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On Stochastic Orders and their Applications: Policy Limits and Deductibles

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Abstract: The paper deals with several types of stochastic order affecting random variables and linear combinations of random variables. We study the problem of finding maximal expected utility for some functionals on insurance portfolios involving some additional (independent) randomization. Applications in policy limits and deductible are obtained, and some relationships with other actuarial main topics (comparison of copulas, individual and collective risk models, reinsurance contracts, etc.) are studied too.

Keywords: Convex order, Comonotonocity, Policy limits, Deductibles

1 Introduction and motivation

Modern actuarial theory and risk theory play a crucial role in the economy and finance. Traditionally, insurance is based on the assumption of independence. But now, the progression and the complexity of insurance products has led to increased actuarial interest in the modeling of dependent risks.

Comparing risks is the very essence of the actuarial profession. This work is innovative in many respects. It integrates the theory of stochastic orders, one of the methodological cornerstones of risk theory and the theory of stochastic dependence, which has become increasingly important as new types of risks emerge. More precisely, risk measures will be used to generate stochastic ordering, by identifying pairs of risks about which a class of risk measures agree. Stochastic ordering are then used to define positive dependence relationships.

In the literature, ordering of optimal allocation of policy limits and deductible were established by maximizing the expected utility of wealth of the policyholder. In this paper, we study the problems of optimal allocation of policy limits and deductible for general model. In addition, by applying the bivariate characterizations of stochastic ordering relations, we reconsider the general model and derive some new results on ordering of optimal allocations and deductible. The results extend the main results in Cheung [1], Hua and Cheung [4] and Zhuang et al. [14]

We consider for the following model :

$$S_N = X_1 f(Y_1) + X_2 f(Y_2) + \ldots + X_n f(Y_n) \qquad \text{(M1)}$$

where :$Y_i = \delta_i T_i$, S_N is total discounted loss, X_i are loss due to the i-th risk, T_i are time of occurrence of i-th insured risk and δ_i are discount rate capture the impact of financial environment (X_i, T_i are independent non-negative random variables and δ_i are non-random numbers). Also, we will make the following assumptions :

1. $f(Y_i) \geq 0; \forall Y_i$ and $\lim\limits_{Y_i \to \infty} f(Y_i) = 0$.
2. $f(Y_i)$ is decreasing and convex function.
3. $Y_1, \ldots, Y_n$ are mutually independent.
4. A policyholder exposed to risks $X_1, X_2, \ldots, X_n$ is granted a total of l dollars $(l > 0)$ as the policy limit with which (s)he can allocate arbitrarily among the n risks.

Remark. A very good property of the model $(M1)$ is that X_i's characterize the scales of the losses while $f(Y_i)$ characterize the chances of the losses.

In this situation, if some risk occurs, the insurer will make the payment right after the event of the loss and the insurance coverage for this risk will terminate. However, the insurance coverage for the other risks is still in effect. If $(l_1, \ldots, l_n)$ are the allocated policy we have $\forall i : l_i \geq 0$ and $\sum\limits_{i=1}^{n} l_i = l$. When l is n-tuple admissible and $\mathscr{A}_n(l)$

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denote the class of all such n-tuples. If $\mathbf{l} = (l_1,...,l_n) \in \mathscr{A}_n(l)$ is chosen, then the discounted value of benefits obtained from the insurer would be

$$\sum_{i=1}^{n} (X_i \wedge l_i) f(Y_i)$$

If we take expected utility of wealth as the criterion for the optimal allocation, then the problem of the optimal allocation of policy limits is

$$\text{Problem } L: \max_{\mathbf{l}\in\mathscr{A}_n(l)} \mathbb{E}\left[u\left(w - \sum_{i=1}^{n} [X_i - (X_i \wedge l_i)] f(Y_i)\right)\right].$$

where u is the utility function of the policyholder and w is the wealth (after premium).

