Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100406

Numerical Analysis for Spread Option Pricing Model in Illiquid underlying Asset Market: Full Feedback Model

A. R. Yazdanian[1,*] *and Traian A. Pirvu*[2]

[1] Faculty of Mathematics, Statistics and Computer Science, Semnan University, Semnan, Iran.
[2] Department of Mathematics and Statistics, McMaster University, Hamilton, Canada.

Received: 19 Mar. 2016, Revised: 5 May 2016, Accepted: 6 May 2016
Published online: 1 Jul. 2016

Abstract: This paper performs the numerical analysis and the computation of a Spread option in a market with imperfect liquidity. The number of shares traded in the stock market has a direct impact on the stock's price. Thus, we consider a *full-feedback model* in which price impact is fully incorporated into the model. The price of a Spread option is characterized by a nonlinear partial differential equation. This is reduced to linear equations by asymptotic expansions. The Peaceman-Rachford scheme, as an alternating direction implicit method, is employed to solve the linear equations numerically. We discuss the stability and the convergence of the numerical scheme. Illustrative examples are included to demonstrate the validity and applicability of the presented method. Finally we provide a numerical analysis of the illiquidity effect in replicating an European Spread option; compared to the Black-Scholes model the price of the option is higher in the model with price impact.

Keywords: Spread option pricing, Price impact, Illiquid markets, Nonlinear finance, Asymptotic analysis, Peaceman-Rachford scheme

1 Introduction

Classical asset pricing theory assumes that traders act as price takers, that is, they have no impact on the prices paid or received. The relaxation of this assumption and its impact on realized returns in asset pricing models is called liquidity risk. Consistent with this discussion, most of the option pricing models assume that an option trader can not affect the price in trading the underlying asset to replicate the option payoff, regardless of her trading size. The papers of Black and Scholes [1], and most of the work undertaken in mathematical finance has been done under this underlying assumption (which is reasonable only in a perfectly liquid market).
In presence of a price impact, the replication of an option becomes more involved. The first issues is whether or not the option is perfectly replicable. Second, one has to find out how the presence of price impact affects the the replicating costs. This encouraged researchers to develop a Black-Scholes model with price impact due to a large trader who is able to move the price by his/her actions.

Let us go over the existing research on this topic. [2] shows how an agent whose trades affect prices of some stock can replicate the payoff of a derivative security written on that stock. They characterize the derivative price in terms of a nonlinear partial differential equation; simulations are used to compare the hedging strategies in their model to classical Black-Scholes model. [3] studies the nonlinear effects of trading strategies to prices. A nonlinear partial differential equation for an option replication strategy is derived and is solved using numerical simulations. [4] analyzes how price impact on an underlying asset modifies the replication of a European option. They found that compared to the Black-Scholes case, a hedger buys more stock to replicate the option. The excess replicating cost over the Black-Scholes price is significant. An excellent survey of these research can be found in [5]. In the context of price impact, [6] solves the nonlinear equation for an option replication strategy by means of semidiscretization technique. Numerical works on this topic were developed in [7,8,9].

All these research works study how the price impact affects the replication of an option written on a single underlying (stock).

The purpose of our paper is to investigate the effects of imperfect liquidity on the replication of an European Spread option by a typical option trader in a full-feedback model (any trade will impact the prices of the

* Corresponding author e-mail: yazdanian@semnan.ac.ir

underlyings). Spread option is a simple example of multi-assets derivative, whose payoff is the difference between the prices of two or more assets; for instance let the prices of two underlying assets at time $t \in [0,T]$ be $S_1(t)$ and $S_2(t)$, then the payoff function of an European Spread option with maturity T is $[S_1(T)-S_2(T)-k]^+$ (here k is the strike of the option and the function x^+ is defined as $x^+ = max(x,0)$). Therefore the holder of an European Spread option has the right but not the obligation to buy the spread $S_1(T)-S_2(T)$ at the prespecified price k and maturity T. In general, there is no any analytical formula for the price of multi-assets options (even in models with perfect liquidity). The only exception is Margrabe formula for exchange options (Spread options with zero strike, see [10]). Margrabe derived a Black and Scholes type solution for this class of options as follows

$$C^M = S_1\Phi(d_+) - S_2\Phi(d_-), \tag{1}$$

where Φ is the standard normal cumulative distribution function and

$$\begin{aligned} d_{\pm} &= \frac{\ln(S_1/S_2)}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}, \\ \sigma &= \sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}. \end{aligned} \tag{2}$$

Here σ_1 and σ_2 are the volatility of two underlying assets $S_1(t)$ and $S_2(t), t \in [0,T]$. Their price dynamics are given by a stochastic differential equations based on two-dimensional standard correlated Brownian motion with correlation ρ. Since a linear combination of correlated lognormals is not lognormal, for non-zero strikes, there is no closed form Spread option valuation formula under the multivariate lognormal model. People rely on approximation formulas and numerical methods for Spreads valuation. Kirk [11] suggested the following analytical approximation for a Spread option with payoff $(S_1(T)-S_2(T)-k)^+$

$$C^M = S_1\Phi(d_+) - (S_2+k)\Phi(d_-), \tag{3}$$

where

$$\begin{aligned} d_{\pm} &= \frac{\ln(S_1/(S_2+k))}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}, \\ \sigma &= \sqrt{\sigma_1^2+(\frac{S_2}{S_2+k})^2\sigma_2^2-2\frac{S_2}{S_2+k}\rho\sigma_1\sigma_2}. \end{aligned} \tag{4}$$

This formula provides a good approximation of Spread option prices when the strike k is not far from zero. All the existing works on spread options assume a model with perfect liquidity.

Several Spread options are traded in the markets. Some popular Spread option products are: fixed Income Spread options and commodity Spread options (including Crush Spread options, Crack Spread options, Spark Spread options). In this work, we will focus our interest on Oil Markets and more specifically one of the most frequently quoted Spread options which are Crack Spreads. A Crack Spread represents the differential between the price of crude oil and petroleum products (gasoline or heating oil). The underlying indexes comprise futures prices of crude oil, heating oil and unleaded gasoline. Details of crack Spread options can be found in the New York Mercantile Exchange (NYMEX) Crack Spread Handbook [12]. In the oil markets with finite liquidity, trading does affect the underlying assets price. In our study we investigate the effects of impact price on Spread option pricing in oil markets, when trading affects the crude oil price, but not petroleum products .

Our Contributions: By the best of our knowledge our work is the first to consider pricing of spread options (options written on two stocks) in a full feedback model of the stocks. The nonlinear partial differential equation (PDE) which characterizes the price of the spread option appeared for the first time in our work. We used the matched asymptotic expansion technique to linearize this nonlinear PDE. The standard alternating direction implicit method (Peaceman-Rachford scheme)was employed to solve the corresponding linear equations numerically [17]. The stability and the convergence of the numerical scheme was established. Numerical experiments have shown that the price of the Spread option is higher in a full feedback model.

This paper is organized as follows: Section 2 discusses the general framework. In Section 3 we apply an asymptotic expansion for the full nonlinear partial differential equation which characterizes the Spread option price. In Section 4, we propose a numerical method for the linearized equation the Peaceman and Rachford numerical scheme. We discuss the stability and convergence of this scheme. In Section 5, we carry out several numerical experiments and provide a numerical analysis of the model for European Spread calls. Section 6 contains the concluding remarks. The paper ends with an Appendix.

