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Modified LQP Method with a New Search Direction for Nonlinear Complimentarity Problems

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Abstract: In this paper, we propose a new LQP method for solving nonlinear complementarity problems. The method uses a new descent direction which differs from the other existing LQP methods and another optimal step length is employed to reach substantial progress in each iteration. We prove the global convergence of the proposed method under the same assumptions as in [A. Bnouhachem *al.*, A note on LQP method for nonlinear complimentarity problems. *Adv. Model. Optim.* **14**(1), 269-283 (2012)]. Some preliminary computational results are given to illustrate the efficiency of the proposed method.

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1 Introduction

The nonlinear complementarity problem (NCP) is to determine a vector $x \in R^n$ such that

$$x \geq 0, \quad F(x) \geq 0 \quad \text{and} \quad x^T F(x) = 0, \tag{1}$$

where F is a nonlinear mapping from R^n into itself. Throughout this paper we assume that F is monotone with respect to R^n_+ and the solution set of (1), denoted by Ω^*, is nonempty.
The interest that attaches to this problem can be measured through its applications in various fields such as operations research transport problems, industry, engineering, optimization, mathematical and physical sciences in a unified framework. Furthermore, NCP is the subject of intense research in order to find a rich and less restrictive theory able to develop suitable iterative methods for its resolution. These iterative methods have emerged in the last decades as a powerful technique for solving NCP effectively. These methods are user friendly and can be implemented easily.
It is well known that the NCP can be alternatively formulated as finding the zero of the maximal monotone operator T

$$T(x) = F(x) + N_{R^n_+}(x) \tag{2}$$

where $N_{R^n_+}(.)$ is the normal cone operator to R^n_+ defined by

$$N_{R^n_+}(x) := \begin{cases} \{y : y^T(v-x) \leq 0, \quad \forall v \in R^n_+\}, & \text{if } x \in R^n_+; \\ \emptyset, & \text{otherwise.} \end{cases}$$

The proximal point algorithm (PPA) is the classical method to solve this problem, it has been proposed proposed by Martinet [18] and further studied by Rockafellar [19,20], for given $x^0 \in R^n$ and for positive real $\beta_k \geq \beta > 0$, find x^{k+1} solution of the following iterative problem

$$0 \in \beta_k T(x) + \nabla_x Q(x, x^k) \tag{3}$$

Rockafellar [19,20] gave the approximate proximal point algorithm, which is more practical and attractive than the exact one. The inexact version of the proximal point algorithm generates iteratively sequence $\{x^k\}$ satisfying

$$\xi^k \in \beta_k T(x) + \nabla_x q(x, x^k) \tag{4}$$

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where $\xi^k \in \mathscr{R}^n$ is the error term and

$$q(x,x^k) = \frac{1}{2}\|x-x^k\|^2. \qquad (5)$$

Several papers [3,4,6,7,10,12,14,17,21,22] have focused to improve the PPA via the generalization of this problem by replacing the usual linear term $x-x^k$ with some nonlinear functions $r(x,x^k)$. Auslender *et al.* [1,2] proposed a new type of proximal interior method through replacing the linear term by

$$x-(1-\mu)x^k-\mu X_k^2 x^{-1} \qquad (6)$$

or

$$x-x^k+\mu X_k log(\frac{x}{x^k}) \qquad (7)$$

where $\mu \in (0,1)$ is a given constant, $X_k = diag(x_1^k, x_2^k, \dots, x_n^k)$ and x^{-1} is an n-vector whose jth elements is $\frac{1}{x_j}$. Auslender *et al.* [1] proved that the sequence $\{x^k\}$ generated by (4) converges under the following conditions:

$$\sum_{k=1}^{+\infty}\|\xi^k\| < +\infty \quad \text{and} \quad \sum_{k=1}^{+\infty}\langle \xi^k, x^k\rangle \text{ exists and is finite.} \qquad (8)$$

Note that (8) implies that (4) should be solved exactly. To release this difficulty Burachik and Svaiter [11] presented a meaningful modification of the inexact LQP method with attractive characteristic that the relative error $\frac{\|\xi^k\|}{\|x^k-x^{k+1}\|}$ can be fixed on a constant.
It is easy to see that, at the kth iteration, solving (1) by the LQP method is equivalent to finding the approximate positive solution to the following system of nonlinear equations

$$\beta_k F(x)+x-(1-\mu)x^k-\mu X_k^2 x^{-1} = \xi^k \qquad (9)$$

or

$$\beta_k F(x)+x-x^k+\mu X_k log(\frac{x}{x^k}) = \xi^k. \qquad (10)$$

In order to improve the LQP method and make it more practical, He *et al.* [16] , Xu *et al.* [23], Bnouhachem *et al.* [5] and Bnouhachem [8] proposed some new LQP based prediction-correction methods via solving the system of nonlinear equation (9) or (10) under significantly relaxed accuracy criterion than (8). The above results have motivated us to propose a new LQP method for solving nonlinear complementarity problems. The method uses a new searching direction which differs from the other existing LQP methods, and another optimal step length is employed to reach substantial progress in each iteration. We proved the global convergence of the proposed method under the same assumptions as in [8]. Some preliminary computational results are given to illustrate the efficiency of the proposed method.