Similarly, instead of policy limits, the policyholder may be granted a total of d dollars $(d > 0)$ as the policy deductible with which (s)he can allocate arbitrarily among the n risks. If $\mathbf{d} = (d_1,...,d_n) \in \mathscr{A}_n(d)$ are the allocated deductible, $\forall i : d_i \geq 0$, $\sum_{i=1}^{n} d_i = d$, and the discounted value of benefits obtained from the insurer would be

$$\sum_{i=1}^{n} (X_i - d_i)_+ f(Y_i)$$

Then the problem of the optimal allocation of policy deductible is

$$\text{Problem } D: \max_{\mathbf{d}\in\mathscr{A}_n(d)} \mathbb{E}\left[u\left(w - \sum_{i=1}^{n} \left[X_i - (X_i - d_i)_+\right] f(Y_i)\right)\right].$$

The paper is organized as follows : In section 2 we introduce the preliminaries and the notations and we will recall some basic concepts and lemmas which will be used in later sections. Section 3 is devoted to state the main results and its proofs. Finally, we give some examples and application of the theory of ordering risks in modern actuarial.

2 Preliminaries and notations

In this section, we will collect some basic definitions and facts that are useful in the sequel. Notations and conventions used throughout the paper will also be fixed.

In the following, we define $\mathscr{I}_n = \{(a_1,...,a_n) \in \mathbb{R}^n : a_1 \leq \cdots \leq a_n\}$ and $\mathscr{D}_n = \{(a_1,...,a_n) \in \mathbb{R}^n : a_1 \geq \cdots \geq a_n\}$. The notation $x_{[i]}$ and $x_{(i)}$ are the i-th largest and the i-th smallest element of $\mathbf{x}$ respectively. For any vector $\mathbf{x} = (x_1,...,x_n) \in \mathbb{R}^n$, the increasing rearrangement $(x_{(1)},...,x_{(n)}) \in \mathscr{I}_n$ will be denoted as $\mathbf{x}\uparrow$, and the decreasing rearrangement $(x_{[1]},...,x_{[n]}) \in \mathscr{D}_n$ will be denoted as $\mathbf{x}\downarrow$. If τ represents a permutation of the set $\{1,2,...,n\}$, then the permuted vector $(x_{\tau_{(1)}},...,x_{\tau_{(n)}})$ will be denoted as $\mathbf{x}\circ\tau$.

2.1 Stochastic order

We use the following references for this subsection, which Denuit et al. [2,3], Kaas et al. [5,6], Muller and Stoyan [9], Shaked and Shanthikumar [11,12], Zeghdoudi and Remita [13], Zhuang et al. [14]. Throughout this paper, all the random variables considered are defined on a common probability space $(\Omega,\mathscr{F},\mathbb{P})$ and that all the expectations mentioned exist. Also, we introduce some important definitions, known proposition and properties, which lay good foundations for the next section.

Definition 1([3]). *Let X and Y be two random variables,*

1.X is said to be smaller than Y in the usual stochastic order (resp.increasing convex order,decreasing convex order,convex order), denoted by $X \leq_{sl} Y$ (resp.$X \leq_{icx} Y$, $X \leq_{dcx} Y$, $X \leq_{cx} Y$), if

$$\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$$

for all increasing (resp. increasing convex, decreasing convex, convex) function ϕ.

2.X is said to be smaller than Y in the likelihood ratio order, denoted by $X \leq_{lr} Y$, if

$$f_X(x)g_Y(y) \geq f_X(y)g_Y(x) \text{ for all } x \leq y$$

where f_X and g_Y are the density functions of X and Y, respectively.

2.1.1 Convex ordering random variables

In the continuation, we will consider random variables with finite mean. In the actuarial literature it is often to submit a random variable by a ”less attractive” random variable which has a simpler structure, making it easier to determine its distribution function.

The stop-loss premium is defined by $\mathbb{E}[(X-d)_+] = \int_d^{\infty}(1-F_X(x))dx, -\infty < d < +\infty$. And the notation S will be used for the sum of the random vector $(X_1,X_2,...,X_n)$: $S = X_1 + X_2 + ... + X_n$.

Now, we define the stop-loss order between random variables.

Definition 2([2]). *(Stop-loss order). Consider two random variables X and Y then X is said to precede Y in the stop-loss order sense, notation $X \leq_{sl} Y$ if and only if X has lower stop-loss premium then Y:*

$$\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+];\ -\infty < d < +\infty$$

with $(x-d)_+ = max(x-d,0)$.