2 Statement of the problem

In this section we describe the setup used for pricing Spread options. Our model of a financial market, based on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbf{P})$ that satisfies the usual conditions, and consists of two risky assets (stocks). Their prices are modeled by a two-dimensional diffusion process $S(t) = (S_1(t), S_2(t)), t \in [0,T]$. All the stochastic processes in this work are assumed to be $\{\mathscr{F}_t\}_{t\geq 0}$-adapted. Their dynamics are given by the following stochastic differential equations, in which $W(t) = (w_1(t), w_2(t))$ is defined a two–dimensional standard correlated Brownian motion with correlation ρ, and $\{\mathscr{F}_t\}_{t\in[0,T]}$ is its natural filtration augment by all

P-null sets:

$$\frac{dS_i(t)}{S_i(t)} = \mu_i(t,S_i(t))dt + \sigma_i(t,S_i(t))dw_i(t);\ i=1,2. \quad (5)$$

Here $\mu_i(t,S_i(t))$ and $\sigma_i(t,S_i(t))$ are the expected return and the volatility of stock i in the absence of price impact. It is possible to add a forcing term, $\lambda(t,S_1(t))$, i.e.,

$$\begin{aligned} dS_1(t) &= \mu_1(t,S_1(t))S_1(t)dt + \sigma_1(t,S_1(t))S_1(t)dw_1(t) \\ &\quad + \lambda(t,S_1(t))d\Delta_1(t), \\ dS_2(t) &= \mu_2(t,S_2(t))S_2(t)dt + \sigma_2(t,S_2(t))S_2(t)dw_2(t). \end{aligned} \quad (6)$$

Here $\lambda(t,S_1(t))$ is the price impact function on the first stock, S_1. The term $\lambda(t,S_1(t))d\Delta_1(t)$ represents the price impact of the trading strategy $\Delta_1(t)$; at time t, $\Delta_1(t)$ denotes the number of shares of the first stock bought ($\Delta_1(t)\geq 0$), or sold ($\Delta_1(t)\leq 0$). Since the price per share of S_1 depends on the number of shares bought or sold this model is reffered to as full-feedback model. We note that the classical Black-Scholes model is a special case of this model with $\lambda(t,S_1(t))=0$. Our aim is to price a Spread option in this illiquid market model. The option's payoff at maturity T is:

$$h(S_1(T),S_2(T)) = (S_1(T)-S_2(T)-k)^+, \quad (7)$$

where k is the strike price.

Let $V(t,S_1,S_2)$ denote the price (value) at time t of the spread option with payoff $[S_1(T)-S_2(T)-k]^+$ in our full-feedback model given that $S_1(t)=S_1, S_2(t)=S_2$. Then $V(t,S_1,S_2)$ solves the following nonlinear partial differential equation (for details on this see The Appendix):

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2(1-\lambda\frac{\partial^2 V}{\partial S_1^2})^2}(\sigma_1^2S_1^2 + \lambda^2\sigma_2^2S_2^2(\frac{\partial^2 V}{\partial S_1\partial S_2})^2 \\ &+ 2\rho\sigma_1\sigma_2S_1S_2\lambda\frac{\partial^2 V}{\partial S_1\partial S_2})\frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2S_2^2\frac{\partial^2 V}{\partial S_2^2} \\ &+ \frac{1}{1-\lambda\frac{\partial^2 V}{\partial S_1^2}}(\sigma_1\sigma_2\rho S_1S_2 + \lambda\sigma_2^2S_2^2\frac{\partial^2 V}{\partial S_1\partial S_2})\frac{\partial^2 V}{\partial S_1\partial S_2} \\ &+ r(S_1\frac{\partial V}{\partial S_1} + S_2\frac{\partial V}{\partial S_2}) - rV = 0, \quad 0<S_1,S_2<\infty, 0\leq t<T \end{aligned} \quad (8)$$

with the terminal condition at T,

$$V(T,S_1,S_2) = h(S_1,S_2), \quad 0<S_1,S_2<\infty.$$

Notice that the classical Black-Scholes partial differential equation which characterizes the price of the spread option in a model without price impact is obtained by setting $\lambda = 0$ in (8).

3 Matched Asymptotic Expansion

In this section we use a matched asymptotic expansion technique to linearize (8). For this purpose we let $\lambda(t,S_1(t)) = \varepsilon\widehat{\lambda}(t,S_1(t))$, so that (8) becomes

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2(1-\varepsilon\widehat{\lambda}\frac{\partial^2 V}{\partial S_1^2})^2}(\sigma_1^2S_1^2 + (\varepsilon\widehat{\lambda})^2\sigma_2^2S_2^2(\frac{\partial^2 V}{\partial S_1\partial S_2})^2 \\ &+ 2\rho\sigma_1\sigma_2S_1S_2\varepsilon\widehat{\lambda}\frac{\partial^2 V}{\partial S_1\partial S_2})\frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2S_2^2\frac{\partial^2 V}{\partial S_2^2} \\ &+ \frac{1}{1-\varepsilon\widehat{\lambda}\frac{\partial^2 V}{\partial S_1^2}}(\sigma_1\sigma_2\rho S_1S_2 + \varepsilon\widehat{\lambda}\sigma_2^2S_2^2\frac{\partial^2 V}{\partial S_1\partial S_2})\frac{\partial^2 V}{\partial S_1\partial S_2} \\ &+ r(S_1\frac{\partial V}{\partial S_1} + S_2\frac{\partial V}{\partial S_2}) - rV = 0. \end{aligned} \quad (9)$$

By replacing $V(t,S_1,S_2)$ in the equation (9) with

$$V(t,S_1,S_2) = V^0(t,S_1,S_2) + \varepsilon V^1(t,S_1,S_2) + o(\varepsilon^2), \quad (10)$$

we get

$$\begin{aligned} &\frac{\partial(V^0+\varepsilon V^1)}{\partial t} + \frac{\frac{\partial^2(V^0+\varepsilon V^1)}{\partial S_1^2}}{2(1-\varepsilon\widehat{\lambda}\frac{\partial^2(V^0+\varepsilon V^1)}{\partial S_1^2})^2}(\sigma_1^2S_1^2 \\ &+ (\varepsilon\widehat{\lambda})^2\sigma_2^2S_2^2(\frac{\partial^2(V^0+\varepsilon V^1)}{\partial S_1\partial S_2})^2 \\ &+ 2\rho\sigma_1\sigma_2S_1S_2\varepsilon\widehat{\lambda}\frac{\partial^2(V^0+\varepsilon V^1)}{\partial S_1\partial S_2}) + \frac{1}{2}\sigma_2^2S_2^2\frac{\partial^2(V^0+\varepsilon V^1)}{\partial S_2^2} \\ &+ \frac{(\sigma_1\sigma_2\rho S_1S_2 + \varepsilon\widehat{\lambda}\sigma_2^2S_2^2\frac{\partial^2(V^0+\varepsilon V^1)}{\partial S_1\partial S_2})\frac{\partial^2(V^0+\varepsilon V^1)}{\partial S_1\partial S_2}}{1-\varepsilon\widehat{\lambda}\frac{\partial^2(V^0+\varepsilon V^1)}{\partial S_1^2}} \\ &+ r(S_1\frac{\partial(V^0+\varepsilon V^1)}{\partial S_1} + S_2\frac{\partial(V^0+\varepsilon V^1)}{\partial S_2}) \\ &- r(V^0+\varepsilon V^1) + o(\varepsilon^2) = 0, \end{aligned} \quad (11)$$

By using Taylor series expansion one gets

$$\begin{aligned} \frac{1}{2(1-\varepsilon\widehat{\lambda}\frac{\partial^2(V^0+\varepsilon V^1)}{\partial S_1^2})^2} &= \frac{1}{2} + \varepsilon\widehat{\lambda}\frac{\partial^2 V^0}{\partial S_1^2} + o(\varepsilon^2) \\ \frac{1}{1-\varepsilon\widehat{\lambda}\frac{\partial^2(V^0+\varepsilon V^1)}{\partial S_1^2}} &= 1 + \varepsilon\widehat{\lambda}\frac{\partial^2 V^0}{\partial S_1^2} + o(\varepsilon^2). \end{aligned} \quad (12)$$

Therefore the following linear equations for $V^0(t,S_1,S_2)$ and $V^1(t,S_1,S_2)$ can be derived

$$\begin{aligned}&\frac{\partial V^0}{\partial t}+\frac{\sigma_1^2S_1^2}{2}\frac{\partial^2V^0}{\partial S_1^2}+\frac{\sigma_2^2S_2^2}{2}\frac{\partial^2V^0}{\partial S_2^2}+\sigma_1\sigma_2S_1S_2\rho\frac{\partial^2V^0}{\partial S_1\partial S_2}\\&+r[S_1\frac{\partial V^0}{\partial S_1}+S_2\frac{\partial V^0}{\partial S_2}]-rV^0=0,\\&V^0(T,S_1,S_2)=h(S_1,S_2),\qquad 0<S_1,S_2<\infty,\end{aligned}\tag{13}$$

$$\begin{aligned}&\frac{\partial V^1}{\partial t}+\frac{\sigma_1^2S_1^2}{2}\frac{\partial^2V^1}{\partial S_1^2}+\frac{\sigma_2^2S_2^2}{2}\frac{\partial^2V^1}{\partial S_2^2}+\sigma_1\sigma_2S_1S_2\rho\frac{\partial^2V^1}{\partial S_1\partial S_2}\\&+r[S_1\frac{\partial V^1}{\partial S_1}+S_2\frac{\partial V^1}{\partial S_2}]-rV^1=G,\\&V^1(T,S_1,S_2)=0,\qquad 0<S_1,S_2<\infty.\end{aligned}\tag{14}$$