2 Preliminaries

In this section, we present some known results which will be used in the sequel.

Lemma 1. *Let $K \subset R_+^n$ be a nonempty closed convex set and $P_K[.]$ denotes the projection on K under the Euclidean norm, that is, $P_K[z] = \arg\min\{\|z-x\| : x \in K\}$. Then the following statements hold:*

$$(z-P_K[z])^T(P_K[z])-v) \geq 0, \quad \forall z \in R_+^n, v \in K. \qquad (11)$$

$$\|P_K[z]-v\|^2 \leq \|z-v\|^2 - \|z-P_K[z]\|^2, \forall z \in R_+^n,\ v \in K. \qquad (12)$$

Definition 1. *The operator $F : R^n \to R^n$ is said to be monotone, if*

$$\forall u,v \in R^n, \qquad (v-u)^T(F(v)-F(u)) \geq 0.$$

The following lemma is similar to Lemma 2 in [1]. Hence the proof will be omitted.

Lemma 2.[16,23] *For given $x^k > 0$ and $q \in R^n$, let x be the positive solution of the following equation:*

$$q+x-(1-\mu)x^k-\mu X_k^2 x^{-1} = 0, \qquad (13)$$

then for any $y \geq 0$ we have

$$\langle y-x, q\rangle \geq \tfrac{1+\mu}{2}\left(\|x-y\|^2-\|x^k-y\|^2\right)+\tfrac{1-\mu}{2}\|x^k-x\|^2. \qquad (14)$$

In [8], Bnouhachem used LQP method to suggest and analyze the following algorithm for solving problem (1). For given $x^k > 0$ and $\beta_k > 0$, each iteration of Bnouhachem's method consists of three steps, the first step offers $\tilde{x}^k$, the second step makes $\bar{x}^k$ and the third step produces the new iterate x^{k+1}.
First step: Find an approximate solution $\tilde{x}^k$ of (9), such that

$$0 \approx \beta_k F(\tilde{x}^k)+\tilde{x}^k-(1-\mu)x^k-\mu X_k^2(\tilde{x}^k)^{-1} = \xi^k \qquad (15)$$

and ξ^k satisfies

$$\|\xi^k\| \leq \eta\|x^k-\tilde{x}^k\|, \qquad 0 < \mu, \eta < 1. \qquad (16)$$

Second step: $\bar{x}^k(\alpha_k)$ is defined by

$$\bar{x}^k(\alpha_k) = P_{R_+^n}\left[x^k - \frac{\alpha_k\beta_k}{1+\mu}F(\tilde{x}^k)\right], \qquad (17)$$

where

$$\alpha_k = \frac{\varphi_k}{\|d^k\|^2}, \quad \varphi_k = \frac{1}{1+\mu}\|x^k-\tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k-\tilde{x}^k)^T\xi^k$$

and

$$d^k = (x^k-\tilde{x}^k)+\frac{1}{1+\mu}\xi^k.$$

Third step: For $0 < \rho < 1$, the new iterate $x^{k+1}(\tau_k)$ is defined by

$$x^{k+1}(\tau_k) = \rho x^k + (1-\rho)P_{R_+^n}\left[x^k - \tau_k g(x^k)\right]. \qquad (18)$$

where

$$g(x^k) = x^k - \bar{x}^k(\alpha_k), \quad \tau_k = \frac{\phi(x^k)}{\|g(x^k)\|^2},$$

$$\phi(x^k) = \frac{\|x^k - \bar{x}^k(\alpha_k)\|^2 + \Phi(\alpha_k)}{2}$$

and

$$\Phi(\alpha_k) = \alpha_k \varphi_k.$$

The third step of this method can be alternatively interpreted as follows: starting from x^k, Bouhachem's method moves along the direction $-g(x^k)$ with the step size $\tau_k = \frac{\phi(x^k)}{\|g(x^k)\|^2}$. A natural question is whether there exists another new profitable direction with a better optimal step size than τ_k. In Han [9] and many other papers, the authors considered $g(x)$ as a profitable direction and $\phi(x)$ as an error measure function, which measures how much x fails to be a solution of NCP. In this sense, many efforts have been devoted to constructing new profitable functions satisfying

$$\|g^{new}(x)\| \leq \|g(x)\| \tag{19}$$

and

$$(x - x^*)^T g^{new}(x) \geq (x - x^*)^T g(x) \geq \phi(x) \geq 0 \tag{20}$$

where $\phi(x)$ is a continuous function from R^n to R_+, and

$$\phi(x) = 0 \Longleftrightarrow x \quad \text{is a solution of NCP.} \tag{21}$$

The basic of our idea is very simple and can be summarized as follows. In the following section, we propose a new LQP method by using a new descent direction. The new descent direction can be viewed as combination of the descent direction of the existing LQP correction step and the descent direction used in the last iteration.