Definition 3([2]). *(convex order). Consider two random variables X and Y such that $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$, for all convex functions ϕ,provided expectation exit. Then X is said to be smaller than Y in the convex order denoted as $X \leq_{cx} Y$.*

Proposition 1([2]). *(Convex order characterization using stop-loss pemrium). Consider two random variables X and Y . Then X is said to precede Y in convex order sense if and only if*

$$\mathbb{E}(X) = \mathbb{E}(Y)$$
$$\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+];\ -\infty < d < +\infty$$

Definition 4([6]). *(Majorization Order). Given any two vectors $\mathbf{a},\mathbf{b} \in \mathbb{R}^n$,*
1) $\mathbf{b}$ is said to be majorized by $\mathbf{a}$ (denoted by $\mathbf{b} \prec \mathbf{a}$), if

$$\begin{cases} \sum_{i=1}^n b_{[i]} = \sum_{i=1}^n a_{[i]} \\ \sum_{i=1}^m b_{[i]} \leq \sum_{i=1}^m a_{[i]},\ m = 1,\dots,n-1 \end{cases}$$

2) $\mathbf{b}$ is said to be weakly majorized by $\mathbf{a}$ (denoted by $\mathbf{b} \prec\prec \mathbf{a}$), if

$$\sum_{i=1}^m b_{[i]} \leq \sum_{i=1}^m a_{[i]},\ m = 1,\dots,n.$$

2.1.2 Properties of Convex Ordering of Random Variables

1. If X precedes Y in convex order sense i.e if $X \leq_{cx} Y$, then $\mathbb{E}[X] = \mathbb{E}[Y]$ and $V[X] \leq V[Y]$, where $V[X]$ is variance of X.(See [2])
2. If $X \leq_{cx} Y$ and Z is independent of X and Y then $X + Z \leq_{cx} Y + Z$.(See [2])
3. Let X and Y be two random variable, then $X \leq_{cx} Y \Rightarrow -X \leq_{cx} -Y$. (See [3])
4. Let X and Y be two random variable such that $\mathbb{E}(\ X) = \mathbb{E}(Y)$.Then $X \leq_{cx} Y$ if and only if $\mathbb{E}|X-a| \leq \mathbb{E}|Y-a|, \forall a \in \mathbb{R}$.(See [3])
5. The convex order is closed under mixtures: Let X, Y and Z be random variables such that $[X \mid Z = z] \leq_{cx} [Y \mid Z = z]$ $\forall z$ in the support of Z. Then $X \leq_{cx} Y$.(See [6])
6. The convex order is closed under convolution: let $X_1, X_2, \dots, X_m$ be a set of independent random variable and $Y_1, Y_2, \dots, Y_m$ be another set of independent random variables. If $X_j \leq_{cx} Y_j$, for $j = 1, \dots, m$, then $\sum_{j=1}^m X_j \leq_{cx} \sum_{j=1}^m Y_j$.(See [2])
7. Let X be a random variable with finite mean. Then $X + \mathbb{E}(X) \leq_{cx} 2X$.(It suffices to use the proposition 1)
8. Let $X_1, X_2, ..X_n$ and Y be $(n+1)$ random variables. If $X_i \leq_{cx} Y$, $i = 1,\dots,n$, then $\sum_{i=1}^n a_i X_i \leq_{cx} Y$, whenever $a_i \geq 0, i = 1,\dots,n$ and $\sum_{i=1}^n a_i = 1$.(It suffices to use the property 6)
9. Let X and Y be two independent random variables. Then $X \leq_{cx} Y$ if and only if $\mathbb{E}[\Phi(X,Y)] \leq \mathbb{E}[\Phi(Y,X)]$, where

$$\forall \Phi \in \Psi_{cx} = \left\{ \begin{array}{c} \Phi : \mathbb{R}^2 \to \mathbb{R} : \Phi(X,Y) - \Phi(Y,X) \\ \text{is convex for all } x \in y \end{array} \right\}.$$

(It suffices to use the proposition 1)