Here

$$\begin{aligned}G=&-\widehat{\lambda}(2\rho\sigma_1\sigma_2S_1S_2\frac{\partial^2V^0}{\partial S_1\partial S_2}\frac{\partial^2V^0}{\partial S_1^2}+\sigma_1^2S_1^2(\frac{\partial^2V^0}{\partial S_1^2})^2\\&+\sigma_2^2S_2^2(\frac{\partial^2V^0}{\partial S_1\partial S_2})^2).\end{aligned}\tag{15}$$

Let us notice that PDE (13) is the standard Black-Scholes PDE, so $V^0(t,S_1,S_2)$ is the Black-Scholes price of the European Spread option in a model without price impact. In sequence we propose a numerical scheme for computing $V^0(t,S_1,S_2)$ and $V^1(t,S_1,S_2)$.

4 Alternating Direction Implicit Method

4.1 Peaceman-Rachford scheme

In this section, we present a numerical method for solving the following PDEs:

$$\begin{aligned}&\frac{\partial V^0}{\partial t}+\frac{\sigma_1^2x^2}{2}\frac{\partial^2V^0}{\partial x^2}+\frac{\sigma_2^2y^2}{2}\frac{\partial^2V^0}{\partial y^2}+\sigma_1\sigma_2xy\rho\frac{\partial^2V^0}{\partial x\partial y}\\&+r[x\frac{\partial V^0}{\partial x}+y\frac{\partial V^0}{\partial y}]-rV^0=0,\\&V^0(T,x,y)=h(x,y),\qquad 0<x,y<\infty,\end{aligned}\tag{16}$$

and

$$\begin{aligned}&\frac{\partial V^1}{\partial t}+\frac{\sigma_1^2x^2}{2}\frac{\partial^2V^1}{\partial x^2}+\frac{\sigma_2^2y^2}{2}\frac{\partial^2V^1}{\partial y^2}+\rho\sigma_1\sigma_2xy\frac{\partial^2V^1}{\partial x\partial y}\\&+r[x\frac{\partial V^1}{\partial x}+y\frac{\partial V^1}{\partial y}]-rV^1=G,\\&V^1(T,x,y)=0,\quad 0<x,y<\infty,\end{aligned}\tag{17}$$

The functions $V^0(t,x,y)$ and $V^1(t,x,y)$ are defined on $[0,T]\times[0,\infty)\times[0,\infty)$. For simplicity, we write:

$$L=\frac{\partial}{\partial t}+A_x+A_y+A_{xy},\tag{18}$$

where

$$\begin{aligned}A_x&=\frac{1}{2}\sigma_1^2x^2\frac{\partial^2}{\partial x^2}+rx\frac{\partial}{\partial x}-r\Theta,\\A_y&=\frac{1}{2}\sigma_2^2y^2\frac{\partial^2}{\partial y^2}+ry\frac{\partial}{\partial y}-r(1-\Theta),\\A_{xy}&=\sigma_1\sigma_1xy\rho\frac{\partial^2}{\partial x\partial y},\end{aligned}\tag{19}$$

and $0\le\Theta\le1$. While symmetry considerations might speak for an $\Theta=\frac{1}{2}$, it is computationally simpler to use $\Theta=0$ or $\Theta=1$, i.e., include the $rV-$ term fully in one of the two operators. Hence, we can write

$$\begin{cases}LV^0(t,x,y)=0,\\LV^1(t,x,y)=G,\end{cases}\tag{20}$$

where

$$\begin{aligned}G=&-\widehat{\lambda}(2\rho\sigma_1\sigma_2xy\frac{\partial^2V^0}{\partial x\partial y}\frac{\partial^2V^0}{\partial x^2}+\sigma_1^2x^2(\frac{\partial^2V^0}{\partial x^2})^2\\&+\sigma_2^2y^2(\frac{\partial^2V^0}{\partial x\partial y})^2).\end{aligned}\tag{21}$$

In order to find a numerical solution for these equations, we need to truncate the spatial domain to a bounded domain as: $\{(x,y);0\le x\le x_{max},0\le y\le y_{max}\}$. Let us introduce a grid of points in the time interval and in the truncated spatial domain as:

$$\begin{aligned}&t_l=l\Delta t,\ \ l=0,1,...L,\ \ \Delta t=\frac{T}{L},\\&x_m=m\Delta x,\ \ m=0,1,...M,\ \ \Delta x=\frac{x_{max}}{M},\\&y_n=n\Delta y,\ \ n=0,1,...N,\ \ \Delta y=\frac{y_{max}}{N}.\end{aligned}\tag{22}$$

Without loss of generality let $x_{max}=y_{max}$ and $\Delta x=\Delta y$. The functions $V^0(t,x,y)$ and $V^1(t,x,y)$ evaluated at a point on the grid are denoted as $V^{0,l}_{mn}=V^0(t_l,x_m,y_n)$ and $V^{1,l}_{mn}=V^1(t_l,x_m,y_n)$. If we need to refer to the solution at a specific time point, we will use notation $V^{0,l}=V^0(t_l,x_m,y_n)$ and $V^{1,l}=V^1(t_l,x_m,y_n)$. Furthermore, let symbols A_{dx},A_{dy} and A_{dxdy} denote second-order approximations to the operators A_x,A_y and A_{xy}. Since the differential operator can be split as in (19) we can use Alternating Direction Implicit (ADI) method. The general idea is to split a time step into two and consider one operator or one space coordinate at a time [16]. In other words, ADI involves the reduction of the problem to several one-dimensional implicit problems by factoring the scheme. We implement the Peaceman-Rachford scheme[17]. Let us begin by discretizing (16) in the time-direction:

$$
\begin{aligned}
V_t^0((l+1/2)\Delta t,x,y) &= \frac{V^{0,l+1}-V^{0,l}}{\Delta t}+O(\Delta t)^2)\\
(A_x+A_y+A_{xy})V^{0,l} &= \frac{1}{2}A_x(V^{0,l+1}+V^{0,l})\\
&+\frac{1}{2}A_y(V^{0,l+1}+V^{0,l})\\
&+\frac{1}{2}A_{xy}(V^{0,l+1}+V^{0,l})+O((\Delta t)^2).
\end{aligned} \tag{23}
$$

Using (23) in the equation (18) yields:

$$
\begin{aligned}
(I-\frac{1}{2}\Delta tA_x-\frac{1}{2}\Delta tA_y)V^{0,l} &= (I+\frac{1}{2}\Delta tA_x+\frac{1}{2}\Delta tA_y)V^{0,l+1}\\
&+\frac{1}{2}\Delta tA_{xy}(V^{0,l+1}+V^{0,l})\\
&+O((\Delta t)^3),
\end{aligned} \tag{24}
$$

where I denotes the identity operator. If we add $\frac{1}{4}(\Delta t)^2A_xA_yV^{0,l}$ on the left hand and $\frac{1}{4}(\Delta t)^2A_xA_yV^{0,l+1}$ on the right hand then we commit an error which is $O((\Delta t)^3)$ and therefore:

$$
\begin{aligned}
&(I-\frac{1}{2}\Delta tA_x)(I-\frac{1}{2}\Delta tA_y)V^{0,l} =\\
&(I+\frac{1}{2}\Delta tA_x)(I+\frac{1}{2}\Delta tA_y)V^{0,l+1}\\
&+\frac{1}{2}\Delta tA_{xy}(V^{0,l+1}+V^{0,l})\\
&+O((\Delta t)^3).
\end{aligned} \tag{25}
$$

We now discretize in the space coordinates replacing A_x by A_{dx}, A_y by A_{dy} and A_{xy} by A_{dxdy}

$$
\begin{aligned}
&(I-\frac{1}{2}\Delta tA_{dx})(I-\frac{1}{2}\Delta tA_{dy})V^{0,l} =\\
&(I+\frac{1}{2}\Delta tA_{dx})(I+\frac{1}{2}\Delta tA_{dy})V^{0,l+1}\\
&+\frac{1}{2}\Delta tA_{dxdy}(V^{0,l+1}+V^{0,l})\\
&+O((\Delta t)^3)+O(\Delta t(\Delta x)^2).
\end{aligned} \tag{26}
$$