The following results are the key to prove the convergence of the proposed method.

Theorem 1.[8] *For given $x^k > 0$, let $\bar{x}^k(\alpha_k)$ be defined by (17). Then for any $x^* \in \Omega^*$, we have*

$$\|x^k - x^*\|^2 - \|\bar{x}^k(\alpha_k) - x^*\|^2 \geq \Phi(\alpha_k). \tag{22}$$

Theorem 2.[8] *For given $x^k > 0$ and $\tilde{x}^k$ satisfies (15), then we have the following*

$$\alpha_k \geq \frac{1-\eta}{2(1+\mu)} \tag{23}$$

and

$$\Phi(\alpha_k) \geq \frac{(1-\eta)^2}{2(1+\mu)^2} \|x^k - \tilde{x}^k\|^2. \tag{24}$$

3 Iterative method and convergence results

Now, we suggest the following algorithm for solving problem (1). Given $x^1 > 0$ and $D_0 = 0$, the proposed method consists of three steps, the first step offers $\tilde{x}^k$, the second step makes $\bar{x}^k$ and the third step produces the new iterate x^{k+1}.

First step: Find an approximate solution $\tilde{x}^k$ of (9), such that

$$0 \approx \beta_k F(\tilde{x}^k) + \tilde{x}^k - (1-\mu)x^k - \mu X_k^2 (\tilde{x}^k)^{-1} = \xi^k \tag{25}$$

and $\xi^k := \beta_k (F(\tilde{x}^k) - F(x^k))$ satisfies

$$\|\xi^k\| \leq \eta \|x^k - \tilde{x}^k\|, \qquad 0 < \mu, \eta < 1.$$

Second step: $\bar{x}^k(\alpha_k)$ is defined by

$$\bar{x}^k(\alpha_k) = P_{R_+^n}\left[x^k - \frac{\alpha_k \beta_k}{1+\mu} F(\tilde{x}^k)\right],$$

where

$$\alpha_k = \frac{\varphi_k}{\|d^k\|^2}.$$

Third step: For $0 < \rho < 1$, compute

$$\lambda_k = \max\left(0, \frac{-g(x^k)^T D_{k-1}}{\|D_{k-1}\|^2}\right) \tag{26}$$

and

$$D_k = g(x^k) + \lambda_k D_{k-1}, \tag{27}$$

the new iterate $x^{k+1}(\delta_k)$ is defined by

$$x^{k+1}(\delta_k) = \rho x^k + (1-\rho) P_{R_+^n}\left[x^k - \delta_k D_k\right] \tag{28}$$

where

$$\delta_k = \frac{\phi(x^k)}{\|D_k\|^2} \quad \text{and} \quad \phi(x^k) = \frac{\|x^k - \bar{x}^k(\alpha_k)\|^2 + \Phi(\alpha_k)}{2}. \tag{29}$$

Remark 1.

– The solution of (25) can be componentwise obtained by

$$\tilde{x}_j^k = \frac{(1-\mu)x_j^k - \beta_k F_j(x^k) + \sqrt{[(1-\mu)x_j^k - \beta_k F_j(x^k)]^2 + 4\mu (x_j^k)^2}}{2}. \tag{30}$$

Moreover for any $x^k > 0$ we have always $\tilde{x}^k > 0$.

– The proposed method can be viewed as an extension for some well-known results, for example the following.
  – If $\lambda_k = 0$, the new method reduces to the method proposed in [8].
  – If $\lambda_k = 0$ and $\delta_k = 1$, the new method reduces to the method proposed in [25].

For the convergence analysis, we need the following results.

Lemma 3. *Let $x^* \in \Omega^*, x^k > 0$. Then, we have*

$$(x^k - x^*)^T g(x^k) \geq \phi(x^k) \geq 0.$$

Proof: Using the following identity

$$(x^*-\bar{x}^k(\alpha_k))^T(x^k-\bar{x}^k(\alpha_k))=\frac{1}{2}\left(\|\bar{x}^k(\alpha_k)-x^*\|^2-\|x^k-x^*\|^2\right)+\frac{1}{2}\|x^k-\bar{x}^k(\alpha_k)\|^2$$

which implies that

$$\|g(x^k)\|^2-2(x^*-\bar{x}^k(\alpha_k))^T(x^k-\bar{x}^k(\alpha_k))=\|x^k-x^*\|^2-\|\bar{x}^k(\alpha_k)-x^*\|^2. \quad (31)$$

