{K}\mathscr{D}_6}+\frac{\eta_1\mathscr{A}_6\mathscr{K}_6}{1+\alpha_1\mathscr{K}_6}-\frac{\eta_1\mathscr{A}_6\mathscr{K}_6}{(1+\alpha_1\mathscr{K}_6)}\frac{\mathscr{K}}{\mathscr{K}_6}-\mu_1\mathscr{W}_6\mathscr{D}+\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}_6-\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}\\
&=-\alpha\frac{(\mathscr{A}-\mathscr{A}_6)^2}{\mathscr{A}}-\frac{\alpha_2(\mathscr{U}-\mathscr{U}_6)^2}{\mathscr{U}_6(1+\alpha_2\mathscr{U})(1+\alpha_2\mathscr{U}_6)}-\frac{\alpha_1(\mathscr{K}-\mathscr{K}_6)^2}{\mathscr{K}_6(1+\alpha_1\mathscr{K})(1+\alpha_1\mathscr{K}_6)}\\
&+\frac{\eta_1\mathscr{A}_6\mathscr{K}_6}{1+\alpha_1\mathscr{K}_6}\left(4-\frac{\mathscr{A}_6}{\mathscr{A}}-\frac{\mathscr{A}\mathscr{K}\mathscr{D}_6(1+\alpha_1\mathscr{K}_6)}{\mathscr{A}_6\mathscr{K}_6\mathscr{D}(1+\alpha_1\mathscr{K})}-\frac{\mathscr{K}_6\mathscr{D}}{\mathscr{K}\mathscr{D}_6}-\frac{1+\alpha_1\mathscr{K}}{1+\alpha_1\mathscr{K}_6}\right)\\
&+\frac{\eta_2\mathscr{A}_6\mathscr{U}_6}{1+\alpha_2\mathscr{U}_6}\left(3-\frac{\mathscr{A}_6}{\mathscr{A}}-\frac{\mathscr{A}(1+\alpha_2\mathscr{U}_6)}{\mathscr{A}_6(1+\alpha_2\mathscr{U})}-\frac{1+\alpha_2\mathscr{U}}{1+\alpha_2\mathscr{U}_6}\right)+\frac{\mu_2}{\varphi}\left(\mathscr{U}_6-\frac{\varsigma_2}{\sigma_2}\right)\mathscr{Z}.\end{aligned}$$

We have

$$\mathscr{U}_6-\frac{\varsigma_2}{\sigma_2}=\frac{\mu_2(\mathbb{Q}_1\xi\varsigma_1(\sigma_2+\varsigma_2\alpha_2)+\varepsilon\sigma_1(\alpha\sigma_2+\mathbb{Q}_2\varsigma_2)+\xi\alpha_1\eta_2\varsigma_1\varsigma_2)}{\varphi(\xi\varsigma_1\alpha_1\eta_2+\mathbb{Q}_1\xi\varsigma_1\alpha_2+\mathbb{Q}_2\varepsilon\sigma_1)}(\mathrm{R}_{10}-1).$$

Then

$$\begin{aligned}\frac{dF_6}{dt}&=-\alpha\frac{(\mathscr{A}-\mathscr{A}_6)^2}{\mathscr{A}}-\frac{\alpha_2(\mathscr{U}-\mathscr{U}_6)^2}{\mathscr{U}_6(1+\alpha_2\mathscr{U})(1+\alpha_2\mathscr{U}_6)}-\frac{\alpha_1(\mathscr{K}-\mathscr{K}_6)^2}{\mathscr{K}_6(1+\alpha_1\mathscr{K})(1+\alpha_1\mathscr{K}_6)}\\
&+\frac{\eta_1\mathscr{A}_6\mathscr{K}_6}{1+\alpha_1\mathscr{K}_6}\left(4-\frac{\mathscr{A}_6}{\mathscr{A}}-\frac{\mathscr{A}\mathscr{K}\mathscr{D}_6(1+\alpha_1\mathscr{K}_6)}{\mathscr{A}_6\mathscr{K}_6\mathscr{D}(1+\alpha_1\mathscr{K})}-\frac{\mathscr{K}_6\mathscr{D}}{\mathscr{K}\mathscr{D}_6}-\frac{1+\alpha_1\mathscr{K}}{1+\alpha_1\mathscr{K}_6}\right)\\
&+\frac{\eta_2\mathscr{A}_6\mathscr{U}_6}{1+\alpha_2\mathscr{U}_6}\left(3-\frac{\mathscr{A}_6}{\mathscr{A}}-\frac{\mathscr{A}(1+\alpha_2\mathscr{U}_6)}{\mathscr{A}_6(1+\alpha_2\mathscr{U})}-\frac{1+\alpha_2\mathscr{U}}{1+\alpha_2\mathscr{U}_6}\right)\\
&+\frac{\mu_2(\mathbb{Q}_1\xi\varsigma_1(\sigma_2+\varsigma_2\alpha_2)+\varepsilon\sigma_1(\alpha\sigma_2+\mathbb{Q}_2\varsigma_2)+\xi\alpha_1\eta_2\varsigma_1\varsigma_2)}{\varphi(\xi\varsigma_1\alpha_1\eta_2+\mathbb{Q}_1\xi\varsigma_1\alpha_2+\mathbb{Q}_2\varepsilon\sigma_1)}(\mathrm{R}_{10}-1)\mathscr{Z}.\end{aligned}$$

Hence, if $\mathrm{R}_6>1$, $\mathrm{R}_7>1$ and $\mathrm{R}_{10}\le 1$, then $\frac{dF_6}{dt}\le 0$ for each $\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z}>0$, and $\frac{dF_6}{dt}=0$ when $\mathscr{A}(t)=\mathscr{A}_6$, $\mathscr{D}(t)=\mathscr{D}_6$, $\mathscr{K}(t)=\mathscr{K}_6$ and $\mathscr{Z}(t)=0$. The system (1)-(6) solutions converge to Υ_6' which contains element with

$\mathscr{A}(t)=\mathscr{A}_6$, $\mathscr{U}(t)=\mathscr{U}_6$, $\mathscr{D}(t)=\mathscr{D}_6$, $\mathscr{K}(t)=\mathscr{K}_6$ and $\mathscr{Z}(t)=0$. Hence, $\dot{\mathscr{A}}=\dot{\mathscr{D}}=0$ and from the system (1)-(6) second equations, we have

$$0=\dot{\mathscr{D}}=\frac{\eta_1\mathscr{A}(t)\mathscr{K}_6}{1+\alpha_1\mathscr{K}_6}-a\mathscr{D}(t)-\mu_1\mathscr{D}(t)\mathscr{W}(t),$$

which gives $\mathscr{W}(t)=\mathscr{C}_6^{\mathscr{D}}$ for all t. Therefore, $\Upsilon_6'=\{\mathfrak{U}_6\}$. Applying L-LAS we get $\mathfrak{U}_6$ is G.A.S.

Theorem 8. If $\mathrm{R}_8>1$, $\mathrm{R}_9>1$and $\mathrm{R}_{11}\le 1$, then $\mathfrak{U}_7$ is G.A.S .
Proof. Define $\digamma_7(\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z})$ as:

$$\digamma_7=\mathscr{A}_7\mathfrak{D}\left(\frac{\mathscr{A}}{\mathscr{A}_7}\right)+\mathscr{D}_7\mathfrak{D}\left(\frac{\mathscr{D}}{\mathscr{D}_7}\right)+\frac{1}{\varphi}\mathscr{U}_7\mathfrak{D}\left(\frac{\mathscr{U}}{\mathscr{U}_7}\right)+\frac{\eta_1\mathscr{A}_7}{\varepsilon(1+\alpha_1\mathscr{K}_7)}\mathscr{K}_7\mathfrak{D}\left(\frac{\mathscr{K}}{\mathscr{K}_7}\right)+\frac{\mu_1}{\sigma_1}\mathscr{W}+\frac{\mu_2}{\varphi\sigma_2}\mathscr{Z}_7\mathfrak{D}\left(\frac{\mathscr{Z}}{\mathscr{Z}_7}\right).$$

Calculating $\frac{d\digamma_7}{dt}$ as:

$$\begin{aligned}\frac{d\digamma_7}{dt}&=\left(1-\frac{\mathscr{A}_7}{\mathscr{A}}\right)\left(\eth-\alpha\mathscr{A}-\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}-\frac{\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}}\right)+\left(1-\frac{\mathscr{D}_7}{\mathscr{D}}\right)\left(\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}-a\mathscr{D}-\mu_1\mathscr{W}\mathscr{D}\right)\\&+\frac{1}{\varphi}\left(1-\frac{\mathscr{U}_7}{\mathscr{U}}\right)\left(\frac{\varphi\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}}-\delta\mathscr{U}-\mu_2\mathscr{Z}\mathscr{U}\right)+\frac{\eta_1\mathscr{A}_7}{\varepsilon(1+\alpha_1\mathscr{K}_7)}\left(1-\frac{\mathscr{K}_7}{\mathscr{K}}\right)(\xi\mathscr{D}-\varepsilon\mathscr{K})\\&+\frac{\mu_1}{\sigma_1}(\sigma_1\mathscr{W}\mathscr{D}-\varsigma_1\mathscr{W})+\frac{\mu_2}{\varphi\sigma_2}\left(1-\frac{\mathscr{Z}_7}{\mathscr{Z}}\right)(\sigma_2\mathscr{Z}\mathscr{U}-\varsigma_2\mathscr{Z})\\&=\left(1-\frac{\mathscr{A}_7}{\mathscr{A}}\right)(\eth-\alpha\mathscr{A})+\frac{\eta_1\mathscr{A}_5\mathscr{K}}{1+\alpha_1\mathscr{K}}+\frac{\eta_2\mathscr{A}_7\mathscr{U}}{1+\alpha_2\mathscr{U}}-a\mathscr{D}-\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}\frac{\mathscr{D}_7}{\mathscr{D}}+a\mathscr{D}_7\\&+\mu_1\mathscr{W}\mathscr{D}_7-\frac{\delta}{\varphi}\mathscr{U}-\frac{\eta_2\mathscr{A}\mathscr{U}_7}{1+\alpha_2\mathscr{U}}-\frac{\eta_1\mathscr{A}_7\mathscr{K}}{(1+\alpha_1\mathscr{K}_7)}+\frac{\delta}{\varphi}\mathscr{U}_7+\frac{\mu_2}{\varphi}\mathscr{Z}\mathscr{U}_7+\frac{\eta_1\mathscr{A}_7}{(1+\alpha_1\mathscr{K}_7)}\frac{\xi\mathscr{D}}{\varepsilon}\\&-\frac{\eta_1\mathscr{A}_7\mathscr{K}_7}{1+\alpha_1\mathscr{K}_7}\frac{\xi\mathscr{D}}{\varepsilon\mathscr{K}}+\frac{\eta_1\mathscr{A}_7\mathscr{K}_7}{1+\alpha_1\mathscr{K}_7}-\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}-\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}-\frac{\mu_2}{\varphi}\mathscr{Z}_7\mathscr{U}+\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}_7.\end{aligned}$$