This leads to the Peaceman-Rachford method (see [13])

$$
\begin{aligned}
(I-\frac{\Delta t}{2}A_{dx})V^{0,l+1/2} &= (I+\frac{\Delta t}{2}A_{dy})V^{0,l+1}+\alpha,\\
(I-\frac{\Delta t}{2}A_{dy})V^{0,l} &= (I+\frac{\Delta t}{2}A_{dx})V^{0,l+1/2}+\beta,
\end{aligned} \tag{27}
$$

where the auxiliary function $V^{0,l+1/2}$ links the above equations. We have introduced the values α and β to take into account the mix derivative term because it is not obvious how this term should be split. To align (27) with (26), we require that

$$
(I+\frac{\Delta t}{2}A_{dx})\alpha+(I-\frac{\Delta t}{2}A_{dx})\beta = \frac{1}{2}\Delta tA_{dxdy}(V^{0,l+1}+V^{0,l}), \tag{28}
$$

where a discrepancy of order $O((\Delta t)^3)$ may be allowed with reference to a similar term in (25). One possible choice for α and β is

$$
\alpha = \frac{\Delta t}{2}A_{dxdy}V^{0,l+1}, \quad \beta = \frac{\Delta t}{2}A_{dxdy}V^{0,l+1/2}. \tag{29}
$$

Finally, the Peaceman-Rachford scheme for V^0 in (16) is obtained as follows

$$
\begin{aligned}
(I-\frac{\Delta t}{2}A_{dx})V^{0,l+1/2} &= (I+\frac{\Delta t}{2}A_{dy})V^{0,l+1}+\frac{\Delta t}{2}A_{dxdy}V^{0,l+1},\\
(I-\frac{\Delta t}{2}A_{dy})V^{0,l} &= (I+\frac{\Delta t}{2}A_{dx})V^{0,l+1/2}+\frac{\Delta t}{2}A_{dxdy}V^{0,l+1/2}.
\end{aligned} \tag{30}
$$

In a first step we calculate $V^{0,l+1/2}$ using $V^{0,l+1}$. This step is implicit with respect to x. In a second step, defined by equations (30), we use $V^{0,l+1/2}$ to calculate $V^{0,l}$. This step is implicit in the direction of y. The Peaceman-Rachford scheme for V^1 in (17) is obtained as follows:

$$
\begin{aligned}
(I-\frac{\Delta t}{2}A_{dx})V^{1,l+1/2} &= (I+\frac{\Delta t}{2}A_{dy})V^{1,l+1}+\alpha,\\
(I-\frac{\Delta t}{2}A_{dy})V^{1,l} &= (I+\frac{\Delta t}{2}A_{dx})V^{1,l+1/2}+\beta,
\end{aligned} \tag{31}
$$

where auxiliary function $V^{1,l+1/2}$ links above equations. To align (31) with (26) we require that

$$
\begin{aligned}
(I+\frac{\Delta t}{2}A_{dx})\alpha+(I-\frac{\Delta t}{2}A_{dx})\beta &= \frac{1}{2}\Delta tA_{dxdy}(V^{1,l+1}+V^{1,l})\\
&-\frac{1}{2}\Delta t(G^{l+1}+G^{l}).
\end{aligned} \tag{32}
$$

One of the possible choice for α and β is

$$
\begin{aligned}
\alpha &= \frac{\Delta t}{2}A_{dxdy}V^{1,l+1}-\frac{\Delta t}{2}G^{l+1},\\
\beta &= \frac{\Delta t}{2}A_{dxdy}V^{0,l+1/2}-\frac{\Delta t}{2}G^{l}.
\end{aligned} \tag{33}
$$

The Peaceman-Rachford scheme for V^1 of (17) is obtained as follows:

$$
\begin{aligned}
(I-\frac{\Delta t}{2}A_{dx})V^{1,l+1/2} &= (I+\frac{\Delta t}{2}A_{dy})V^{1,l+1}\\
&+\frac{\Delta t}{2}A_{dxdy}V^{1,l+1}-\frac{\Delta t}{2}G^{l+1},\\
(I-\frac{\Delta t}{2}A_{dy})V^{1,l} &= (I+\frac{\Delta t}{2}A_{dx})V^{1,l+1/2}\\
&+\frac{\Delta t}{2}A_{dxdy}V^{0,l+1/2}-\frac{\Delta t}{2}G^{l}.
\end{aligned} \tag{34}
$$

In a first step calculate $V^{1,l+1/2}$ using $V^{1,l+1}$. This step is implicit with respect to x. In a second step, defined by equations (34), we use $V^{1,l+1/2}$ to calculate $V^{1,l}$. This step is implicit with respect to y.
Notice that due to the use of centered approximations of the derivatives, at $x_0 = y_0 = 0$, $x_m = x_{max}$ and $y_n = y_{max}$, there appear external fictitious nodes $x_{-1} = -\Delta x$, $y_{-1} = -\Delta y$, $x_{M+1} = (M+1)\Delta x$ and $y_{N+1} = (N+1)\Delta y$. The approximations of $V^{0,l}_{-1,n}, V^{0,l}_{M+1,n}, V^{0,l}_{m,-1}$ and $V^{0,l}_{m,N+1}$ are obtained by using linear interpolation. Thus we have the following relations

$$V^{0,l}_{-1,n} = 2V^{0,l}_{0,n} - V^{0,l}_{1,n}, V^{0,l}_{M+1,n} = 2V^{0,l}_{M,n} - V^{0,l}_{M-1,n}; n = 1(1)N,$$
$$V^{0,l}_{m,-1} = 2V^{0,l}_{m,0} - V^{0,l}_{m,1}, V^{0,l}_{m,N+1} = 2V^{0,l}_{m,N} - V^{0,l}_{m,N-1}; m = 1(1)M. \tag{35}$$

Similarly, we can write the same relations in terms of $V^{1,l}$. Now all values $V^{0,l}_{m,n}$ and $V^{1,l}_{m,n}$ are available. By repeating this procedure for $l = L-1, L-2, ..., 0$ we obtain $V^0_{m,n}$ and $V^1_{m,n}$ at all time points. The price of a Spread option at time $t_0 = 0$ can be approximated as :

$$V(t_0,x,y) \approx V^0(t_0,x,y) + \varepsilon V^1(t_0,x,y). \tag{36}$$

4.2 Stability Analysis of the Numerical Scheme

In this section, we discuss stability and convergence of the numerical schemes introduced in Section 4.1. First we analyze the stability of the Peaceman-Rachford scheme. In this case, we can use the Von Neumann analysis to establish the conditions for stability. This approach was described in [13](Chapter 2.2). The Von Neumann analysis is based on calculating the amplification factor of a scheme and deriving conditions under which it is less than one in absolute value.

Theorem 1.*A one-step finite difference scheme (with constant coefficients) is stable in a stability region Λ (any bounded nonempty region of the first octant of R^3 that has the origin as an accumulation point) if and only if there exist a constant c (independent of θ, ϕ, dt, dx and dy) such that*

$$|g(\theta,\phi,dt,dx,dy)| \leq 1 + cdt. \tag{37}$$

Here $g(\theta,\phi,dt,dx,dy)$ is the amplification factor of scheme with $(dt,dx,dy) \in \Lambda$. If $g(\theta,\phi,dt,dx,dy)$ is independent of dx, dy and dt, the above stability condition can be replaced with the restricted stability condition

$$|g(\theta,\phi)| \leq 1. \tag{38}$$

*Proof.*See [13].

*Remark.*This Theorem shows that to determine the stability of a finite difference scheme with constant coefficients, we only need to consider the amplification factor g. This Theorem does not apply directly to problems with variable coefficients. Nonetheless, the stability conditions obtained for constant coefficient schemes can be used to give stability conditions for the same scheme applied to equations with variable coefficients. The general procedure is that one considers each of the frozen coefficient problems arising from the scheme. The frozen coefficient problems are the constant coefficient problems obtained by fixing the coefficients at their values attained at each point in the domain of the computation. If each frozen coefficient problem is stable, then the variable coefficient problem is also stable (see [14,15]).