Then

$$\begin{aligned} g(x^k)^T(x^k-x^*) &= (x^k-\bar{x}^k(\alpha_k))^T(x^k-x^*) \\ &= \|x^k-\bar{x}^k(\alpha_k)\|^2+(x^k-\bar{x}^k(\alpha_k))^T(\bar{x}^k(\alpha_k)-x^*) \\ &= \|x^k-\bar{x}^k(\alpha_k)\|^2-(x^k-\bar{x}^k(\alpha_k))^T(x^*-\bar{x}^k(\alpha_k)) \\ &= \|x^k-\bar{x}^k(\alpha_k)\|^2 \\ &\quad +\frac{1}{2}\left(\|x^k-x^*\|^2-\|\bar{x}^k(\alpha_k)-x^*\|^2\right) \\ &\quad -\frac{1}{2}\|x^k-\bar{x}^k(\alpha_k)\|^2 \\ &\geq \frac{1}{2}(\|x^k-\bar{x}^k(\alpha_k)\|^2+\Phi(\alpha_k)) \\ &= \phi(\alpha_k) \\ &\geq 0 \end{aligned}$$

where the first inequalities and second inequality follow from Theorem 1 and Theorem 2, respectively. □

Lemma 4.[9] *Using the definitions of $g(x^k)$ and D_k, we get*

$$\|D_k\|\leq\|g(x^k)\|$$

Proof:
If $\lambda_k=0$, we have $\|D_k\|=\|g(x^k)\|$.
If $\lambda_k=\frac{-g(x^k)^TD_{k-1}}{\|D_{k-1}\|^2}$, we have

$$\begin{aligned} \|D_k\|^2 &= \|g(x^k)+\lambda_kD_{k-1}\|^2 \\ &= \|g(x^k)\|^2-\frac{(g(x^k)^TD_{k-1})^2}{\|D_{k-1}\|^2} \\ &\leq \|g(x^k)\|^2 \end{aligned}$$

which implies that

$$\|D_k\|\leq\|g(x^k)\|. \qquad \square$$

Lemma 5.[9] *For any $k\geq 1$, we have $D_{k-1}^T(x^k-x^*)\geq 0$.*

Proof: Note that this is trivially true for $k=1$ since $D_0=0$. By induction, consider any $k\geq 2$ and assume that

$$D_{k-2}^T(x^{k-1}-x^*)\geq 0.$$

Using the definition of D_{k-1}, we have

$$\begin{aligned} D_{k-1}^T(x^k-x^*) &= D_{k-1}^T(x^{k-1}-x^*)+D_{k-1}^T(x^k-x^{k-1}) \\ &= g(x^{k-1})^T(x^{k-1}-x^*)+\lambda_{k-1}D_{k-2}^T(x^{k-1}-x^*) \\ &\quad +D_{k-1}^T(x^k-x^{k-1}) \\ &\geq g(x^{k-1})^T(x^{k-1}-x^*)+D_{k-1}^T(x^k-x^{k-1}) \\ &\geq \phi(x^{k-1}) \\ &\quad -\|D_{k-1}\|\|P_{R^n_+}\left[x^{k-1}-\delta_{k-1}D_{k-1}\right]-x^{k-1}\| \\ &\geq \phi(x^{k-1})-\|D_{k-1}\|\|\delta_{k-1}D_{k-1}\| \\ &= 0 \end{aligned}$$

□

Using lemma 3 and lemma 5, we have

$$(x^k-x^*)^TD_k\geq(x^k-x^*)^Tg(x^k)\geq\phi(x^k)\geq 0.$$

To ensure that $x^{k+1}(\delta_k)$ is closer to the solution set than x^k. For this purpose, we define

$$\Upsilon(\delta_k)=\|x^k-x^*\|^2-\|x^{k+1}(\delta_k)-x^*\|^2. \quad (32)$$

Theorem 3. *Let $x^*\in\Omega^*$, then we have*

$$\Upsilon(\delta_k)\geq(1-\rho)(\delta_k\{\|g(x^k)\|^2+\|x^k-x^*\|^2-\|\bar{x}^k(\alpha_k)-x^*\|^2\} -\delta_k^2\|D_k\|^2). \quad (33)$$

Proof: Since $x^*\in\Omega^*\subset R^n_+$ and let $x^k_*(\delta_k)=P_{R^n_+}\left[x^k-\delta_kD_k\right]$ it follows from (12) that

$$\|x^k_*(\delta_k)-x^*\|^2\leq\|x^k-\delta_kD_k-x^*\|^2-\|x^k-\delta_kD_k-x^k_*(\delta_k)\|^2. \quad (34)$$