Utilizing the $\mathfrak{U}_7$ equilibrium conditions:

$$\eth=\alpha\mathscr{A}_7+\frac{\eta_1\mathscr{A}_7\mathscr{K}_7}{1+\alpha_1\mathscr{K}_7}+\frac{\eta_2\mathscr{A}_7\mathscr{U}_7}{1+\alpha_2\mathscr{U}_7},\ \frac{\phi\eta_2\mathscr{A}_7\mathscr{U}_7}{1+\alpha_2\mathscr{U}_7}=\delta\mathscr{U}_7+\mu_2\mathscr{U}_7\mathscr{Z}_7,\ \mathscr{K}_7=\frac{\xi}{\varepsilon}\mathscr{D}_7,\ \frac{\eta_1\mathscr{A}_7\mathscr{K}_7}{1+\alpha_1\mathscr{K}_7}=a\mathscr{D}_7,\ \mathscr{U}_7=\frac{\varsigma_2}{\sigma_2},$$

we obtain

$$\begin{aligned}\frac{d\digamma_7}{dt}&=\left(1-\frac{\mathscr{A}_7}{\mathscr{A}}\right)(\alpha\mathscr{A}_7-\alpha\mathscr{A})+\left(1-\frac{\mathscr{A}_7}{\mathscr{A}}\right)\left(\frac{\eta_1\mathscr{A}_7\mathscr{K}_7}{1+\alpha_1\mathscr{K}_7}+\frac{\eta_2\mathscr{A}_7\mathscr{U}_7}{1+\alpha_2\mathscr{U}_7}\right)+\frac{\eta_1\mathscr{A}_5\mathscr{K}}{1+\alpha_1\mathscr{K}}+\frac{\eta_2\mathscr{A}_7\mathscr{U}}{1+\alpha_2\mathscr{U}}\\&-\frac{\eta_1\mathscr{A}_7\mathscr{K}_7}{1+\alpha_1\mathscr{K}_7}\frac{\mathscr{D}}{\mathscr{D}_7}-\frac{\eta_1\mathscr{A}_7\mathscr{K}_7}{1+\alpha_1\mathscr{K}_7}\frac{\mathscr{A}\mathscr{K}\mathscr{D}_7(1+\alpha_1\mathscr{K}_7)}{\mathscr{A}_7\mathscr{K}_7\mathscr{D}(1+\alpha_1\mathscr{K})}+\frac{\eta_1\mathscr{A}_7\mathscr{K}_7}{1+\alpha_1\mathscr{K}_7}+\mu_1\mathscr{W}\mathscr{D}_7-\frac{\eta_2\mathscr{A}_7\mathscr{U}_7}{1+\alpha_2\mathscr{U}_7}\frac{\mathscr{U}}{\mathscr{U}_7}+\frac{\mu_2}{\phi}\mathscr{U}\mathscr{Z}_7\\&-\frac{\eta_2\mathscr{A}_7\mathscr{U}_7}{1+\alpha_2\mathscr{U}_7}\frac{\mathscr{A}(1+\alpha_2\mathscr{U}_7)}{\mathscr{A}_7(1+\alpha_2\mathscr{U})}-\frac{\eta_1\mathscr{A}_7\mathscr{K}_7}{(1+\alpha_1\mathscr{K}_7)}\frac{\mathscr{K}}{\mathscr{K}_7}+\frac{\eta_2\mathscr{A}_7\mathscr{U}_7}{1+\alpha_2\mathscr{U}_7}-\frac{\mu_2}{\phi}\mathscr{U}_7\mathscr{Z}_7+\frac{\mu_2}{\varphi}\mathscr{Z}\mathscr{U}_7+\frac{\eta_1\mathscr{A}_7\mathscr{K}_7}{(1+\alpha_1\mathscr{K}_7)}\frac{\mathscr{D}}{\mathscr{D}_7}\\&-\frac{\eta_1\mathscr{A}_7\mathscr{K}_7}{1+\alpha_1\mathscr{K}_7}\frac{\mathscr{K}_7\mathscr{D}}{\mathscr{D}_7\mathscr{K}}+\frac{\eta_1\mathscr{A}_7\mathscr{K}_7}{1+\alpha_1\mathscr{K}_7}-\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}-\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}-\frac{\mu_2}{\varphi}\mathscr{U}\mathscr{Z}_7+\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}_7\\&=-\alpha\frac{(\mathscr{A}-\mathscr{A}_7)^2}{\mathscr{A}}-\frac{\alpha_2(\mathscr{U}-\mathscr{U}_7)^2}{\mathscr{U}_7(1+\alpha_2\mathscr{U})(1+\alpha_2\mathscr{U}_7)}-\frac{\alpha_1(\mathscr{K}-\mathscr{K}_7)^2}{\mathscr{K}_7(1+\alpha_1\mathscr{K})(1+\alpha_1\mathscr{K}_7)}\\&+\frac{\eta_1\mathscr{A}_7\mathscr{K}_7}{1+\alpha_1\mathscr{K}_7}\left(4-\frac{\mathscr{A}_7}{\mathscr{A}}+\frac{\mathscr{A}\mathscr{K}\mathscr{D}_7(1+\alpha_1\mathscr{K}_7)}{\mathscr{A}_7\mathscr{K}_7\mathscr{D}(1+\alpha_1\mathscr{K})}-\frac{\mathscr{K}_7\mathscr{D}}{\mathscr{K}\mathscr{D}_7}-\frac{1+\alpha_1\mathscr{K}}{1+\alpha_1\mathscr{K}_7}\right)\\&+\frac{\eta_2\mathscr{A}_7\mathscr{U}_7}{1+\alpha_2\mathscr{U}_7}\left(3-\frac{\mathscr{A}_7}{\mathscr{A}}-\frac{\mathscr{A}(1+\alpha_2\mathscr{U}_7)}{\mathscr{A}_7(1+\alpha_2\mathscr{U})}-\frac{1+\alpha_2\mathscr{U}}{1+\alpha_2\mathscr{U}_7}\right)+\mu_1\left(\mathscr{D}_7-\frac{\varsigma_1}{\sigma_1}\right)\mathscr{W}.\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\frac{d\mathcal{F}_7}{dt} &= -\alpha\frac{(\mathscr{A}-\mathscr{A}_7)^2}{\mathscr{A}} - \frac{\alpha_2(\mathscr{U}-\mathscr{U}_7)^2}{\mathscr{U}_7(1+\alpha_2\mathscr{U})(1+\alpha_2\mathscr{U}_7)} - \frac{\alpha_1(\mathscr{K}-\mathscr{K}_7)^2}{\mathscr{K}_7(1+\alpha_1\mathscr{K})(1+\alpha_1\mathscr{K}_7)}\\ &+ \frac{\eta_1\mathscr{A}_7\mathscr{K}_7}{1+\alpha_1\mathscr{K}_7}\left(4-\frac{\mathscr{A}_7}{\mathscr{A}}+\frac{\mathscr{A}\mathscr{K}\mathscr{D}_7(1+\alpha_1\mathscr{K}_7)}{\mathscr{A}_7\mathscr{K}_7\mathscr{D}(1+\alpha_1\mathscr{K})}-\frac{\mathscr{K}_7\mathscr{D}}{\mathscr{K}\mathscr{D}_7}-\frac{1+\alpha_1\mathscr{K}}{1+\alpha_1\mathscr{K}_7}\right)\\ &+ \frac{\eta_2\mathscr{A}_7\mathscr{U}_7}{1+\alpha_2\mathscr{U}_7}\left(3-\frac{\mathscr{A}_7}{\mathscr{A}}-\frac{\mathscr{A}(1+\alpha_2\mathscr{U}_7)}{\mathscr{A}_7(1+\alpha_2\mathscr{U})}-\frac{1+\alpha_2\mathscr{U}}{1+\alpha_2\mathscr{U}_7}\right)\\ &+ \frac{\mu_1(\xi\varsigma_1\mathbb{Q}_1(\sigma_2+\varsigma_2\alpha_2)+\varepsilon\sigma_1(\alpha\sigma_2+\varsigma_2\mathbb{Q}_2)+\xi\alpha_1\eta_2\varsigma_1\varsigma_2)}{\xi\sigma_1(\mathbb{Q}_1(\sigma_2+\alpha_2\varsigma_2)+\eta_2\varsigma_2\alpha_1)}(\mathrm{R}_{11}-1)\mathscr{W}.\end{aligned}$$

Hence, if $\mathrm{R}_8>1$, $\mathrm{R}_9>1$and $\mathrm{R}_{11}\le 1$, then $\frac{d\mathcal{F}_7}{dt}\le 0$ for each $\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z}>0$, and $\frac{d\mathcal{F}_7}{dt}=0$ when $\mathscr{A}(t)=\mathscr{A}_7$, $\mathscr{D}(t)=\mathscr{D}_7$, $\mathscr{U}(t)=\mathscr{U}_7$, $\mathscr{K}(t)=\mathscr{K}_7$ and $\mathscr{W}(t)=0$. The system (1)-(6) solutions converge to Υ_7'. For each element of Υ_7'we have $\mathscr{A}(t)=\mathscr{A}_7$, $\mathscr{D}(t)=\mathscr{D}_7$, $\mathscr{K}(t)=\mathscr{K}_7$ and hence $\dot{\mathscr{U}}=0$. The of system (1)-(6) third equation implies that

$$0=\dot{\mathscr{U}}=\frac{\phi\eta_2\mathscr{A}_7\mathscr{U}_7}{1+\alpha_2\mathscr{U}_7}-\delta\mathscr{U}_7-\mu_2\mathscr{U}_7\mathscr{Z}$$

which ensures that $\mathscr{Z}(t)=\mathscr{Z}_7$ for all t. Therefore, $\Upsilon_7'=\{\mathfrak{U}_7\}$. Applying L-LAS we get $\mathfrak{U}_7$ is G.A.S.