For finding the amplification factor, a simpler and equivalent procedure is to replace $V^{0,l}_{mn}$ and $V^{1,l}_{mn}$ in the scheme by $g^{-l}e^{im\theta}e^{in\phi}$ for each value of l,n and m. In this scheme the main issue is $V^{0,l+\frac{1}{2}}$, the intermediate quantity that links two separate steps in the scheme. To eliminate all reference to the intermediate quantity $V^{0,l+\frac{1}{2}}$, obtaining an equation for $V_{mn}^{0,l+\frac{1}{2}}$ in terms of $V^{0,l}_{mn}$ for each value of l,n and m, we use an equivalent and simpler procedure, which is to replace all occurrences of $V^{0,l+\frac{1}{2}}$ by $\widehat{g}g^{-l}e^{im\theta}e^{in\theta}$ as well as the usual replacement of $V^{0,l}_{mn}$ by $g^{-l}e^{im\theta}e^{in\phi}$ for each value of l,n and m, where $\widehat{g}$ is a function of θ,ϕ which in general will also depend on $\Delta t, \Delta x$ like g. Following [13] replace $V^{0,l+1/2}_{mn}$ and $V^{0,l}_{mn}$ by $\widehat{g}g^{-l}e^{im\theta}e^{in\phi}$ and $g^{-l}e^{im\theta}e^{in\phi}$ respectively to get

$$\begin{aligned}
&\frac{\Delta t}{2}A_{dx}V^{0,l+1/2}_{m,n} = \widehat{g}g^{-l}e^{im\theta}e^{in\phi}(-a_1 sin^2\frac{1}{2}\theta + b_1 i sin\theta)\\
&\frac{\Delta t}{2}A_{dy}V^{0,-l}_{m,n} = g^{-l}e^{im\theta}e^{in\phi}(-a_2 sin^2\frac{1}{2}\phi + b_2 i sin\phi - c_1)\\
&\frac{\Delta t}{2}A_{dxdy}V^{0,l+1/2}_{m,n} = -\widehat{g}g^{-l}e^{im\theta}e^{in\phi}c_2 sin\theta sin\phi\\
&\frac{\Delta t}{2}A_{dxdy}V^{0,l}_{m,n} = -g^{-l}e^{im\theta}e^{in\phi}c_2 sin\theta sin\phi.
\end{aligned} \tag{39}$$

Here

$$\begin{aligned}
&a_1(x_m) = \frac{\Delta t \sigma_1^2 x_m^2}{\Delta x^2}, a_2(y_n) = \frac{\Delta t \sigma_2^2 y_n^2}{(\Delta y)^2},\\
&b_1(x_m) = \frac{\Delta t r x_m}{2\Delta x}, b_2(y_n) = \frac{\Delta t r y_n}{2\Delta y}\\
&c_1 = \frac{r\Delta t}{2}, c_2(x_m,y_n) = \frac{\Delta t \sigma_1 \sigma_2 \rho x_m y_n}{2\Delta x \Delta y}.
\end{aligned} \tag{40}$$

Also, by replacing $V_{mn}^{1,l+1/2}$ and $V_{mn}^{1,l}$ by $\widehat{g}g^{-l}e^{im\theta}e^{in\phi}$ and $g^{-l}e^{im\theta}e^{in\phi}$ respectively, one gets

$$\begin{aligned}
\frac{\Delta t}{2}A_{dx}V_{m,n}^{1,l+1/2} &= \widehat{g}g^{-l}e^{im\theta}e^{in\phi}(-a_1 sin^2\frac{1}{2}\theta + b_1 isin\theta)\\
\frac{\Delta t}{2}A_{dy}V_{m,n}^{1,l} &= g^{-l}e^{im\theta}e^{in\phi}(-a_2 sin^2\frac{1}{2}\phi + b_2 isin\phi - c_1)\\
\frac{\Delta t}{2}A_{dxdy}V_{m,n}^{1,l+1/2} &= -\widehat{g}g^{-l}e^{im\theta}e^{in\phi}c_2 sin\theta sin\phi\\
\frac{\Delta t}{2}A_{dxdy}V_{m,n}^{1,l+1} &= -g^{-l}e^{im\theta}e^{in\phi}c_2 sin\theta sin\phi.
\end{aligned} \tag{41}$$

According to "Duhamel's principle" we ignore the G^{l+1} and G^{l} terms in stability analysis (for more details on this see [13]). We obtain the amplification factor

$$g = \frac{1 - a_2 sin^2\frac{1}{2}\phi + b_2 isin\phi - c_1 - c_2 sin\theta sin\phi}{(1 + a_1 sin^2\frac{1}{2}\theta - b_1 isin\theta)\widehat{g}}, \tag{42}$$

with

$$\widehat{g} = \frac{1 + a_2 sin^2\frac{1}{2}\phi - b_2 isin\phi + c_1}{1 - a_1 sin^2\frac{1}{2}\theta + b_1 isin\theta - c_2 sin\theta sin\phi}. \tag{43}$$

This factor can simplify to

$$\begin{aligned}
g &= \frac{[1 - a_1 sin^2\frac{1}{2}\theta - c_2 sin\theta sin\phi + (b_1 sin\theta)i]}{[1 + a_1 sin^2\frac{1}{2}\theta - (b_1 sin\theta)i]}\\
&\times \frac{[1 - a_2 sin^2\frac{1}{2}\phi - c_1 - c_2 sin\theta sin\phi + (b_2 sin\phi)i]}{[1 + a_2 sin^2\frac{1}{2}\phi + c_1 - (b_2 sin\phi)i]}.
\end{aligned} \tag{44}$$

Thus

$$\begin{aligned}
g^2 &= \frac{[(1 - a_1 sin^2\frac{1}{2}\theta - c_2 sin\theta sin\phi)^2 + b_1^2 sin^2\theta]}{[(1 + a_1 sin^2\frac{1}{2}\theta)^2 + b_1^2 sin^2\theta]}\\
&\times \frac{[(1 - a_2 sin^2\frac{1}{2}\phi - c_1 - c_2 sin\theta sin\phi)^2 + b_2^2 sin^2\phi]}{[(1 + a_2 sin^2\frac{1}{2}\phi + c_1)^2 + b_2^2 sin^2\phi]}
\end{aligned} \tag{45}$$

According to (40), we can write $a_2 = Ca_1$, $c_2 = \widehat{C}a_1$where C and $\widehat{C}$ are constants. Moreover $\frac{b_1}{a_1} \to 0$, as $\Delta x \to 0$. Since $\Delta x = \Delta y$, then $b_1 = b_2 = c_1 = \xi a_1$, with $\xi \to 0$ as $\Delta x \to 0$. In light of this, taking the limit in (45) one gets

$$\begin{aligned}
\lim_{\xi\to 0} g^2 &= \frac{(1 - a_1 sin^2\frac{1}{2}\theta - \widehat{C}a_1 sin\theta sin\phi)^2}{(1 + a_1 sin^2\frac{1}{2}\theta)^2}\\
&\times \frac{(1 - Ca_1 sin^2\frac{1}{2}\phi - \widehat{C}a_1 sin\theta sin\phi)^2}{(1 + Ca_1 sin^2\frac{1}{2}\phi)^2}.
\end{aligned} \tag{46}$$

It is enough to find conditions so that

$$\begin{aligned}
&\frac{(1 - a_1 sin^2\frac{1}{2}\theta - \widehat{C}a_1 sin\theta sin\phi)^2}{(1 + a_1 sin^2\frac{1}{2}\theta)^2}\\
&\times \frac{(1 - Ca_1 sin^2\frac{1}{2}\phi - \widehat{C}a_1 sin\theta sin\phi)^2}{(1 + Ca_1 sin^2\frac{1}{2}\phi)^2} \le 1.
\end{aligned} \tag{47}$$

Notice that

$$\begin{aligned}
&a_1 sin^2\frac{1}{2}\theta + \widehat{C}a_1 sin\theta sin\phi\\
&\le a_1|sin^2\frac{1}{2}\theta| + \widehat{C}a_1|sin\theta sin\phi|\\
&\le a_1|sin\frac{1}{2}\theta|[|sin\frac{1}{2}\theta| + 2\widehat{C}|cos\frac{1}{2}\theta sin\phi|]\\
&\le a_1[1 + 2\widehat{C}].
\end{aligned} \tag{48}$$