On the other hand, we have

$$\begin{aligned} \|x^{k+1}(\delta_k)-x^*\|^2 &= \|\rho(x^k-x^*)+(1-\rho)(x^k_*(\delta_k)-x^*)\|^2 \\ &= \rho^2\|x^k-x^*\|^2+(1-\rho)^2\|x^k_*(\delta_k)-x^*\|^2 \\ &\quad +2\rho(1-\rho)(x^k-x^*)^T(x^k_*(\delta_k)-x^*). \end{aligned}$$

Using the following identity

$$2(a+b)^Tb=\|a+b\|^2-\|a\|^2+\|b\|^2$$

for $a=x^k-x^k_*(\delta_k), b=x^k_*(\delta_k)-x^*$ and (34), and using $0<\rho<1$, we obtain

$$\begin{aligned} \|x^{k+1}(\delta_k)-x^*\|^2 &= \rho^2\|x^k-x^*\|^2+(1-\rho)^2\|x^k_*(\delta_k)-x^*\|^2 \\ &\quad +\rho(1-\rho)\{\|x^k-x^*\|^2-\|x^k-x^k_*(\delta_k)\|^2 \\ &\quad +\|x^k_*(\delta_k)-x^*\|^2\} \\ &= \rho\|x^k-x^*\|^2+(1-\rho)\|x^k_*(\delta_k)-x^*\|^2 \\ &\quad -\rho(1-\rho)\|x^k-x^k_*(\delta_k)\|^2 \\ &\leq \rho\|x^k-x^*\|^2+(1-\rho)\|x^k-\delta_kD_k-x^*\|^2 \\ &\quad -(1-\rho)\|x^k-\delta_kD_k-x^k_*(\delta_k)\|^2 \\ &\quad -\rho(1-\rho)\|x^k-x^k_*(\delta_k)\|^2 \\ &= \|x^k-x^*\|^2-(1-\rho)\{\|x^k-x^k_*(\delta_k)-\delta_kD_k\|^2 \\ &\quad +\rho\|x^k-x^k_*(\delta_k)\|^2-\delta_k^2\|D_k\|^2+2\delta_k(x^k-x^*)^TD_k\} \\ &\leq \|x^k-x^*\|^2-(1-\rho)\{2\delta_k(x^k-x^*)^T(g(x^k)+\lambda_kD_{k-1}) \\ &\quad -\delta_k^2\|D_k\|^2\} \\ &= \|x^k-x^*\|^2-(1-\rho)\{2\delta_k(x^k-x^*)^Tg(x^k) \\ &\quad +2\delta_k(x^k-x^*)^T\lambda_kD_{k-1}-\delta_k^2\|D_k\|^2\} \\ &\leq \|x^k-x^*\|^2-(1-\rho)\{2\delta_k(x^k-x^*)^Tg(x^k) \\ &\quad -\delta_k^2\|D_k\|^2\} \end{aligned}$$

where the last inequality follows from Lemma 5. Using the definition of $\Upsilon(\delta_k)$, we get

$$\begin{aligned} \Upsilon(\delta_k) &\geq (1-\rho)\{2\delta_k(x^k-x^*)^Tg(x^k)-\delta_k^2\|D_k\|^2\} \\ &= (1-\rho)(2\delta_k\{\|g(x^k)\|^2-(x^*-\bar{x}^k(\alpha_k))^Tg(x^k)\} \\ &\quad -\delta_k^2\|D_k\|^2). \end{aligned} \quad (35)$$

Using (31), we get the assertion of this theorem. □

Remark 2. By using Theorem 3 and Theorem 1, we get

$$\Upsilon(\delta_k) \geq (1-\rho)\Gamma(\delta_k), \tag{36}$$

where

$$\Gamma(\delta_k) = \delta_k\{\|g(x^k)\|^2 + \Phi(\alpha_k)\} - \delta_k{}^2\|g(x^k)\|^2 = \frac{\delta_k\{\|g(x^k)\|^2+\Phi(\alpha_k)\}}{2}. \tag{37}$$

And from [8], we have

$$\Upsilon'(\tau_k) \geq (1-\rho)\Lambda(\delta_k), \tag{38}$$

where

$$\Lambda(\tau_k) = \frac{\tau_k\{\|g(x^k)\|^2 + \Phi(\alpha_k)\}}{2}. \tag{39}$$

$\Upsilon(\delta_k)$ and $\Upsilon'(\tau_k)$ measure the progresses made by the new iterates generated by the proposed algorithm and the algorithm presented in [8], respectively.