Theorem 9. If $\mathrm{R}_{11}>1$, then $\mathfrak{U}_8$ is G.A.S .
Proof. Define $\mathcal{F}_8(\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z})$ as:

$$\mathcal{F}_8=\mathscr{A}_8\mathfrak{D}\left(\frac{\mathscr{A}}{\mathscr{A}_8}\right)+\mathscr{D}_8\mathfrak{D}\left(\frac{\mathscr{D}}{\mathscr{D}_8}\right)+\frac{1}{\varphi}\mathscr{U}_8\mathfrak{D}\left(\frac{\mathscr{U}}{\mathscr{U}_8}\right)+\frac{\eta_1\mathscr{A}_8}{\varepsilon(1+\alpha_1\mathscr{K}_8)}\mathscr{K}_8\mathfrak{D}\left(\frac{\mathscr{K}}{\mathscr{K}_8}\right)+\frac{\mu_1}{\sigma_1}\mathscr{W}_8\mathfrak{D}\left(\frac{\mathscr{W}}{\mathscr{W}_8}\right)+\frac{\mu_2}{\varphi\sigma_2}\mathscr{Z}_8\mathfrak{D}\left(\frac{\mathscr{Z}}{\mathscr{Z}_8}\right).$$

Calculating $\frac{d\mathcal{F}_8}{dt}$ as:

$$\begin{aligned}\frac{d\mathcal{F}_8}{dt} &= \left(1-\frac{\mathscr{A}_8}{\mathscr{A}}\right)\left(\eth-\alpha\mathscr{A}-\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}-\frac{\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}}\right)+\left(1-\frac{\mathscr{D}_8}{\mathscr{D}}\right)\left(\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}-a\mathscr{D}-\mu_1\mathscr{W}\mathscr{D}\right)\\ &+\frac{1}{\varphi}\left(1-\frac{\mathscr{U}_8}{\mathscr{U}}\right)\left(\frac{\varphi\eta_2\mathscr{A}\mathscr{U}}{1+\alpha_2\mathscr{U}}-\delta\mathscr{U}-\mu_2\mathscr{Z}\mathscr{U}\right)+\frac{\eta_1\mathscr{A}_8}{\varepsilon(1+\alpha_1\mathscr{K}_8)}\left(1-\frac{\mathscr{K}_8}{\mathscr{K}}\right)(\xi\mathscr{D}-\varepsilon\mathscr{K})\\ &+\frac{\mu_1}{\sigma_1}\left(1-\frac{\mathscr{W}_8}{\mathscr{W}}\right)(\sigma_1\mathscr{W}\mathscr{D}-\varsigma_1\mathscr{W})+\frac{\mu_2}{\varphi\sigma_2}\left(1-\frac{\mathscr{Z}_8}{\mathscr{Z}}\right)(\sigma_2\mathscr{Z}\mathscr{U}-\varsigma_2\mathscr{Z})\\ &=\left(1-\frac{\mathscr{A}_8}{\mathscr{A}}\right)(\eth-\alpha\mathscr{A})+\frac{\eta_1\mathscr{A}_8\mathscr{K}}{1+\alpha_1\mathscr{K}}+\frac{\eta_2\mathscr{A}_8\mathscr{U}}{1+\alpha_2\mathscr{U}}-a\mathscr{D}-\frac{\eta_1\mathscr{A}\mathscr{K}}{1+\alpha_1\mathscr{K}}\frac{\mathscr{D}_8}{\mathscr{D}}+a\mathscr{D}_8\\ &+\mu_1\mathscr{D}_8\mathscr{W}-\frac{\delta}{\varphi}\mathscr{U}-\frac{\eta_2\mathscr{A}\mathscr{U}_8}{1+\alpha_2\mathscr{U}}+\frac{\delta}{\varphi}\mathscr{U}_8+\frac{\mu_2}{\varphi}\mathscr{Z}\mathscr{U}_8+\frac{\eta_1\mathscr{A}_8}{1+\alpha_1\mathscr{K}_8}\frac{\xi\mathscr{D}}{\varepsilon}\\ &-\frac{\eta_1\mathscr{A}_8\mathscr{K}}{1+\alpha_1\mathscr{K}_8}-\frac{\eta_1\mathscr{A}_8\mathscr{K}_8}{1+\alpha_1\mathscr{K}_8}\frac{\xi\mathscr{D}}{\varepsilon\mathscr{K}}+\frac{\eta_1\mathscr{A}_8\mathscr{K}_8}{1+\alpha_1\mathscr{K}_8}-\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}-\mu_1\mathscr{D}\mathscr{W}_8\\ &+\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}_8-\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}-\frac{\mu_2}{\varphi}\mathscr{Z}_8\mathscr{U}+\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}_8.\end{aligned}$$

Utilizing the $\mathfrak{U}_8$ equilibrium conditions:

$$\eth=\alpha\mathscr{A}_8+\frac{\eta_1\mathscr{A}_8\mathscr{K}_8}{1+\alpha_1\mathscr{K}_8}+\frac{\eta_2\mathscr{A}_8\mathscr{U}_8}{1+\alpha_2\mathscr{U}_8},\quad \frac{\phi\eta_2\mathscr{A}_8\mathscr{U}_8}{1+\alpha_2\mathscr{U}_8}=\delta\mathscr{U}_8+\mu_2\mathscr{U}_8\mathscr{Z}_8,$$
$$\mathscr{K}_8=\frac{\xi}{\varepsilon}\mathscr{D}_8,\ \frac{\eta_1\mathscr{A}_8\mathscr{K}_8}{1+\alpha_1\mathscr{K}_8}=a\mathscr{D}_8+\mu_1\mathscr{D}_8\mathscr{W}_8,\quad \mathscr{U}_8=\frac{\varsigma_2}{\sigma_2},\quad \mathscr{D}_8=\frac{\varsigma_1}{\sigma_1},$$

we obtain

$$\frac{d\mathcal{F}_8}{dt}=\left(1-\frac{\mathscr{A}_8}{\mathscr{A}}\right)(\alpha\mathscr{A}_8-\alpha\mathscr{A})+\left(1-\frac{\mathscr{A}_8}{\mathscr{A}}\right)\left(\frac{\eta_1\mathscr{A}_8\mathscr{K}_8}{1+\alpha_1\mathscr{K}_8}+\frac{\eta_2\mathscr{A}_8\mathscr{U}_8}{1+\alpha_2\mathscr{U}_8}\right)+\frac{\eta_1\mathscr{A}_8\mathscr{K}}{1+\alpha_1\mathscr{K}}+\frac{\eta_2\mathscr{A}_8\mathscr{U}}{1+\alpha_2\mathscr{U}}$$

$$
\begin{aligned}
&+\mu_1\mathscr{D}\mathscr{W}_8-\frac{\eta_1\mathscr{A}_8\mathscr{K}_8}{1+\alpha_1\mathscr{K}_8}\frac{\mathscr{D}}{\mathscr{D}_8}-\frac{\eta_1\mathscr{A}_8\mathscr{K}_8}{1+\alpha_1\mathscr{K}_8}\frac{\mathscr{A}\mathscr{K}\mathscr{D}_8(1+\alpha_1\mathscr{K}_8)}{\mathscr{A}_8\mathscr{K}_8\mathscr{D}(1+\alpha_1\mathscr{K})}+\frac{\eta_1\mathscr{A}_8\mathscr{K}_8}{1+\alpha_1\mathscr{K}_8}-\frac{\varsigma_1\mu_1}{\sigma_1}\mathscr{Z}_8\\
&+\frac{\varsigma_1\mu_1}{\sigma_1}\mathscr{W}-\frac{\eta_2\mathscr{A}_8\mathscr{U}_8}{1+\alpha_2\mathscr{U}_8}\frac{\mathscr{U}}{\mathscr{U}_8}+\frac{\mu_2}{\phi}\mathscr{U}\mathscr{Z}_8-\frac{\eta_2\mathscr{A}_8\mathscr{U}_8}{1+\alpha_2\mathscr{U}_8}\frac{\mathscr{A}(1+\alpha_2\mathscr{U}_8)}{\mathscr{A}_8(1+\alpha_2\mathscr{U})}+\frac{\eta_2\mathscr{A}_8\mathscr{U}_8}{1+\alpha_2\mathscr{U}_8}\\
&-\frac{\mu_2\varsigma_2}{\phi\sigma_2}\mathscr{Z}_8+\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}+\frac{\eta_1\mathscr{A}_8\mathscr{K}_8}{(1+\alpha_1\mathscr{K}_8)}\frac{\mathscr{D}}{\mathscr{D}_8}-\frac{\eta_1\mathscr{A}_8\mathscr{K}_8}{(1+\alpha_1\mathscr{K}_8)}\frac{\mathscr{K}}{\mathscr{K}_8}-\frac{\eta_1\mathscr{A}_8\mathscr{K}_8}{1+\alpha_1\mathscr{K}_8}\frac{\mathscr{K}_8\mathscr{D}}{\mathscr{D}_8\mathscr{K}}\\
&+\frac{\eta_1\mathscr{A}_8\mathscr{K}_8}{1+\alpha_1\mathscr{K}_8}-\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}-\mu_1\mathscr{D}\mathscr{W}_8+\frac{\mu_1\varsigma_1}{\sigma_1}\mathscr{W}_8-\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}-\frac{\mu_2}{\varphi}\mathscr{U}\mathscr{Z}_8+\frac{\mu_2\varsigma_2}{\varphi\sigma_2}\mathscr{Z}_8\\
&=-\alpha\frac{(\mathscr{A}-\mathscr{A}_8)^2}{\mathscr{A}}-\frac{\alpha_2(\mathscr{U}-\mathscr{U}_8)^2}{\mathscr{U}_8(1+\alpha_2\mathscr{U})(1+\alpha_2\mathscr{U}_8)}-\frac{\alpha_1(\mathscr{K}-\mathscr{K}_8)^2}{\mathscr{K}_8(1+\alpha_1\mathscr{K})(1+\alpha_1\mathscr{K}_8)}\\
&+\frac{\eta_1\mathscr{A}_8\mathscr{K}_8}{1+\alpha_1\mathscr{K}_8}\left(4-\frac{\mathscr{A}_8}{\mathscr{A}}+\frac{\mathscr{A}\mathscr{K}\mathscr{D}_8(1+\alpha_1\mathscr{K}_8)}{\mathscr{A}_8\mathscr{K}_8\mathscr{D}(1+\alpha_1\mathscr{K})}-\frac{\mathscr{K}_8\mathscr{D}}{\mathscr{K}\mathscr{D}_8}-\frac{1+\alpha_1\mathscr{K}}{1+\alpha_1\mathscr{K}_8}\right)\\
&+\frac{\eta_2\mathscr{A}_8\mathscr{U}_8}{1+\alpha_2\mathscr{U}_8}\left(3-\frac{\mathscr{A}_8}{\mathscr{A}}-\frac{\mathscr{A}(1+\alpha_2\mathscr{U}_8)}{\mathscr{A}_8(1+\alpha_2\mathscr{U})}-\frac{1+\alpha_2\mathscr{U}}{1+\alpha_2\mathscr{U}_8}\right).
\end{aligned}
$$

Hence, if $\frac{d\Gamma_8}{dt}\leq 0$ for each $\mathscr{A},\mathscr{D},\mathscr{U},\mathscr{K},\mathscr{W},\mathscr{Z}>0$, Moreover, $\frac{d\Gamma_8}{dt}=0$ when$\mathscr{A}(t)=\mathscr{A}_8,\mathscr{U}(t)=\mathscr{U}_8$, $\mathscr{D}(t)=\mathscr{D}_8$ and $\mathscr{K}(t)=\mathscr{K}_8$. The system (1)-(6) solutions tend to Υ_8' it includes elements $\mathscr{A}(t)=\mathscr{A}_8,\mathscr{U}(t)=\mathscr{U}_8$, $\mathscr{D}(t)=\mathscr{D}_8$ and $\mathscr{K}(t)=\mathscr{K}_8$. It follows that $\dot{\mathscr{D}}=\dot{\mathscr{U}}=0$ and from the system (1)-(6) second and third equations, we have

$$
\begin{aligned}
0=\dot{\mathscr{D}}&=\frac{\eta_1\mathscr{A}_8\mathscr{K}_8}{1+\alpha_1\mathscr{K}_8}-a\mathscr{D}_8-\mu_1\mathscr{D}_8\mathscr{W}_8,\\
0=\dot{\mathscr{U}}&=\frac{\phi\eta_2\mathscr{A}_8\mathscr{U}_8}{1+\alpha_2\mathscr{U}_8}-\delta\mathscr{U}_8-\mu_2\mathscr{U}_8\mathscr{Z}_8,
\end{aligned}
$$

which ensure that $\mathscr{W}(t)=\mathscr{W}_8$ and $\mathscr{Z}(t)=\mathscr{Z}_8$ for all t. The result is $\Upsilon_8'=\{\mathfrak{U}_8\}$ and applying L-LAS we get $\mathfrak{U}_8$ is G.A.S. using.□

6 Numerical simulations

In this section, we clarify the results of **Theorems 1-9** by carrying out the numerical simulations. In order to solve the system (1)-(6) numerically, several parameters' values will be fixed and taken from the literature (see Table 2). To ensure that the night equilibria described in **Theorems 1-9**, We alter a few factors that have an impact on the threshold parameter values, which in turn regulate the existence and stability of the equilibria.