Thus

$$1 - a_1 sin^2\frac{1}{2}\theta - \widehat{C}a_1 sin\theta sin\phi \ge 0,$$

provided that $a_1[1 + 2\widehat{C}] \le 1$. Furthermore we have

$$\begin{aligned}
&Ca_1 sin^2\frac{1}{2}\phi + \widehat{C}a_1 sin\theta sin\phi\\
&\le Ca_1|sin^2\frac{1}{2}\phi| + \widehat{C}a_1|sin\theta sin\phi|\\
&\le a_1|sin\frac{1}{2}\phi|[Csin\frac{1}{2}\phi + 4\widehat{C}|cos\frac{1}{2}\phi sin\theta|\\
&\le a_1[C + 2\widehat{C}].
\end{aligned} \tag{49}$$

Then

$$1 - Ca_1 sin^2\frac{1}{2}\phi - \widehat{C}a_1 sin\theta sin\phi \ge 0,$$

provided that $a_1[C + 2\widehat{C}] \le 1$. Therefore we should find conditions so that

$$\begin{aligned}
&\frac{(1 - a_1 sin^2\frac{1}{2}\theta - \widehat{C}a_1 sin\theta sin\phi)}{(1 + a_1 sin^2\frac{1}{2}\theta)}\\
&\times \frac{(1 - Ca_1 sin^2\frac{1}{2}\phi - \widehat{C}a_1 sin\theta sin\phi)}{(1 + Ca_1 sin^2\frac{1}{2}\phi)} \le 1,
\end{aligned} \tag{50}$$

or equivalently

$$a_1(sin^2\frac{1}{2}\theta + \widehat{C}sin\theta sin\phi + Csin^2\frac{1}{2}\phi)(-2 + a_1\widehat{C}sin\theta sin\phi) \le 0. \tag{51}$$

Since $|y| \le 1$, then for any $x \in R$, $xy \ge -|x|$, and by $C \ge 4\widehat{C}^2$, we have that

$$\begin{aligned}
&sin^2\frac{1}{2}\theta + \widehat{C}sin\theta sin\phi + Csin^2\frac{1}{2}\phi \ge\\
&|sin\frac{1}{2}\theta|^2 - 4\widehat{C}|sin\frac{1}{2}\theta sin\frac{1}{2}\phi| + 4\widehat{C}^2|sin\frac{1}{2}\phi|^2 =\\
&(|sin\frac{1}{2}\theta| - 2\widehat{C}|sin\frac{1}{2}\phi|)^2 \ge 0.
\end{aligned} \tag{52}$$

Thus (50) is satisfied if $a_1 \le \frac{2}{C}$ and $|g(\theta,\phi)| \le 1$ holds true if

$$\begin{aligned}
&a_1 \le A = min\{\frac{2}{\widehat{C}}, \frac{1}{1+2\widehat{C}}, \frac{1}{4\widehat{C}^2 + 2\widehat{C}}\} \ \ or\\
&\frac{\Delta t}{(\Delta x)^2} \le \frac{A}{\sigma_1^2.x_{max}^2}, \ \frac{\Delta t}{(\Delta y)^2} \le \frac{A}{\sigma_2^2.y_{max}^2}.
\end{aligned} \tag{53}$$

Since $\Delta x = \Delta y$ and $x_{max} = y_{max}$, a sufficient condition for stability of the scheme is

$$\frac{\Delta t}{(\Delta x)^2} \leq \frac{A}{max\{\sigma_1^2, \sigma_2^2\} x_{max}^2}. \tag{54}$$

Thus, the Peaceman-Rachford scheme is stable if L the number of steps in the time interval, and M, N the number of steps in the spatial domain satisfy inequality (54). This condition is a consequence of the cross-derivative terms. In the absence of these terms, the scheme would be unconditionally stable.

The remaining issue we need to address is the convergence of the numerical method to the true value of the problem. According to [13] , this scheme is first-order accurate in time and space and due to its stability the scheme is convergent. Results of this convergence are summarized in the next section.

5 Numerical Results

Let us fix the values of the parameters of the marginal dynamical equations according to Table 1. We also assume the following form for price impact

$$\lambda(t) = \begin{cases} \varepsilon(1 - e^{-\beta(T-t)^{3/2}}), & \underline{S} \leqslant S_1 \leqslant \overline{S}, \\ 0, & otherwise, \end{cases}$$

where ε is a constant price impact coefficient, $T - t$ is time to expiry, β is a decay coefficient, $\underline{S}$ and $\overline{S}$ represent respectively, the lower and upper limit of the stock price within which there is a impact price.
We consider $\underline{S} = 60, \overline{S} = 140, \varepsilon = 0.01$ and $\beta = 100$ for the subsequent numerical analysis. Choosing a different value for $\beta, \underline{S}$ and $\overline{S}$ will change the magnitude of the subsequent results, however, the main qualitative results remain valid.

Table 1: Model data together with $r = 0.04$

	$S(t_0)$	σ	S_{min}	S_{max}
Asset 1	112	0.15	0	200
Asset 2	104	0.10	0	200

Convergence of Numerical Scheme. As we mentioned in Section 1, the exact option values for the option in illiquid market are unknown. Since $\lambda = 0$ leads to the standard Black-Scholes model, we compare the results obtained from the numerical method (with $\lambda = 0$ and strike 0) with the Margrabe's closed formula for exchange options (i.e. Spread Option with strike 0). We fix the values of the parameters according to Table 1, and vary the values of the correlation coefficient ρ. Results of this convergence study are summarized in Table 2. We can see from the table that the agreement is excellent. We plot the absolute error of our approximation (using $\lambda = 0$, strike 0 and Margrabe's closed formula as benchmark) against the stocks in Fig 1. Results of the numerical method for Spread option in illiquid market are stated in Table 3.

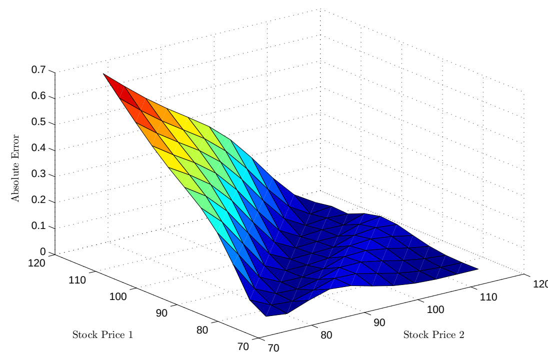

Fig. 1: Absolute errors between our approximation and Margrabe's closed formula, with $\sigma_1 = 0.15, \sigma_2 = 0.10, r = 0.05, \rho = 0.7, T = 0.7$ year, $m = 50$ and $l = 100$.

Table 2: Convergence of the Peaceman-Rachford method to Magrabe formula. Data are given in Table 1.

	m	l	$T = 0.1$	$T = 0.3$	$T = 0.5$	$T = 0.7$	$T = 1$
$\rho = 0.1$	50	100	8.1979	9.1570	10.0519	10.8369	11.8622
	100	100	8.2110	9.1892	10.0930	10.8757	11.9579
	200	200	8.2153	9.2373	10.1607	10.9727	12.0041
Margrabe			*8.2323*	*9.2462*	*10.1723*	*10.9892*	*12.0666*
$\rho = 0.5$	50	100	8.0088	8.5425	9.1276	9.6662	10.5095
	100	100	8.0591	8.5983	9.1961	9.7205	10.5405
	200	200	8.0687	8.6222	9.2209	9.7843	10.5636
Margrabe			*8.0692*	*8.6235*	*9.2294*	*9.7949*	*10.5648*
$\rho = 0.7$	50	100	7.9195	8.2199	8.6180	9.0019	9.5315
	100	210	7.9734	8.2509	8.6296	9.0929	9.6244
	200	200	7.9950	8.3023	8.7106	9.1035	9.6728
Margrabe			*8.0186*	*8.3128*	*8.7115*	*9.1110*	*9.6775*
$\rho = 0.9$	50	100	7.9252	7.9803	8.1740	8.3417	8.6412
	100	100	7.9310	7.9852	8.1894	8.3532	8.6498
	200	200	7.9938	8.0515	8.2032	8.3686	8.6571
Margrabe			*8.0005*	*8.0588*	*8.2015*	*8.3799*	*8.6675*

Replicating Cost. Next investigate the effects of the price impact (full feedback model) on the replication cost of Spread option. We investigate the excess price which is the difference between the call price in the full feedback model and the corresponding Black-Scholes price.
These figures 2, 3 and 4 indicate that the Spread option price in the full feedback model is higher than the classical Spread option price.