10. Let X_1 and X_2 be a pair of independent random variables and let Y_1 and Y_2 be another pair of independent random variables. If $X_i \leq_{cx} Y_i$, $i = 1,2$ then $X_1 X_2 \leq_{cx} Y_1 Y_2$.(See [2])
11. Let $X \leq_{cx} Y$ if and only $\mathbb{E}[\upsilon(X)] \leq \mathbb{E}[\upsilon(Y)]$ for all convex function υ, provided expectation exit.(See [6])
12. Let X, Y and Z be random variables such that $X \leq_{cx} Y$ and $Y \leq_{cx} Z$,then $X \leq_{cx} Z$.(It suffices to use the proposition 1)
13. If $X \leq_{lr} Y$ and ϕ is any decreasing function, then $\phi(X) \geq_{lr} \phi(Y)$ See [12].
14. Let $X \in \mathbb{R}_+^n$ and $X_1 \leq_{lr} \dots \leq_{lr} X_n$ are mutually independent. If $\mathbf{b}$ is weakly majorities by $\mathbf{a}$ (denoted by $\mathbf{b} \prec\prec \mathbf{a}$) and $\mathbf{a} \in \mathscr{I}_n$, then $\sum_{i=1}^n b_i x_i \leq_{icx} \sum_{i=1}^n a_i x_i$ (see [4]).

2.2 Arrangement Increasing

Definition 5. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be arrangement increasing [decreasing], if for all i and j such that : $1 \leq i < j \leq n$*

$$(x_i - x_j)\{f(x_1,\dots,x_i,\dots,x_j,\dots,x_n) - f(x_1,\dots,x_j,\dots,x_i,\dots,x_n)\} \leq [\geq] 0.$$

One major example is given by the joint density function of mutually independent random variables that are ordered by the likelihood ratio order.

Lemma 1. *If $X_1,\dots,X_n$ are mutually independent and $X_1 \leq_{lr} \dots \leq_{lr} X_n$, then the joint density function of $(X_1,\dots,X_n)$ is arrangement increasing.*

Proof. see [1].

Definition 6. *A function $g(\mathbf{x},\lambda) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to be an arrangement increasing (AI) function if*

1. *g is permutation invariant, i.e., $g(\mathbf{x},\lambda) = g(\mathbf{x} \circ \tau, \lambda \circ \tau)$ for any permutation τ, and*
2. *g exhibits permutation order, i.e., $g(\mathbf{x}\downarrow, \lambda\uparrow) \leq g(\mathbf{x}\downarrow, \lambda \circ \tau) \leq g(\mathbf{x}\downarrow, \lambda\downarrow)$ for any permutation τ .*

The following lemma give us two examples of *AI* functions. Proofs can be found in [1].

Lemma 2. *The function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by*

$$g(\mathbf{x},\lambda) = -\sum_{i=1}^n (x_i - \lambda_i)_+ \text{ and } g(\mathbf{x},\lambda) = \sum_{i=1}^n (x_i \wedge \lambda_i)$$

are an AI function.

Proofs of the following lemmas can be found in [4].

Lemma 3. *Suppose that the function $\phi(x,\lambda) : \mathbb{R}^2 \to \mathbb{R}$ is increasing both in x and λ . If the function*

$$g(\mathbf{x},\lambda) = \sum_{i=1}^n \phi(x_i,\lambda_i)$$

from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ is an AI function, then

$$\hat{\phi}(\mathbf{x}\downarrow, \lambda\uparrow) \prec\prec \hat{\phi}(\mathbf{x}\downarrow, \lambda \circ \tau) \prec\prec \hat{\phi}(\mathbf{x}\downarrow, \lambda\downarrow)$$

for any permutation τ.

Lemma 4.*Suppose that the function $\phi(x,\lambda):\mathbb{R}^2\to\mathbb{R}$ is increasing in one variable and decreasing in the other. If the function*

$$g(\mathbf{x},\lambda)=\sum_{i=1}^{n}\phi(x_i,\lambda_i)$$

from $\mathbb{R}^n\times\mathbb{R}^n$ to $\mathbb{R}$ is an AI function, then

$$-\hat{\phi}(\mathbf{x}\downarrow,\lambda\downarrow)\prec\prec-\hat{\phi}(\mathbf{x}\downarrow,\lambda\circ\tau)\prec\prec\hat{\phi}(\mathbf{x}\downarrow,\lambda\uparrow)$$

for any permutation τ.