It follows from Lemma 4 that

$$\begin{aligned}\delta_k &= \frac{\|g(x^k)\|^2 + \Phi(\alpha_k)}{2\|D_k\|^2}\\ &\geq \frac{\|g(x^k)\|^2 + \Phi(\alpha_k)}{2\|g(x^k)\|^2}\\ &= \tau_k\end{aligned} \tag{40}$$

and

$$\begin{aligned}\Gamma(\delta_k) &= \frac{\delta_k\{\|g(x^k)\|^2 + \Phi(\alpha_k)\}}{2}\\ &\geq \frac{\tau_k\{\|g(x^k)\|^2 + \Phi(\alpha_k)\}}{2}\\ &= \Lambda(\tau_k).\end{aligned} \tag{41}$$

Note that if $\lambda_k = 0$ and $\delta_k = 1$ the proposed method reduces to the method in [25]. Since δ_k is to maximize the profit function $\Gamma(\delta_k)$, we have

$$\Gamma(\delta_k) \geq \Gamma(1). \tag{42}$$

Inequalities (41) and (42) show theoretically that the proposed method is expected to make more progress than those in [8] and [25] at each iteration, and so it explains theoretically that the proposed method outperforms the methods in [8] and [25].

Since $\Phi(\alpha_k) > 0$, and from the definition of δ_k it is easy to prove that

$$\delta_k \geq \tau_k \geq \frac{1}{2}. \tag{43}$$

Since δ_k is to maximize the profit function $\Gamma(\delta_k)$ and from (33), Theorems 1 and 2, we have

$$\begin{aligned}\Gamma(\delta_k) &\geq \Gamma(1)\\ &\geq \Phi(\alpha_k)\\ &\geq \frac{(1-\eta)^2}{2(1+\mu)^2}\|x^k - \tilde{x}^k\|^2.\end{aligned} \tag{44}$$

From the computational point of view, a relaxation factor $\gamma \in [1,2)$ is preferable in the new iteration. Through simple manipulations, we obtain

$$\begin{aligned}\Gamma(\gamma\delta_k) &= \gamma\delta_k\{\|g(x^k)\|^2 + \Phi(\alpha_k)\} - (\gamma^2\delta_k)(\delta_k\|D_k\|^2)\\ &= \gamma(2-\gamma)\Gamma(\delta_k)\end{aligned} \tag{45}$$

It follows from (32), (36), (44) and (45) that there is a constant $c > 0$ such that

$$\|x^{k+1}(\gamma\delta_k) - x^*\|^2 \leq \|x^k - x^*\|^2 - c\|x^k - \tilde{x}^k\|^2 \qquad \forall x^* \in \Omega^*. \tag{46}$$

The following result can be proved by similar arguments as those in [3,16,23]. Hence the proof will be omitted.

Theorem 4.*[3,16,23] If $\inf_{k=1}^{\infty}\beta_k = \beta > 0$, then the sequence $\{x^k\}$ generated by the proposed method converges to some x^∞ which is a solution of NCP.*

The detailed algorithm is as follows.

Step 0. Let $\beta_1 > 0$, $\varepsilon > 0$, $0 < \mu < 1$, $0 < \eta < 1$, $0 < \rho < 1$, $1 \leq \gamma < 2$, $x^1 > 0$, $D_0 = 0$ and set $k := 1$.

Step 1. If $\|\min(x^k, F(x^k))\|_\infty \leq \varepsilon$, then stop. Otherwise, go to Step 2.

Step2. $s := (1-\mu)x^k - \beta_k F(x^k)$, $\tilde{x}_i^k := \left(s_i + \sqrt{(s_i)^2 + 4\mu(x_i^k)^2}\right)/2$, $\xi^k := \beta_k(F(\tilde{x}^k) - F(x^k))$, $r := \|\xi^k\|/\|x^k - \tilde{x}^k\|$.

while $(r > \eta)$

$\beta_k := \beta_k * 0.8/r$, $s := (1-\mu)x^k - \beta_k F(x^k)$, $\tilde{x}_i^k := \left(s_i + \sqrt{(s_i)^2 + 4\mu(x_i^k)^2}\right)/2$, $\xi^k := \beta_k(F(\tilde{x}^k) - F(x^k))$, $r := \|\xi^k\|/\|x^k - \tilde{x}^k\|$.

end while

Step 3. Compute

$$\bar{x}^k(\alpha_k) = P_{R^n_+}[x^k - \frac{\alpha_k\beta_k}{1+\mu}F(\tilde{x}^k)],$$

where

$$\alpha_k = \frac{\gamma\varphi_k}{\|d^k\|^2}$$

with $\varphi_k = \frac{1}{1+\mu}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T\xi^k$ and $d^k = (x^k - \tilde{x}^k) + \frac{1}{1+\mu}\xi^k$.

Step 4. Compute $g(x^k) = x^k - \bar{x}^k(\alpha_k)$, $\lambda_k = \max(0, \frac{-g(x^k)^T D_{k-1}}{\|D_{k-1}\|^2})$ and $D_k = g(x^k) + \lambda_k D_{k-1}$.