Table 2: the model(1)-(6) parameters values and their sources.

Parameter	Value	Source	Parameter	Value	Source	Parameter	Value	Source
$\eth$	10	[14],[15]	ς_1	0.1	[16]	ξ	5	[9]
α	0.01	[17],[18]	ς_2	0.1	[9]	δ	0.2	[19]
a	0.5	[20]	μ_2	0.2	[19]	ε	2	[9]
φ	0.2	[21]	μ_1	0.2	[9]			

6.1 Stability of the equilibria

In this subsection, we illustrate our global stability results given in **Theorems 1-9**. Therefore, for systems (1)-(6), we select the following three beginning conditions:

Initial-1 : $(\mathscr{A}(0),\mathscr{D}(0),\mathscr{U}(0),\mathscr{K}(0),\mathscr{W}(0),\mathscr{Z}(0))=(600,1.5,1,5,1,0.2)$,

Initial-2: $(\mathscr{A}(0),\mathscr{D}(0),\mathscr{U}(0),\mathscr{K}(0),\mathscr{W}(0),\mathscr{Z}(0)) = (500,1,1.5,2,2,0.1)$,
Initial-3: $(\mathscr{A}(0),\mathscr{D}(0),\mathscr{U}(0),\mathscr{K}(0),\mathscr{W}(0),\mathscr{Z}(0)) = (300,0.5,2,1.5,3,0.05)$.
Choosing the values of η_1, η_2, σ_1, σ_2, α_1 and α_2 as the following scenarios:

Scenario 1: $\eta_1 = 0.0001$, $\eta_2 = 0.0002$, $\sigma_1 = \sigma_2 = 0.2, \alpha_1 = 0.05$ and $\alpha_2 = 0.07$. For this set of parameters, we have $R_0 = 0.5 < 1$ and $R_1 = 0.2 < 1$. Figure 1 shows that the trajectories starting from Initial-1,Initial-2 and Initial-3 achieve the equilibrium $\mathfrak{U}_0 = (1000,0,0,0,0,0)$. This proves Theorem 1 statement that $\mathfrak{U}_0$ is G.A.S. Both HIV and HTLV-I will be eliminated in this scenario.

Scenario 2: $\eta_1 = 0.001$, $\eta_2 = 0.0001$, $\sigma_1 = 0.003$, $\sigma_2 = 0.2, \alpha_1 = 0.00002$ and $\alpha_2 = 0.00001$. With such choice we get $R_0 = 5 \geq 1$, $R_1 = 0.1 < 1$ and $R_2 = 0.5 < 1$. Figure 2 illustrates that the trajectories starting from Initials 1, 2, and 3 tend to $\mathfrak{U}_1 = (200.16,15.99,0,40.02,0,0)$. As a result, Theorem 2 is supported by the numerical findings. This example shows a CTL-mediated immune response to a chronic HIV mono-infection that is not triggered.

Scenario 3: $\eta_1 = 0.0001$, $\eta_2 = 0.0023$, $\sigma_1 = 0.001$, $\sigma_2 = 0.01$, $\alpha_1 = 0.00002$ and $\alpha_2 = 0.00001$. Then, we calculate $R_0 = 0.5 < 1$, $R_1 = 2.3 \geq 1$and then $R_3 = 0.69 < 1$. Figure 3 illustrates that the trajectories starting from Initials 1, 2, and 3 tend to $\mathfrak{U}_2 = (434.80,0,5.65,0,0,0)$. Thus, Theorem 3 can be supported by the numerical results. This evidence indicates that there is a chronic CTL-mediated immunological response to HTLV mono-infection.

Scenario 4: $\eta_1 = 0.001$, $\eta_2 = 0.00032$, $\sigma_1 = \sigma_2 = 0.01, \alpha_1 = 0.00002$ and $\alpha_2 = 0.00001$. Then, we calculate $R_1 = 0.32 \leq 1$ and $R_2 = 1.42 \geq 1$. Figure 4 shows that the trajectories starting from Initials 1, 2, and 3 tend to tend to $\mathfrak{U}_3 = (285.81,9.968,0,24.92,1.072,0)$.$\mathfrak{U}_3$ is therefore G.A.S which is consistent with Theorem 4. With an HIV-specific CTL-mediated immune response, a chronic HIV mono-infection has been obtained.

Scenario 5: $\eta_1 = 0.000025$, $\eta_2 = 0.005$, $\sigma_1 = 0.05$, $\sigma_2 = 0.02, \alpha_1 = 0.00002$ and $\alpha_2 = 0.00001$. Then, we calculate $R_0 = 0.125 < 1$ and $R_3 = 1.42 > 1$. Thus, $\mathfrak{U}_4$ exists with $\mathfrak{U}_4 = (285.71,0,4.998,0,0,0.42)$. In Figure 5, We demonstrate that the trajectory with the three initials tends to $\mathfrak{U}_4$, and that G.A.S. then conforms with Theorem 5. As a result, a chronic HTLV-I mono-infection with an immunological response mediated by CTLs unique to HTLV is achieved.

Scenario 6: $\eta_1 = 0.0004$, $\eta_2 = 0.0025$, $\sigma_1 = 0.0001$, $\sigma_2 = 0.002, \alpha_1 = 0.001$ and $\alpha_2 = 0.2$. Then, we calculate $R_4 = 1.33 > 1$,$R_5 = 1.28 > 1, R_7 = 0.027 < 1$and $R_9 = 0.11 < 1$. Hence, $\mathfrak{U}_5$ exists with $\mathfrak{U}_5 = (508.877,7.098,1.3609,17.7711,0,0)$. In Figure 6, We demonstrate that the initial-1, initial-2, and initial-3 trajectories tend to $\mathfrak{U}_5$, and that G.A.S. therefore agrees with Theorem 6. There is a chronic co-infection with HIV and HTLV-I as a result, Thus neither the HTLV-specific CTL-mediated immune response nor the immunological response specific to HIV are activated..

Scenario 7: $\eta_1 = 0.002$, $\eta_2 = 0.0026$, $\sigma_1 = 0.043$, $\sigma_2 = 0.001, \alpha_1 = 0.001$ and $\alpha_2 = 0.0017$. Then, we calculate $R_6 = 1.0028 > 1$, $R_7 = 3.83 > 1$,and $R_{10} = 0.0911 < 1$. Theorem 7 is supported by the numerical findings in Figure 7, which reveal that $\mathfrak{U}_6 = (385.71,2.3255,1.683,5.813,7.087,0)$ is real and that it is G.A.S. The HIV-specific the immunological response mediated by CTL is consequently activated but the HTLV-specific CTL-mediated immune response is not triggered, leading to a persistent co-infection with HIV and HTLV-I.

Scenario 8: $\eta_1 = 0.007$, $\eta_2 = 0.005$, $\sigma_1 = 0.005$, $\sigma_2 = 0.1, \alpha_1 = 0.001$ and $\alpha_2 = 0.001$. We compute $R_8 = 2.33 > 1$, $R_9 = 1.46 > 1$ and $R_{11} = 0.68 \leq 1$. Based on these conditions, the equilibrium $\mathfrak{U}_7 = (293.72,11.18,1,27.98,0,0.46)$ exists. Additionally, the numerical outcomes presented in Figure 8 demonstrate that $\mathfrak{U}_7$ is G.A.S which supports Theorem 8. As a result, HIV-specific CTL-mediated immune response is passive, compared to the active immunological response against HTLV, leading to a persistent co-infection with HIV and HTLV-I.

Scenario 9: $\eta_1 = 0.002$, $\eta_2 = 0.00257$, $\sigma_1 = 0.04$, $\sigma_2 = 0.1, \alpha_1 = 0.001$ and $\alpha_2 = 0.001$. These data give $R_{11} = 3.97 > 1$. This condition guarantees that the equilibrium $\mathfrak{U}_8$ is present. The trajectories starting with Initial-1, Initial-2, and Initial-3 tend to $\mathfrak{U}_8 = (400,2.5,1,6.24,7.44,0.036)$., as seen in Figure 9. Figure 9 numerical findings demonstrate that $\mathfrak{U}_8$ is G.A.S based on Theorem 9. HTLV-I and HIV-specific Immune responses through CTLs are both in play in this instance, leading to chronic co-infection.

6.2 Effect of saturation on the dynamics of HIV/HTLV-I co-infection

In this subsection, we investigate the effect of the saturation on HIV/HTLV-I co-infection dynamics. We take the values $\eta_1 = 0.002$, $\eta_2 = 0.00257$, $\sigma_1 = 0.04$ and $\sigma_2 = 0.1$. Figure 10 shows the effect of saturation parameters α_1 and α_2 on the solutions of the system with initial condition Initial-3. We observe that, as α_1 and α_2 are increased, both HIV and HTLV-I infection rates are reduced. Moreover, the concentration of $CD4^{+}T$ cells is increased, while the concentration of infected cells and free HIV-1 particle are decreased.