2.3 Comonotonicity

Definition 7.*A subset $A\in\mathbb{R}^n$ is said to be comonotonic if whenever $\mathbf{x}=(x_1,\dots,x_n)$ and $\mathbf{y}=(y_1,\dots,y_n)$ are elements of A, either $x_i\le y_i$ for all i or $y_i\le x_i$ for all i. A random vector $\mathbf{X}=(X_1,\dots,X_n)\in\mathbb{R}^n$ is said to be comonotonic if there is a comonotonic subset A of $\mathbb{R}^n$ such that $P(\mathbf{X}\in A)=1$.*

Let $F_1,\dots,F_n$ be n univariate distribution functions. We use $\mathscr{R}_n(F_1,\dots,F_n)$ to denote the Fréchet space of all the n-dimensional random vectors whose marginal distributions are $F_1,\dots,F_n$,respectively.

Furthermore, we will use the notation $\tilde{\mathbf{X}}=(\tilde{X}_1,\dots,\tilde{X}_n)$ to indicate a comonotonic random vector belonging to the Fréchet class $\mathscr{R}_n(F_1,\dots,F_n)$. The random vector $\tilde{\mathbf{X}}$ is often called a comonotonic counterpart or a comonotonic modification of $\mathbf{X}$.

Lemma 5.*The following statements are equivalent :*

1. *The random vector $\mathbf{X}=(X_1,\dots,X_n)$ is comonotonic.*
2. *A random vector Z and non-decreasing function $f_1,\dots,f_n$ exist such that*

$$\mathbf{X}\overset{d}{=}(f_1(Z),\dots,f_n(Z))$$

where the notation $\overset{d}{=}$ is used to indicate 'equality in distribution'.

This lemma implies that comonotonicity is preserved under a non-decreasing transform on each component of $\mathbf{X}$.

Lemma 6([2]). *If*

$$(\tilde{X}_1,\dots,\tilde{X}_n)\in\mathscr{R}_n(F_1,\dots,F_n)$$

is comonotonic, then

$$X_1+\dots+X_n\le_{cx}\tilde{X}_1+\dots+\tilde{X}_n$$

for any $(X_1,\dots,X_n)\in\mathscr{R}_n(F_1,\dots,F_n)$.

3 Main Results

The main results of this paper are the following theorem, proposition and lemmas.

3.1 Policy Limits and Deductible

If the sum of policy limits or the sum of deductible is fixed, then $X_i\le_{st}X_j$ implies that $l_i^*\le l_j^*$ and $d_i^*\ge d_j^*$ when $(X_1,X_2,\dots,X_n)$ is comonotonic, where l_i^* and d_i^* are the optimal policy limit and the optimal deductible allocated to i-th risk.

In this section we present the problem of the optimal allocation of policy limits and deductible, where the effects of loss severity and loss frequency are considered separately. For make the new model analytically tractable, we will make the following assumptions :

1. the policyholder is risk-averse, and therefore the utility function is increasing and concave;
2. the random vector $\mathbf{X}=(X_1,\dots,X_n)$, which represents the loss severities, and random vector $\mathbf{Y}=(Y_1,\dots,Y_n)$, which represents the time of occurrence of losses, are independent; moreover, $Y_1,\dots,Y_n$ are mutually independent;
3. dependence structure of the severities of the risks is unknown.

*Remark.*Assumption 3 means that while the marginal distributions of $X_1,\dots,X_n$ are assumed to be known to the policyholder, the joint distribution is not.