The new iterate is defined by

$$x^{k+1} = \rho x^k + (1-\rho)P_{R^n_+}\left[x^k - \gamma\delta_k D_k\right]$$

where

$\delta_k = \frac{\|g(x^k)\|^2+\Phi(\alpha_k)}{2\|D_k\|^2}$, and $\Phi(\alpha_k) = 2\alpha_k\varphi_k - \alpha_k^2\|d^k\|^2$.

Step 5. $\beta_{k+1} = \begin{cases} \frac{\beta_k * 0.7}{r} & \text{if} \quad r \le 0.3; \\ \beta_k & \text{otherwise.} \end{cases}$

Step 6. $k := k+1$; go to Step 1.

4 Preliminary computational results

In this section, we consider two examples to illustrate the efficiency and the performance of the proposed algorithm.

4.1 Numerical experiments I

We consider the nonlinear complementarity problems

$$x \ge 0, \qquad F(x) \ge 0, \qquad x^T F(x) = 0, \tag{47}$$

where

$$F(x) = D(x) + Mx + q,$$

$D(x)$ and $Mx+q$ are the nonlinear part and linear part of $F(x)$ respectively.

We form the linear part in the test problems similarly as in Harker and Pang [13]. The matrix $M = A^TA + B$, where A is an $n \times n$ matrix whose entries are randomly generated in the interval $(-5,+5)$ and a skew-symmetric matrix B is generated in the same way. The vector q is generated from a uniform distribution in the interval $(-500,500)$ or in $(-500,0)$. In $D(x)$, the nonlinear part of $F(x)$, the components are chosen to be $D_j(x) = d_j * \arctan(x_j)$, where d_j is a random variable in $(0,1)$.

In all tests we take the logarithmic proximal parameter $\mu = 0.01, \rho = 0.01, \gamma = 1.9$ and $\eta = 0.9$. All iterations start with $x^1 = (1,...,1)^T$ and $\beta_1 = 1$, and stopped criterion whenever

$$\|\min(x^k, F(x^k))\|_\infty \le 10^{-7}.$$

All codes were written in Matlab, and we compare the proposed method with that in [25]. The test results for problem (47) are reported in Tables 1 and 2. k is the number of iteration and l denotes the number of evaluations of mapping F.

Table 1 Numerical results for problem (47) with $q \in (-500,\ 500)$

	The method in [25]			The proposed method		
n	k	l	CPU(Sec.)	k	l	CPU(Sec.)
200	224	470	0.044	127	279	0.022
300	249	519	0.071	149	323	0.035
400	257	539	0.11	156	338	0.05
500	287	601	0.17	172	374	0.12
700	269	562	0.28	162	354	0.19
1000	262	547	1.28	158	341	0.71

Table 2 Numerical results for problem (47) with $q \in (-500,\ 500)$

	The method in [25]			The proposed method		
n	k	l	CPU(Sec.)	k	l	CPU(Sec.)
200	445	924	0.071	264	572	0.065
300	425	887	0.11	259	561	0.07
400	531	1104	0.17	333	720	0.13
500	556	1155	0.28	336	726	0.18
700	483	1001	0.46	279	605	0.31
1000	523	1088	1.86	295	638	1.17

Tables 1 and 2 show that the proposed method is more efficient. Numerical results indicate that the proposed method can be save about $59 \sim 77$ percent of the number of iterations and about $53 \sim 70$ of the amount of computing the value of function F.

4.2 Numerical experiments II

In this subsection, we apply the new method to a traffic equilibrium problem, which is a classical and important problem in transportation science, see, e.g., [15,24]. The numerical results show that the new method is attractive in practice.

Consider a network $[N,L]$ of nodes N and directed links L, which consists of a finite sequence of connecting links with a certain orientation. Let a,b, etc., denote the links; p,q, etc., denote the paths; ω denote an origin/destination (O/D) pair of nodes of the network; P_ω denotes the set of all paths connecting O/D pair ω; u_p represent the traffic flow on path p; d_ω denote the traffic demand between O/D pair ω, which must satisfy

$$d_\omega = \sum_{p \in P_\omega} u_p,$$

where $u_p \geq 0, \forall p$; and f_a denote the link load on link a, which must satisfy the following conservation of flow equation

$$f_a = \sum_{p \in P} \delta_{ap} u_p,$$

where

$$\delta_{ap} = \begin{cases} 1, & \text{if } a \text{ is contained in path } p; \\ 0, & \text{otherwise.} \end{cases}$$