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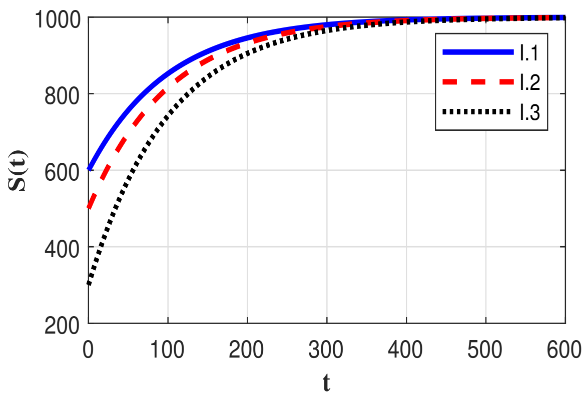
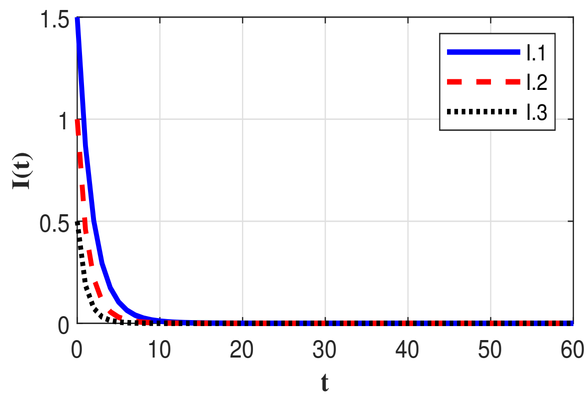
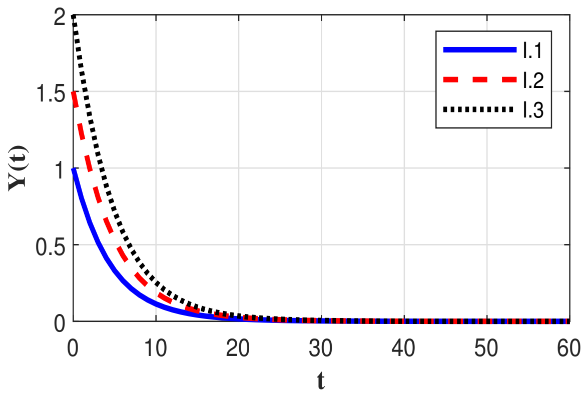
[pos=H]



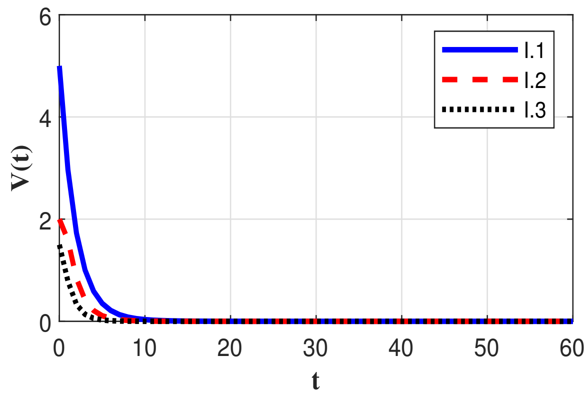
(a) Susceptible CD4$^+$T cells



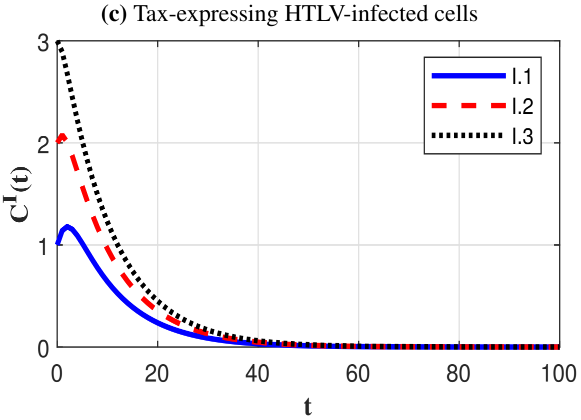
(b) HIV-infected cells



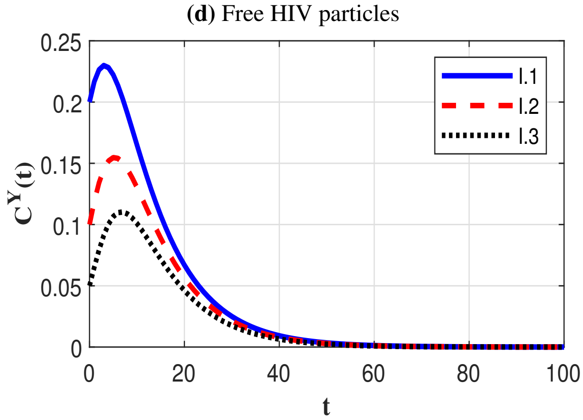
(c) Tax-expressing HTLV-infected cells



(d) Free HIV particles

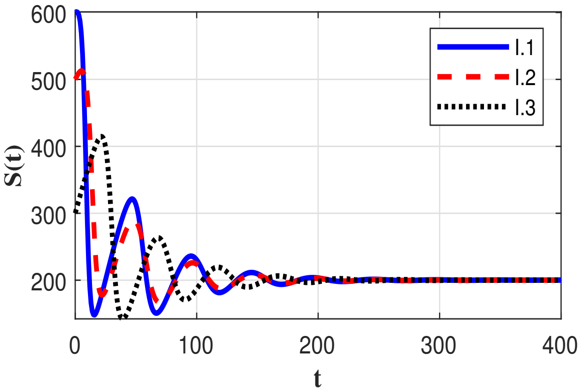


(e) HIV-specific CTLs

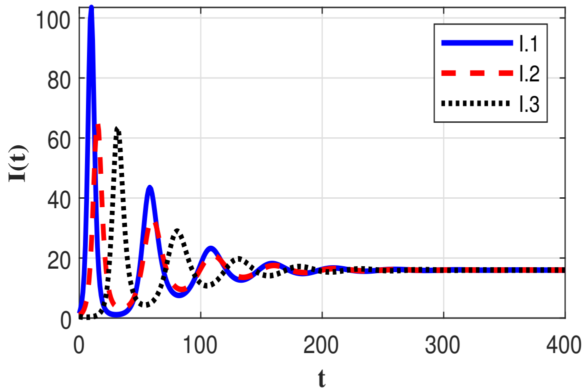


(f) HTLV-specific CTLs

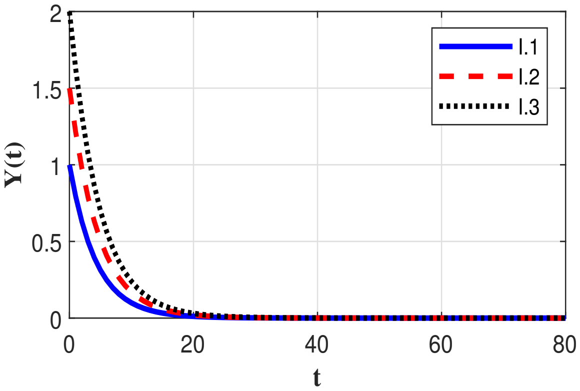
Fig. 1: The steady state $\mathfrak{U}_0 = (1000, 0, 0, 0, 0, 0)$ is G.A.S whenever $\Re_0 \leq 1$ and $\Re_1 \leq 1$.



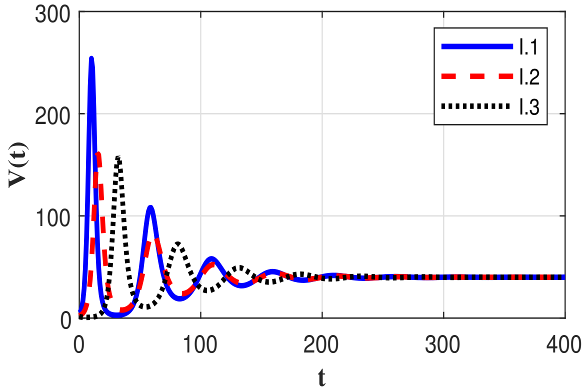
(a) Susceptible CD4^{+}T cells



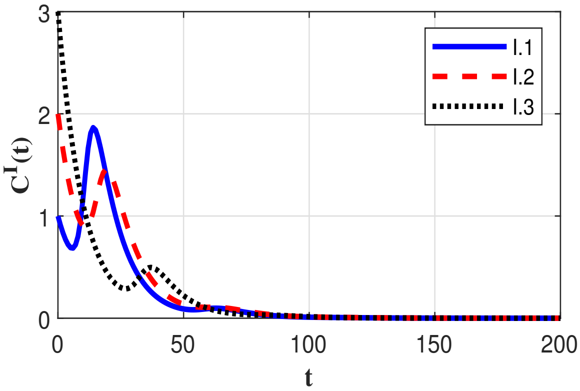
(b) HIV-infected cells



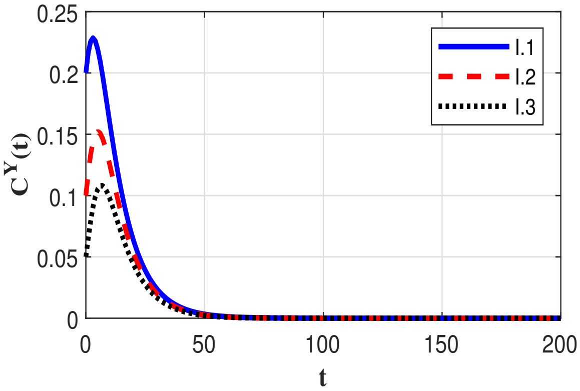
(c) Tax-expressing HTLV-infected cells



(d) Free HIV particles

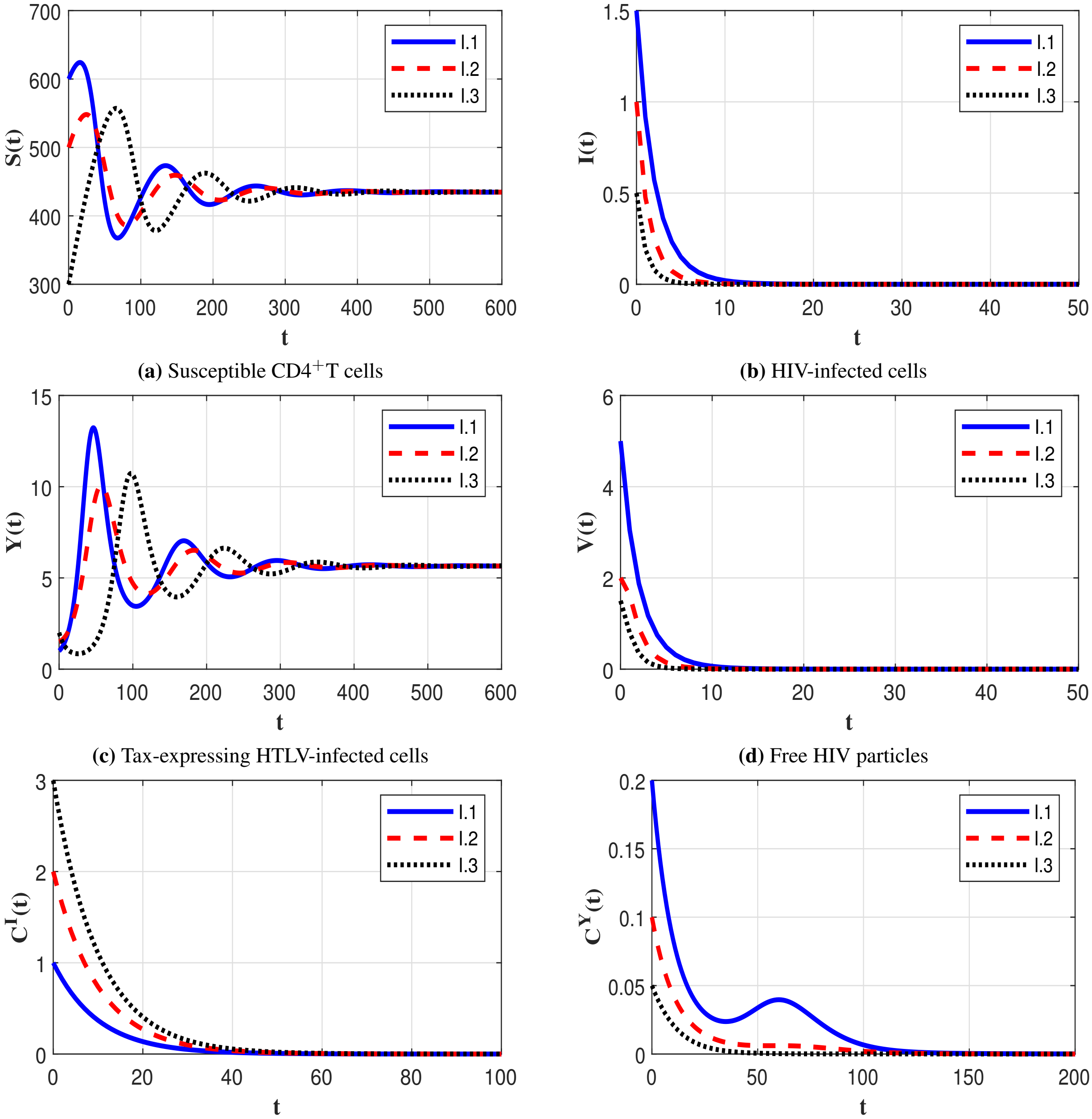


(e) HIV-specific CTLs



(f) HTLV-specific CTLs

Fig. 2: The steady state $\mathfrak{U}_1 = (200.16, 15.99, 0, 40.02, 0, 0)$ is G.A.S whenever $\Re_0 \geq 1$, $\Re_1 \leq 1$ and $\Re_2 \leq 1$.