3.1.1 Policy limits with unknown dependent structures

The first problem to be considered is to maximize the expected utility of wealth:

$$\max_{\mathbf{l}\in\mathscr{A}_n(l)}\min_{\mathbf{X}\in\mathscr{R}}\mathbb{E}\left[u\left(w-\sum_{i=1}^{n}[X_i-(X_i\wedge l_i)]\,f(Y_i)\right)\right]$$

where u and w are the utility function (increasing and concave), the wealth (after premium) respectively and $\tilde{u}$ is an increasing convex function. The problem is equivalent to

$$\min_{\mathbf{l}\in\mathscr{A}_n(l)}\max_{\mathbf{X}\in\mathscr{R}}\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}(X_i-l_i)_+\,f(Y_i)\right)\right]$$

Lemma 7.*If $(\tilde{X}_1,\dots,\tilde{X}_n)\in\mathscr{R}$ is comonotonic, then*

$$\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}(X_i-l_i)_+\,f(Y_i)\right)\right]\le\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}(\tilde{X}_i-l_i)_+\,f(Y_i)\right)\right]$$

for any $(l_1,\dots,l_n)\in\mathscr{A}_n(l)$ and $(X_1,\dots,X_n)\in\mathscr{R}$ independent of $\mathbf{Y}$.

*Proof.*Let $\tilde{X}=(\tilde{X}_1,\dots,\tilde{X}_n)\in\mathscr{R}$ be comonotonic and independent of $\mathbf{Y}$. For any fixed constants $y_1,\dots,y_n$, Lemma 5 implies that

$$\left((\tilde{X}_1-l_1)_+\,f(y_1)\,,\dots,\,(\tilde{X}_n-l_n)_+\,f(y_n)\right)$$

is still comonotonic. Therefore, by Lemma 6 and Theorem 1, we have

$$\sum_{i=1}^{n}(X_i-l_i)_+ f(y_i)\leq_{cx}\sum_{i=1}^{n}\left(\tilde{X}_i-l_i\right)_+ f(y_i)$$

and hence

$$\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}(X_i-l_i)_+ f(y_i)\right)\right]\leq\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}\left(\tilde{X}_i-l_i\right)_+ f(y_i)\right)\right]$$

because $\tilde{u}$ is increasing and convex. Then by the independence of $\mathbf{X}$ and $\mathbf{Y}$,

$$\begin{aligned}&\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}(X_i-l_i)_+ f(Y_i)\right)\right]\\&=\mathbb{E}\left[\mathbb{E}\left\{\tilde{u}\left(\sum_{i=1}^{n}(X_i-l_i)_+ f(Y_i)\right)\mid Y_1,\dots,Y_n\right\}\right]\\&\leq\mathbb{E}\left[\mathbb{E}\left\{\tilde{u}\left(\sum_{i=1}^{n}\left(\tilde{X}_i-l_i\right)_+ f(Y_i)\right)\mid Y_1,\dots,Y_n\right\}\right]\\&=\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}\left(\tilde{X}_i-l_i\right)_+ f(Y_i)\right)\right].\end{aligned}$$

Now, the initial problem becomes
Problem L' : $\left\{\min_{\mathbf{l}\in\mathscr{A}_n(l)}\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}(X_i-l_i)_+ f(Y_i)\right)\right]\right.$

Proposition 2.*Let* $\mathbf{l}^*=(l_1^*,\dots,l_n^*)$ *be the solution to Problem L', then*

$$Y_i\geq_{lr}Y_j, X_i\leq_{st}X_j\Rightarrow l_i^*\leq l_j^*.$$

*Proof.*Assume that $l_i\leq l_j$. Since $x\to f(Y_i)$ is decreasing , by property 13

$$Y_i\geq_{lr}Y_j\ \Rightarrow f(Y_i)\leq_{lr}f(Y_j)$$

Since (X_i,X_j) is comonotonic and $X_i\leq_{st}X_j$, $X_i(\omega)\leq X_j(\omega)$ for any $\omega\in\Omega$. By the independence of $\mathbf{X}$ and $\mathbf{Y}$, we can hereafter fix an outcome of $(X_1,\dots,X_i,\dots,X_j,\dots,X_n)$ as $(x_1,\dots,x_i,\dots,x_j,\dots,x_n)$ with $x_i\leq x_j$. As $g(\mathbf{x},\mathbf{l})=-\sum_{i=1}^{n}(x_i-l_i)_+$ is an *AI* function by Lemma 2 and the function $(x,l)\to-(x-l)_+$ is increasing in l but decreasing in x, then by Lemma 4

$$((x_i-l_i)_+,(x_j-l_j)_+)\prec\prec((x_i-l_j)_+,(x_j-l_i)_+)$$

Since we also have $(x_i-l_j)_+\leq(x_j-l_i)_+$, then by property 14 we have

$$\begin{aligned}&(x_i-l_i)_+f(Y_i)+(x_j-l_j)_+f(Y_j)\\&\leq_{icx}(x_i-l_j)_+f(Y_i)+(x_j-l_i)_+f(Y_j).\end{aligned}$$