Table 3: The values of a 0.4 year European call Spread option based on different correlation, and strikes. Ex-P (Excess Price) shows the difference in call Spread option from Black-Scholes. The values of the parameters used for these runs are $\sigma_1 = 0.15, \sigma_2 = 0.10, r = 0.05$ with $m = l = 100$.

	$k=-15$	$k=-5$	$k=-2$	$k=0$	$k=2$	$k=5$	$k=10$	k=20
$\rho=0.1$	15.0929	7.1600	5.3275	4.2936	3.4027	2.3395	1.1267	0.1905
Ex-P	0.0001	0.0005	0.0005	0.0005	0.0005	0.0005	0.0003	0.00006
$\rho=0.5$	14.7992	6.2972	4.3645	3.3368	2.4486	1.4909	0.5435	0.0426
Ex-P	0.0001	0.0007	0.0009	0.0009	0.0009	0.0007	0.0003	0.00003
$\rho=0.7$	14.7085	5.7956	3.7731	2.7085	1.8642	0.9981	0.2593	0.0055
Ex-P	0.00006	0.0009	0.0013	0.0013	0.0012	0.0009	0.0004	0.00001
$\rho=0.9$	14.6833	5.2299	3.0523	1.9601	1.1531	0.4387	0.0088	0.0029
Ex-P	0.00003	0.0013	0.0020	0.0020	0.0018	0.0012	0.0003	0.00000

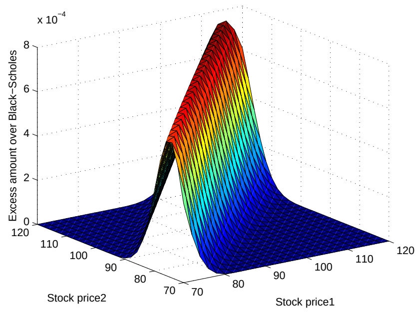

Fig. 2: The call price difference (classical model and full feedback model) as a function of stock price at time 0 against S_1 and S_2. $K = 5, \sigma_1 = 0.3, \sigma_2 = 0.2, r = 0.05, \rho = 0.7, T = 0.1$, and $m = l = 100$.

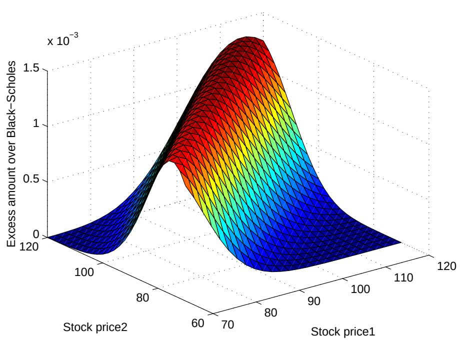

Fig. 3: The call price difference (classical model and full feedback model) as a function of stock price at time 0 against S_1 and S_2. $K = 5, \sigma_1 = 0.3, \sigma_2 = 0.2, r = 0.05, \rho = 0.7, T = 0.4$, and $m = l = 100$.

Excess Cost. Figure 5 shows the numerical results from the excess replicating costs above the corresponding Black-Scholes price for a call as a function of the strike price (with $S_1(t_0) = 100, S_2(t_0) = 110, \sigma_1 = 0.15, \sigma_2 = 0.10, r = 0.05, \rho = 0.7$, $T = 0.4$ year). As the option becomes more and more in the money and out of the money, the excess price converges monotonically to zero.

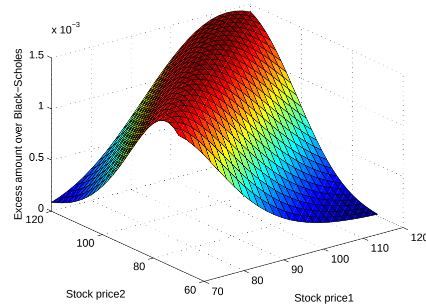

Fig. 4: The call price difference (classical model and full feedback model) as a function of stock price at time 0 against S_1 and S_2. $K = 5, \sigma_1 = 0.3, \sigma_2 = 60.2, r = 0.05, \rho = 0.7, T = 1$, and $m = l = 100$.

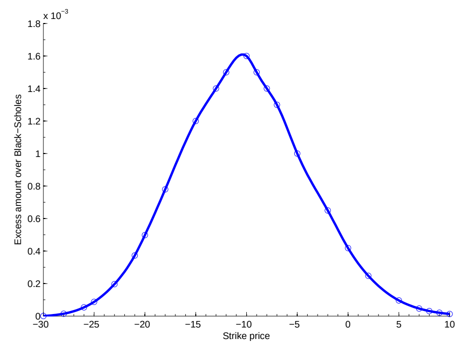

Fig. 5: The call price difference (classical model and full feedback model) against the strike price K. $S_1(t_0) = 100, S_2(t_0) = 110, \sigma_1 = 0.15, \sigma_2 = 0.10, r = 0.05, \rho = 0.7$, $T = 0.4$ year and $m = l = 100$.

6 Conclusion

In this work, we have investigated a model which incorporates illiquidity of the underlying asset into the classical multi-asset Black-Scholes framework. We considered the full feedback model in which the hedger is assumed to be aware of the feedback effect and so would change the hedging strategy accordingly. Since there is no analytical formula for the price of an option within this model, we applied the Matched Asymptotic Expansions technique to linearize the partial differential equation characterizing the price. We applied a standard alternating direction implicit method (Peaceman-Rachford scheme) to solve the corresponding linear equations numerically. We also discussed the stability and the convergence of the numerical scheme. By running a numerical experiment, we investigated the effects of liquidity on the Spread option pricing in the full feedback model. Finally, we found out that the Spread option price in the market with finite liquidity (full feedback model), is higher than the Spread option price in the classical Black-Scholes framework.

Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

Appendix

Consider a portfolio Π, which is long the option and short Δ_1 shares of stock S_1 and Δ_2 shares of stock S_2. The value of this portfolio at time t is

$$\Pi(t) = V(t) - (\Delta_1(t)S_1(t) + \Delta_2(t)S_2(t)).$$

According to the self financing equation

$$d\Pi(t) = dV(t) - \Delta_1(t)dS_1(t) - \Delta_2(t)dS_2(t). \qquad (55)$$

Using Itô's lemma one gets

$$\begin{aligned} dV = {} & \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S_1}dS_1(t) + \frac{\partial V}{\partial S_2}dS_2(t) + \frac{1}{2}\frac{\partial^2 V}{\partial S_1^2}d[S_1,S_1] \\ & + \frac{1}{2}\frac{\partial^2 V}{\partial S_1 \partial S_2}d[S_1,S_2] + \frac{1}{2}\frac{\partial^2 V}{\partial S_2^2}d[S_2,S_2] \\ & + \frac{1}{2}\frac{\partial^2 V}{\partial S_2 \partial S_1}d[S_2,S_1]. \end{aligned} \qquad (56)$$

By substituting equation (56) into (55) one gets

$$\begin{aligned} d\Pi(t) = {} & (\frac{\partial V}{\partial S_1} - \Delta_1(t))dS_1(t) + (\frac{\partial V}{\partial S_2} - \Delta_2(t))dS_2(t) \\ & + \frac{\partial V}{\partial t}dt + \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{1}{2}\frac{\partial^2 V}{\partial S_i \partial S_j}d[S_i,S_j] \end{aligned} \qquad (57)$$

Next set

$$\Delta_1(t) = \frac{\partial V}{\partial S_1}, \Delta_2(t) = \frac{\partial V}{\partial S_2}, \qquad (58)$$

such that the change in value of the portfolio becomes

$$\frac{\partial V}{\partial t}dt + \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{1}{2}\frac{\partial^2 V}{\partial S_i \partial S_j}d[S_i,S_j] = d\Pi(t). \qquad (59)$$

The portfolio is required to have return r, i.e.,

$$d\Pi(t) = r\Pi(t)dt.$$

Plugging this into equation (59) yields

$$\begin{aligned} \frac{\partial V}{\partial t}dt + \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{1}{2}\frac{\partial^2 V}{\partial S_i \partial S_j}d[S_i,S_j] = {} & r(V(t) - \frac{\partial V}{\partial S_1}S_1(t) \\ & - \frac{\partial V}{\partial S_2}S_2(t))dt. \end{aligned} \qquad (60)$$