(a) Susceptible CD4$^+$T cells

(b) HIV-infected cells

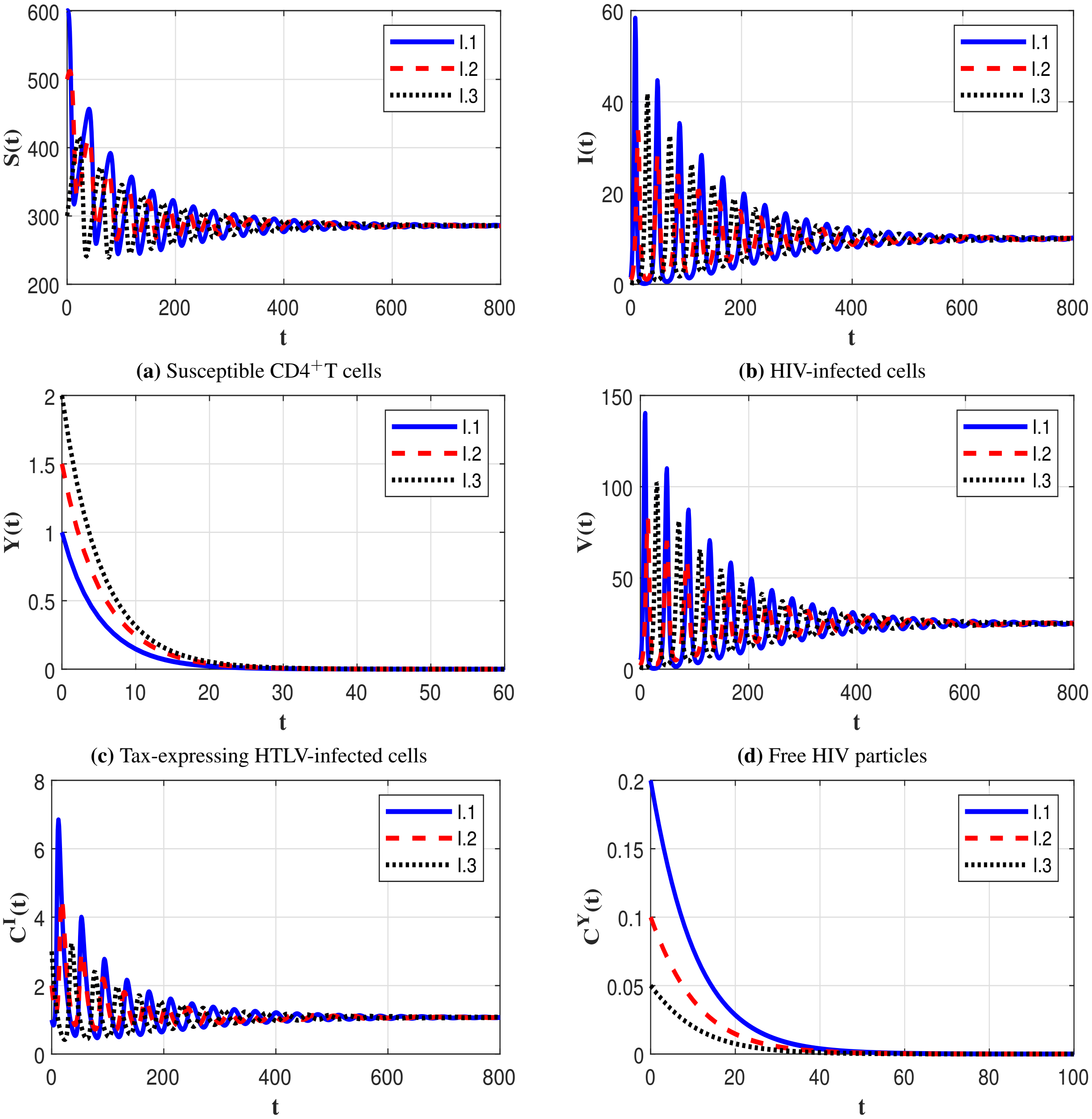
(c) Tax-expressing HTLV-infected cells

(d) Free HIV particles

(e) HIV-specific CTLs

(f) HTLV-specific CTLs

Fig. 3: The steady state $\mathfrak{U}_2 = (434.80, 0, 5.65, 0, 0, 0)$ is G.A.S whenever $\Re_0 \leq 1$, $\Re_1 \geq 1$ and $\Re_3 \leq 1$.



(a) Susceptible CD4^{+}T cells

(b) HIV-infected cells

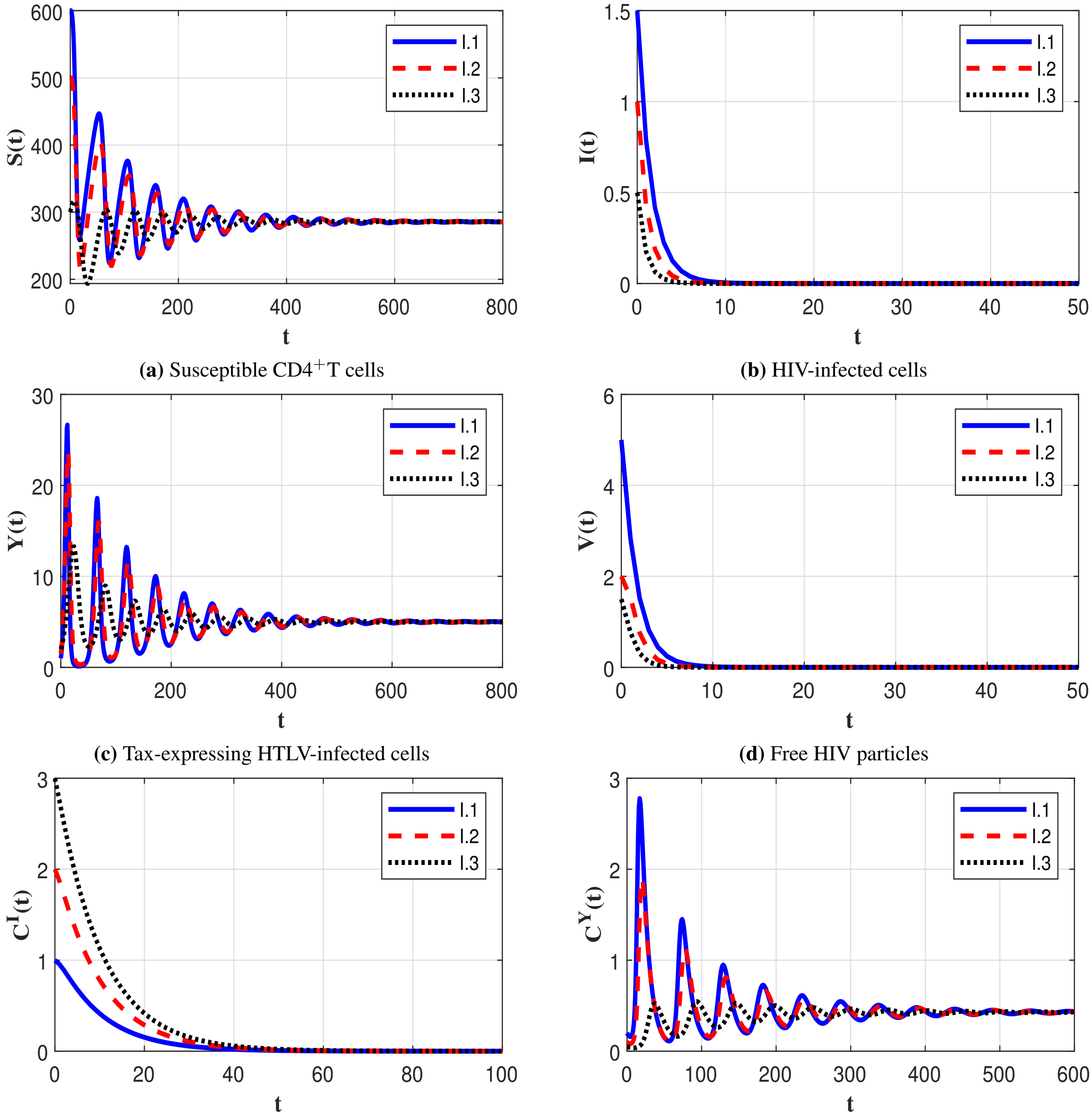
(c) Tax-expressing HTLV-infected cells

(d) Free HIV particles

(e) HIV-specific CTLs

(f) HTLV-specific CTLs

Fig. 4: The steady state $\mathfrak{U}_3 = (285.81, 9.968, 0, 24.92, 1.072, 0)$ is G.A.S whenever $\Re_1 \leq 1$ and $\Re_2 \geq 1$.