By independence convolution and for the increasing convex function $\tilde{u}$, we have

$$\begin{aligned}&\mathbb{E}(\tilde{u}((x_i-l_i)_+f(Y_i)+(x_j-l_j)_+f(Y_j))\\&+\sum_{k\neq i,j}(x_k-l_k)_+f(Y_k)))\\&\leq\mathbb{E}(\tilde{u}((x_i-l_j)_+f(Y_i)+(x_j-l_i)_+f(Y_j))\\&+\sum_{k\neq i,j}(x_k-l_k)_+f(Y_k))).\end{aligned}$$

By taking expectations conditional on $\mathbf{X}$, we obtain
$\mathbb{E}(\tilde{u}((X_i-l_i)_+f(Y_i)+(X_j-l_j)_+f(Y_j))+\sum_{k\neq i,j}(X_k-l_k)_+f(Y_k)))$
$\leq\mathbb{E}(\tilde{u}((X_i-l_j)_+f(Y_i)+(X_j-l_i)_+f(Y_j))+\sum_{k\neq i,j}(X_k-l_k)_+f(Y_k)))$. The result follows.

3.1.2 Policy deductible with unknown dependent structures

Similar to the study of policy limits, now we consider the problem of the optimal allocation of deductible :

$$\max_{\mathbf{d}\in\mathscr{A}_n(d)}\min_{\mathbf{X}\in\mathscr{R}}\mathbb{E}\left[u\left(w-\sum_{i=1}^{n}\left[X_i-(X_i-d_i)_+\right]f(Y_i)\right)\right]$$

which is equivalent to

$$\min_{\mathbf{d}\in\mathscr{A}_n(d)}\max_{\mathbf{X}\in\mathscr{R}}\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}(X_i\wedge d_i)_+ f(Y_i)\right)\right]$$

Lemma 8.*If* $(\tilde{X}_1,\dots,\tilde{X}_n)\in\mathscr{R}$ *is comonotonic and independent of* $\mathbf{Y}$, *then*

$$\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}(X_i\wedge d_i)_+ f(Y_i)\right)\right]\leq\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}\left(\tilde{X}_i\wedge d_i\right)_+ f(Y_i)\right)\right]$$

for any $(d_1,\dots,d_n)\in\mathscr{A}_n(d)$ *and* $(X_1,\dots,X_n)\in\mathscr{R}$ *independent of* $\mathbf{Y}$.

*Proof.*The proof is omitted because it is very similar to the proof of Lemma 7.

From the above lemma, our problem becomes
Problem D' : $\left\{\min_{\mathbf{d}\in\mathscr{A}_n(d)}\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n}(X_i\wedge d_i)_+ f(Y_i)\right)\right]\right.$

Proposition 3.*Let* $\mathbf{d}^*=(d_1^*,\dots,d_n^*)$ *be the solution to Problem D', then*

$$Y_i\geq_{lr}Y_j, X_i\leq_{st}X_j\Rightarrow d_i^*\geq d_j^*.$$

*Proof.*Assume that $d_i\geq d_j$.As in the proof of Proposition 2, we have

$$Y_i\geq_{lr}Y_j\Rightarrow f(Y_i)\leq_{lr}f(Y_j),$$

and we can fix an outcome of $(X_1,\dots,X_i,\dots,X_j,\dots,X_n)$ as $(x_1,\dots,x_i,\dots,x_j,\dots,x_n)$ with $x_i\leq x_j$. As $g(\mathbf{x},\mathbf{d})=\sum_{i=1}^{n}(x_i\wedge d_i)$ is an *AI* function by Lemma 2 and the function $(x,d)\to x\wedge d$ is increasing both in x and d, then by Lemma 3,

$$((x_i\wedge d_i),(x_j\wedge d_j))\prec\prec((x_i\wedge d_j),(x_j\wedge d_i)).$$