In order to proceed further we need the expressions for $d[S_1,S_1], d[S_2,S_2]$ and $d[S_1,S_2]$. Note that from the second equation of (6) one gets

$$d[S_2,S_2](t) = \sigma_2^2(t,S_2(t))S_2^2(t)dt. \qquad (61)$$

Itô's formula applied to $\Delta_1 = \frac{\partial V}{\partial S_1}$ yields

$$\begin{aligned} d\Delta_1(t) = {} & \frac{\partial^2 V}{\partial S_1 \partial t}dt + \frac{\partial^2 V}{\partial S_1^2}dS_1(t) \\ & + \frac{\partial^2 V}{\partial S_1 \partial S_2}dS_2(t) + \sum_{i=1}^{2}\frac{1}{2}\frac{\partial^3 V}{\partial S_1 \partial S_i^2}d[S_i,S_i] \\ & + \frac{\partial^3 V}{\partial S_1^2 \partial S_2}d[S_1,S_2]. \end{aligned} \qquad (62)$$

Substituting this into the first equation of (6) yields

$$\begin{aligned} [1 - \lambda(t,S_1(t))\frac{\partial^2 V}{\partial S_1^2}]dS_1(t) = {} & \sigma_1(t,S_1(t))S_1(t)dw_1(t) \\ & + \lambda(t,S_1(t))\sigma_2(t,S_2(t))S_2(t)\frac{\partial^2 V}{\partial S_1 \partial S_2}dw_2(t) \\ & + \mu_1(t,S_1(t))S_1(t)dt + \lambda(t,S_1(t))\frac{\partial^2 V}{\partial S_1 \partial t}dt \\ & + \lambda(t,S_1(t))\sum_{i=1}^{2}\frac{1}{2}\frac{\partial^3 V}{\partial S_1 \partial S_i^2}d[S_i,S_i], \end{aligned} \qquad (63)$$

By taking quadratic variation and covariation leads to

$$\begin{aligned} \frac{d[S_1,S_1](t)}{dt} = {} & \frac{1}{(1 - \lambda(t,S_1(t))\frac{\partial^2 V}{\partial S_1^2})^2}(\sigma_1^2(t,S_1(t))S_1^2(t) \\ & + \lambda^2(t,S_1(t))\sigma_2^2(t,S_2(t))S_2^2(t)(\frac{\partial^2 V}{\partial S_1 \partial S_2})^2 \\ & + \rho\sigma_1(t,S_1(t))\sigma_2(t,S_2)S_1(t)S_2(t)\lambda(t,S_1(t))\frac{\partial^2 V}{\partial S_1 \partial S_2}), \end{aligned} \qquad (64)$$

and

$$\frac{d[S_1(t),S_2(t)]}{dt} = \frac{(\sigma_1(t,S_1)\sigma_2(t,S_2)\rho S_1 S_2 + \lambda(t,S_1)\sigma_2^2(t,S_2)S_2^2\frac{\partial^2 V}{\partial S_1 \partial S_2})}{1 - \lambda(t,S_1(t))\frac{\partial^2 V}{\partial S_1^2}}. \qquad (65)$$

By substituting (61),(64) and (65) into equation (60) leads to

$$
\begin{aligned}
&\frac{\partial V}{\partial t}+\frac{1}{2(1-\lambda\frac{\partial^2 V}{\partial S_1^2})^2}(\sigma_1^2S_1^2+\lambda^2\sigma_2^2S_2^2(\frac{\partial^2 V}{\partial S_1\partial S_2})^2\\
&+2\rho\sigma_1\sigma_2S_1S_2\lambda\frac{\partial^2 V}{\partial S_1\partial S_2})\frac{\partial^2 V}{\partial S_1^2}+\frac{1}{2}\sigma_2^2S_2^2\frac{\partial^2 V}{\partial S_2^2}\\
&+\frac{1}{1-\lambda\frac{\partial^2 V}{\partial S_1^2}}(\sigma_1\sigma_2\rho S_1S_2+\lambda\sigma_2^2S_2^2\frac{\partial^2 V}{\partial S_1\partial S_2})\frac{\partial^2 V}{\partial S_1\partial S_2}\\
&+r(S_1\frac{\partial V}{\partial S_1}+S_2\frac{\partial V}{\partial S_2})-rV=0, 0<S_1,S_2<\infty, 0\le t<T.
\end{aligned} \tag{66}
$$

References

[1] Black. F and Scholes. M, The pricing of options and corporate liabilities J. Political Economy 81 (1973), pp 637-654.

[2] Frey R., Perfect option replication for a large trader, Finance Stoch., 2 (1998), pp. 115-142.

[3] Schonbucher J., and Wilmott P., The feedback effect of hedging in illiquid markets, SIAM J. Appl. Math., 61(2000), pp. 232-272.

[4] Liu H., and Yong J., Option pricing with an illiquid underlying asset market, J. Econom. Dynam. Control, 29, pp. 2125-2156.

[5] Glover K. J., Duck P. W., and Newton D. P., On Nonlinear Models of Markets with Finite Liquidity: Some Cautionary Notes. SIAM Journal on Applied Mathematics 70:8, (2010), pp 3252-3271.

[6] Company R., Navarro E., and Pintos R. J., Numerical solution of linear and nonlinear BlackScholes option pricing equations, Computer and Mathematics with Applications 50 (2009), pp. 910-920.

[7] Company R., Jodar L., and Pintos R. J., A numerical method for European Option Pricing with transaction costs nonlinear equation, Computer and Mathematics with Applications 56 (2008), pp. 813-821.

[8] Lesmana D. C., Wang S., An upwind finite difference method for a nonlinear BlackScholes equation governing European option valuation under transaction costs, Applied Mathematics and Computation 219 (2013), pp. 8811-8828.

[9] Koleva M. N., Positivity preserving numerical method for non-linear Black-Scholes models, Numerical Analysis and Its Applications, 8236 (2013), pp. 363-370.

[10] Margrabe W., The value of an option to exchange one asset for another, J. Finance 33 (1978) pp 177-86.

[11] Kirk E., Correlation in the energy markets, Managing Energy Price Risk 1st edn (Risk Publications) 1995, pp 71-8.

[12] New York Mercantile Exchange, "Crack Spread handbook." http://partners.futuresource.com/marketcenter/pdfs/crack.pdf, 2011.

[13] Strikwerda, J. C., Finite Difference Schemes and Partial Differential Equations, Chapman & Hall, 1989.

[14] Lax P.D., and Nirenberg L., On stability of difference schemes; a sharp form of Gdrding's inequality, Comm. Pure Appl. Math., 19 (1966), pp. 473-492.

[15] Wade, B. A., Stability and Sharp Convergence Estimates for Symmetrizable Difference Operators, Ph.D. Thesis, University of Wisconsin-Madison, 1987.

[16] Duffy, D. J., Finite Difference Methods in Financial Engineering, Wiley Finance 2006.

[17] Peaceman, D. W. and Rachford H. H., The Numerical Solution of Parabolic and Elliptic Differential Equations, Journal of the Society for Industrial and Applied Mathematics, Vol. 3, No. 1, 1955, 28-41.

AhmadReza Yazdanian is an Assistant Professor in faculty of Mathematics, Statistics & Computer Science at the University of Semnan. Prior to that, Dr Yazdanian obtained his doctorate in Numerical Analysis from Iran University of Science and Technology, while he did research in Finance Mathematics for a sabbatical leave period under the supervision of Dr. Pirvu in Mc. Master University. His main research interests lie on the area of financial mathematics with a special emphasis on optimal investment, risk management, asset pricing in imperfect markets and invers problems in financial models.

Traian A Pirvu is an Associate Professor in Math and Stats Department at McMaster University. He joined McMaster in 2008 after three years of teaching and research in mathematical finance at The University of British Columbia. Prior to that, Dr Pirvu obtained his doctorate in mathematical finance from Carnegie Melon University under the supervision of the well known author and researcher Steven Shreve. Dr. Pirvu has done extensive research on the foundations of mathematical finance, in areas such as the theory of risk measures, optimal consumption and investment with risk limits, time-consistency of decision makers and equilibrium pricing in illiquid markets.