(a) Susceptible CD4$^+$T cells

(b) HIV-infected cells

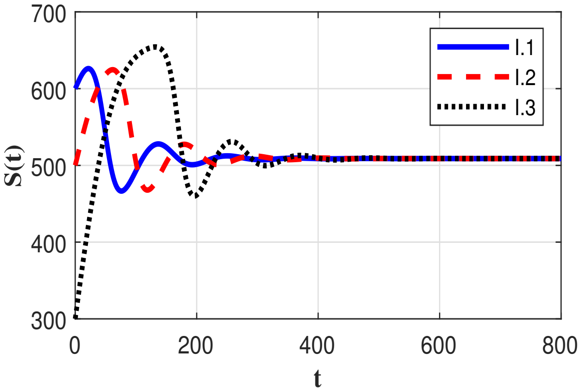
(c) Tax-expressing HTLV-infected cells

(d) Free HIV particles

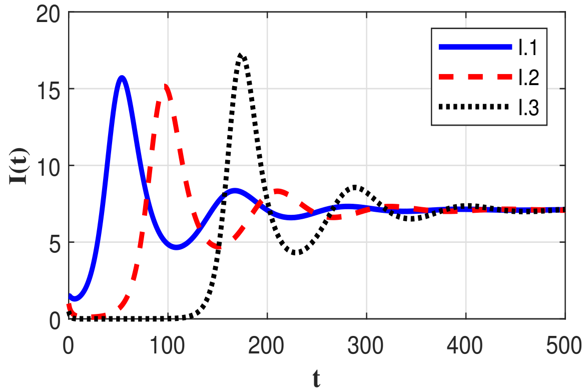
(e) HIV-specific CTLs

(f) HTLV-specific CTLs

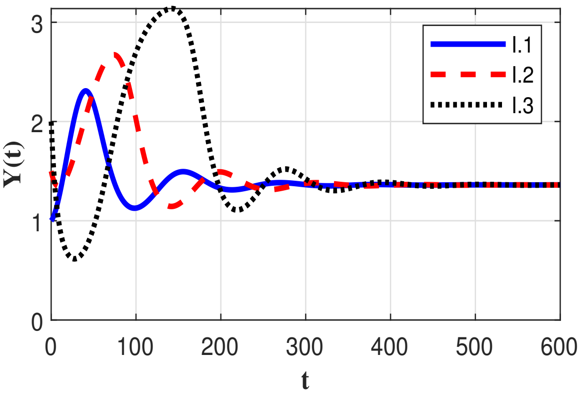
Fig. 5: The steady state $\mathfrak{U}_4 = (285.71, 0, 4.998, 0, 0, 0.42)$ is G.A.S whenever $\Re_0 \le 1$and $\Re_3 \ge 1$.



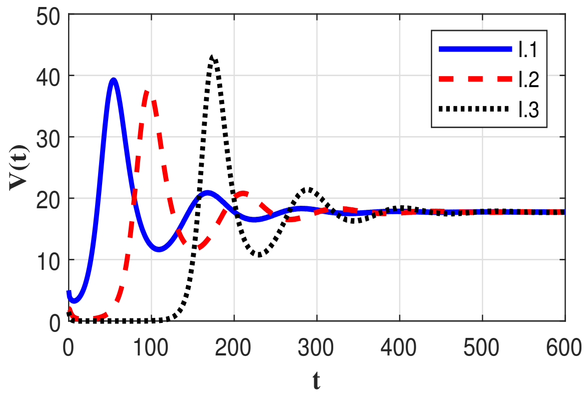
(a) Susceptible CD4$^+$T cells



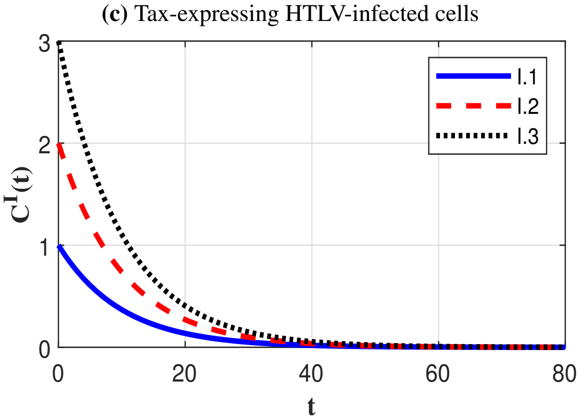
(b) HIV-infected cells



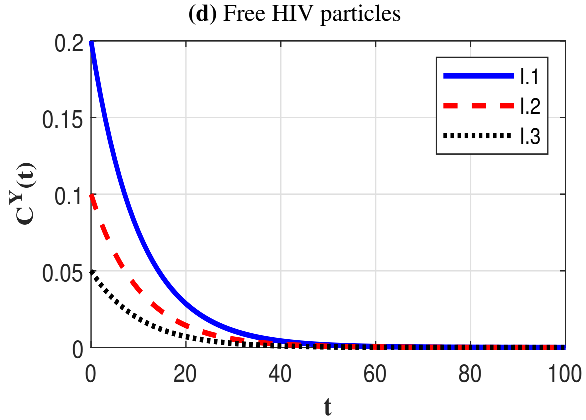
(c) Tax-expressing HTLV-infected cells



(d) Free HIV particles

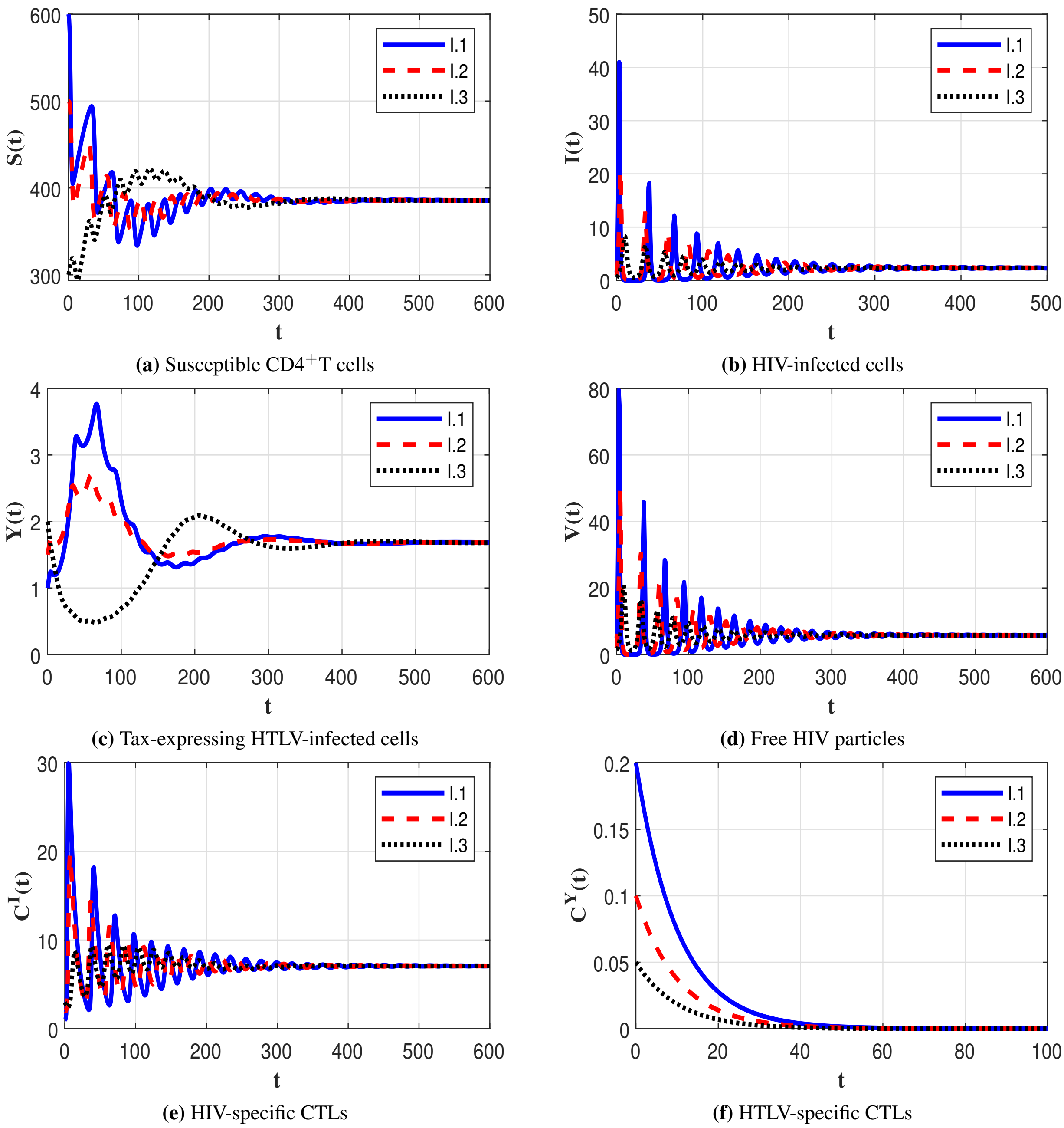


(e) HIV-specific CTLs



(f) HTLV-specific CTLs

Fig. 6: The steady state $\mathfrak{U}_5 = (508.877, 7.098, 1.3609, 17.7711, 0, 0)$ is G.A.S whenever $R_4 > 1$,$R_5 > 1, R_7 \leq 1$and $R_9 \leq 1$.



(a) Susceptible $CD4^{+}T$ cells

(b) HIV-infected cells

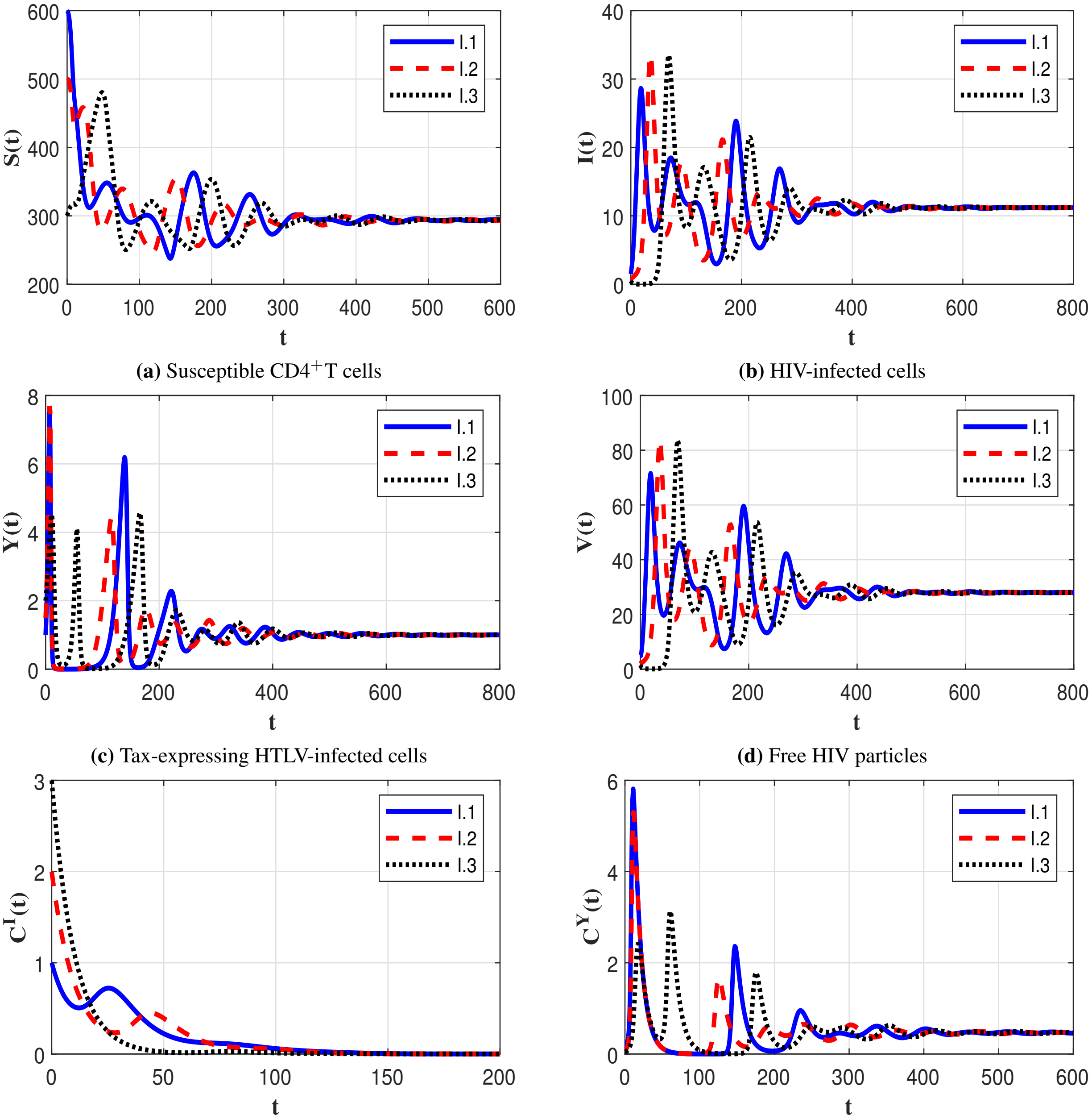
(c) Tax-expressing HTLV-infected cells

(d) Free HIV particles

(e) HIV-specific CTLs

(f) HTLV-specific CTLs

Fig. 7: The steady state $\mathfrak{U}_6 = (385.71, 2.3255, 1.683, 5.813, 7.087, 0)$ is G.A.S whenever $R_6 > 1$,$R_7 > 1$,and $R_{10} \leq 1$.