Let A be the path-arc incidence matrix of the given problem and $f = \{f_a, a \in L\}$ be the vector of the link load. Since u is the path-flow, f is given by

$$f = A^T u. \tag{48}$$

In addition, let $t = \{t_a, a \in L\}$ be the row vector of link costs, with t_a denoting the user cost of travelling link a which is given by

$$t_a(f_a) = t_a^0 \left[1 + 0.15 \left(\frac{f_a}{C_a} \right)^4 \right], \tag{49}$$

where t_a^0 is the free-flow travel cost on link a and C_a is designed capacity of link a. Then t is a mapping of the path-flow u and its mathematical form is

$$t(u) := t(f) = t(A^T u).$$

Note that the travel cost on the path p denoted by θ_p is

$$\theta_p = \sum_{a \in L} \delta_{ap} t_a(f_a).$$

Let P denote the set of all the paths concerned. Let $\theta = \{\theta_p, p \in P\}$ be the vector of (path) travel cost. For given link travel cost vector t, θ is a mapping of the path-flow u, which is given by

$$\theta(u) = At(u) = At(A^T u). \tag{50}$$

Associated with every O/D pair ω, there is a travel disutility $\lambda_\omega(d)$, which is defined as following

$$\lambda_\omega(d) = -m_\omega \log(d_\omega) + q_\omega. \tag{51}$$

Note that both the path costs and the travel disutilities are functions of the flow pattern u. The traffic network equilibrium problem is to seek the path flow pattern u^*, which induces a demand pattern $d^* = d(u^*)$, for every O/D pair ω and each path $p \in P_\omega$,

$$T_p(u) = \theta_p(u) - \lambda_\omega(d(u)). \tag{52}$$

The problem can be reduced to a nonlinear complementarity problem in the space of path-flow pattern u: Find $u \in R^n$ such that

$$\text{Find} \quad u \geq 0 \quad , T(u) \geq 0 \quad \text{and} \quad u^T T(u) = 0. \tag{53}$$

In particular, we test the example studied in [15,24]. The network is depicted in Fig.1.

The free-flow travel cost and the designed capacity of links (49) are given in Table 3, the O/D pairs and the coefficient m and q in the disutility function (51) are given in Table 4. For this example, there are together 12 paths for the 4 given O/D pairs listed in Table 6.

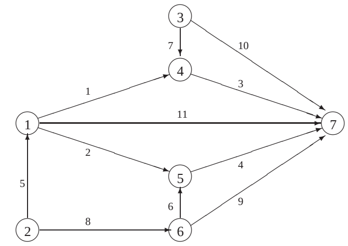

Fig. 1 The network used for the numerical test.

Table 3 The free-flow cost and the designed capacity of links in (49)

Link	Free-flow travel time t_a^0	Capacity C_a	Link	Free-flow travel time t_a^0	Capacity C_a
1	6	200	7	5	150
2	5	200	8	10	150
3	6	200	9	11	200
4	16	200	10	11	200
5	6	100	11	15	200
6	1	100	-	-	-

Table 4 The O/D pairs and the coefficient m and q in (51)

No. of the pair	O/D pair	m_ω	q_ω
1	$(1,7)$	25	$25\log 600$
2	$(2,7)$	33	$33\log 500$
3	$(3,7)$	20	$20\log 500$
4	$(6,7)$	20	$20\log 400$

Table 5 Numerical results for for different ε

Different	The method in [25]		The proposed method	
ε	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)
10^{-4}	136	0.13	95	0.022
10^{-5}	165	0.028	113	0.021
10^{-6}	194	0.033	129	0.022
10^{-7}	220	0.037	148	0.024
10^{-8}	247	0.056	166	0.026

In this example we take the same parameters as in subsection 4.1 and stopped whenever $\|\min(x^k, F(x^k))\|_\infty \leq \varepsilon$. The iteration numbers and the computational time for the proposed method and the method in [25] for different ε are reported in Table 5. For the case $\varepsilon = 10^{-8}$, the optimal path flow and link flow are given in Table 6 and 7, respectively.
The numerical experiments show that the new method is more flexible and efficient to solve the traffic equilibrium problem.

Table 6 The optimal path follow

O/D pair	Path no.	Link of path	Optimal path-flow
	1	$(1,3)$	165.3145
O/D pair (1,7)	2	$(2,4)$	0
	3	(11)	138.5735
	4	$(5,1,3)$	82.5281
	5	$(5,2,4)$	0
O/D pair (2,7)	6	$(5,11)$	55.7871
	7	$(8,6,4)$	0
	8	$(8,9)$	87.0260
O/D pair (3,7)	9	$(7,3)$	19.7549
	10	(10)	229.9747
O/D pair (6,7)	11	(9)	178.5600
	12	$(6,4)$	0

Table 7 The optimal link flow

Link no.	Link flow	Link no.	Link flow	Link no.	Link flow
1	247.8426	4	0	7	19.7549
2	0	5	138.3152	8	87.0260
3	267.5974	6	0	9	265.5860

Link no.	Link flow
10	229.9747
11	194.3606
–	–

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