(a) Susceptible CD4$^+$T cells

(b) HIV-infected cells

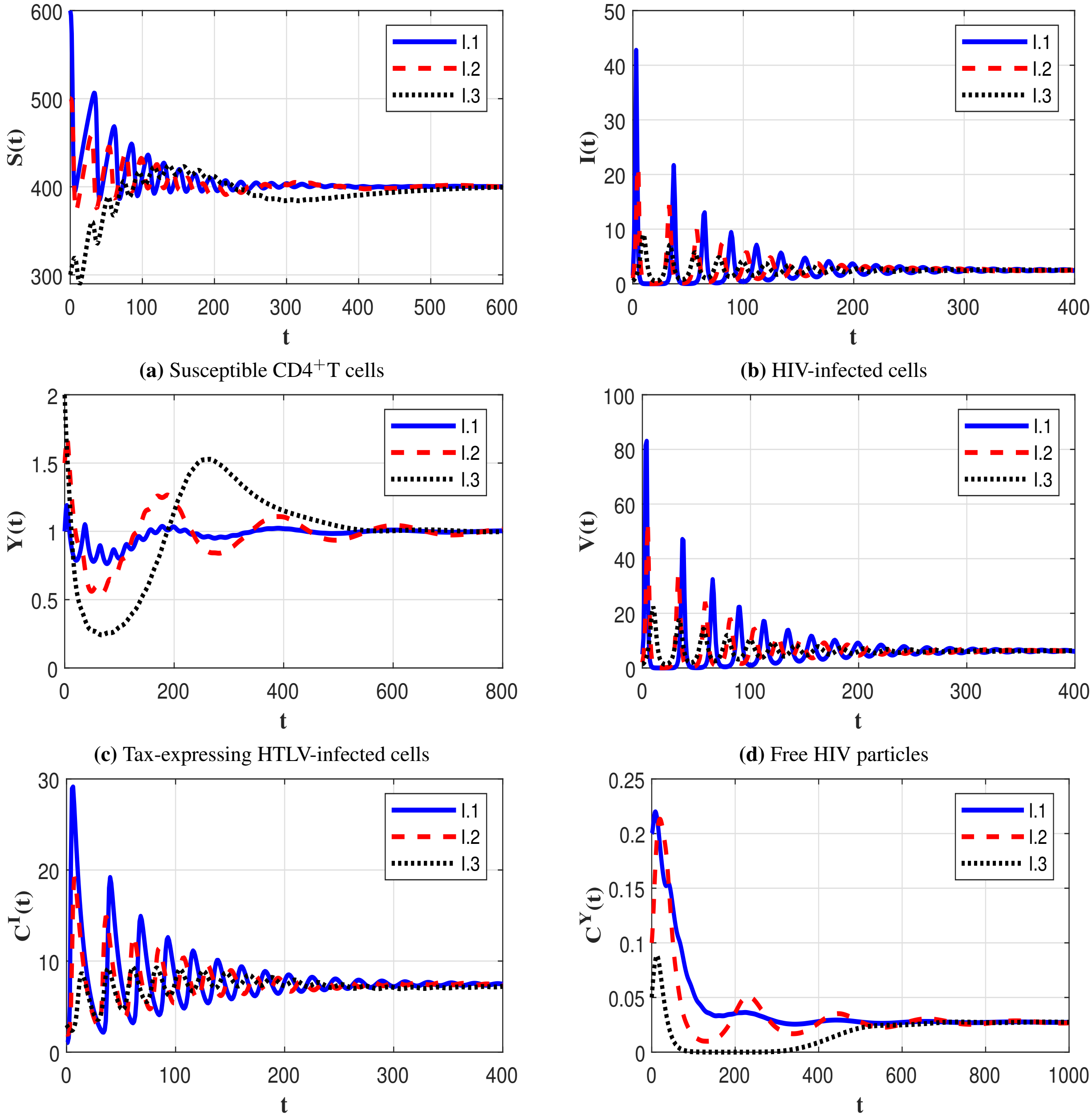
(c) Tax-expressing HTLV-infected cells

(d) Free HIV particles

(e) HIV-specific CTLs

(f) HTLV-specific CTLs

Fig. 8: The steady state $\mathfrak{U}_7 = (293.72, 11.18, 1, 27.98, 0, 0.46)$ is G.A.S whenever $R_8 > 1, R_9 > 1$and $R_{11} \leq 1$.



(a) Susceptible CD4$^+$T cells

(b) HIV-infected cells

(c) Tax-expressing HTLV-infected cells

(d) Free HIV particles

(e) HIV-specific CTLs

(f) HTLV-specific CTLs

Fig. 9: The steady state $\mathfrak{U}_8 = (400, 2.5, 0.99, 6.24, 7.44, 0.036)$ is G.A.S whenever $\mathfrak{R}_{11} > 1$.

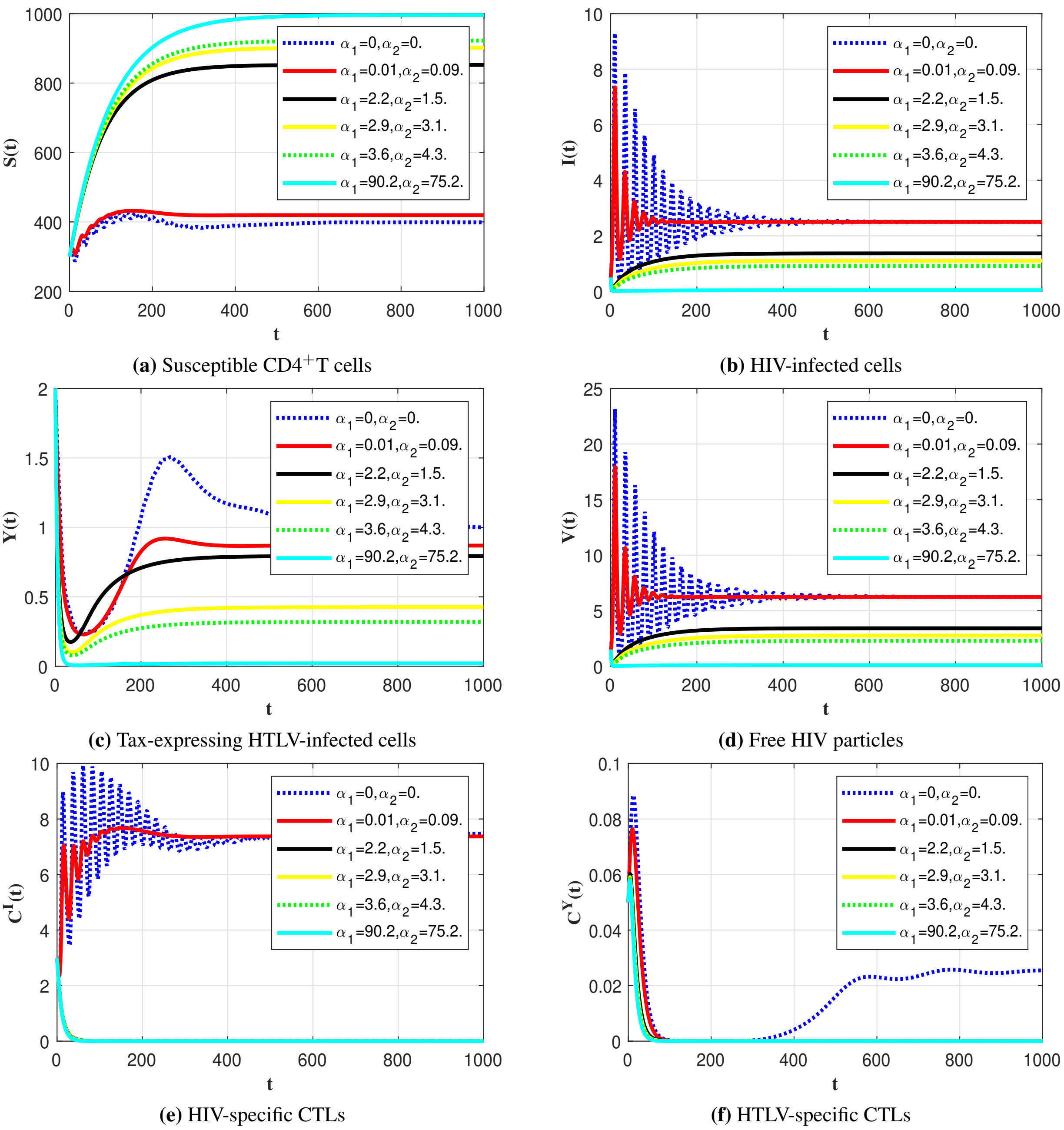


Fig. 10: The evolution of trajectories of model (1)-(6) with different saturation parameters α_1, α_2.

7 Conclusion

In this paper, we proposed an HIV/HTLV-I co-infection model describing the interaction of the HIV/HTLV-I with $CD4^{+}T$ cells taking into account the CTL immune response. The infection rate is given by saturation functional response. We formulated the model via six nonlinear ordinary differential equations. It was demonstrated that the model is both mathematically and biologically acceptable. We found nine stable states in the HIV/HTLV-I co-infection model. The global stability of the nine steady states of the model has been established by constructing suitable Lyapunov functional and using LaSalle's invariant principle. We derived twelve threshold parameters Ri;i = 0, 1, ...,11: which govern the existence and global asymptotic stability of the nine steady states. The numerical simulation was done to verify the viability and applicability of the theoretical hypotheses. The necessity of including a saturated incidence in the HIV/HTLVI dynamics model was discussed.

Conflict of Interest

The authors declare that there is no conflict regarding the publication of this paper.

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