Since we also have $(x_i \wedge d_j) \le (x_j \wedge d_i)$, then by property 14 we have

$$(x_i \wedge d_i) f(Y_i) + (x_j \wedge d_j) f(Y_j) \\ \le_{icx} (x_i \wedge d_j) f(Y_i) + (x_j \wedge d_i) f(Y_j)$$

By independence convolution, we have

$$(x_i \wedge d_i) f(Y_i) + (x_j \wedge d_j) f(Y_j) + \sum_{k \ne i,j} (X_k \wedge d_k)_+ f(Y_k) \\ \le_{icx} (x_i \wedge d_j) f(Y_i) + (x_j \wedge d_i) f(Y_j) + \sum_{k \ne i,j} (X_k \wedge d_k)_+ f(Y_k)$$

Therefore, the increasing convex function $\tilde{u}$,
$\mathbb{E}((\tilde{u}(x_i \wedge d_i) f(Y_i) + (x_j \wedge d_j) f(Y_j) + \sum_{k \ne i,j} (X_k \wedge d_k)_+ f(Y_k)))$
$\le$
$\mathbb{E}((\tilde{u}(x_i \wedge d_j) f(Y_i) + (x_j \wedge d_i) f(Y_j) + \sum_{k \ne i,j} (X_k \wedge d_k)_+ f(Y_k)))$.

3.2 Convex bounds for S_N

In risk theory and finance, one is often interested in distribution of the sums $S = X_1 + ... + X_n$ or the form $S_N = X_1 f(Y_1) + X_2 f(Y_2) + ... + X_n f(Y_n)$(our model) of individual risks of a portfolio $\mathbf{X}$. In this subsection we give a short overview of these stochastic ordering results. For proofs and more details on the presented results, we refer to the overview paper of Dhaene et al. [2] and Zeghdoudi and Remita [13].

Theorem 1.*We note that:*

$$\tilde{S}_N = \tilde{X}_1 f(Y_1) + \tilde{X}_2 f(Y_2) + ... + \tilde{X}_n f(Y_n).$$

For any random vector $X = (X_1, ..., X_n)$ and $f(Y_i), i = 1, ..n$

we have

$$S_N \le_{cx} \tilde{S}_N$$

*Proof.*It is suffices to prove stop-loss order because $\mathbb{E}(S_N) = \mathbb{E}(\tilde{S}_N)$. Hence, we have to prove that

$$\mathbb{E}[(S_N - d)_+] \le \mathbb{E}[(\tilde{S}_N - d)_+]$$

The following holds for all $(X_1 f(Y_1), X_2 f(Y_2), ..., X_n f(Y_n))$ when $d_1 + d_2 + ... + d_n = d$

$$\begin{aligned} & (X_1 f(Y_1) + ... + X_n f(Y_n) - d)_+ \\ = & (X_1 f(Y_1) - d_1 + ... + X_n f(Y_n) - d_n)_+ \\ \le & \left((X_1 f(Y_1) - d_1) + ... + (X_n f(Y_n) - d_n)_+\right)_+ \\ = & (X_1 f(Y_1) - d_1)_+ + ... + (X_n f(Y_n) - d_n)_+ \end{aligned} \quad (1)$$

Now taking expectations, we get that

$$\mathbb{E}\left[(X_1 f(Y_1) + ... + X_n f(Y_n) - d)_+\right] \le \Sigma_{i=1}^n \mathbb{E}\left[(X_i f(Y_i) - d_i)_+\right]$$

According to [3] we have

$$\mathbb{E}[(\tilde{S}_N - d)_+] = \sum_{i=1}^n \mathbb{E}\left[(X_i f(Y_i) - d_i)_+\right]$$

Then,

$$S_N \le_{cx} \tilde{S}_N.$$

Proposition 4.*For any random vector $X = (X_1, ..., X_n)$, any random variable Λ and for $U \backsim Uniform(0,1)$, which is assumed to be a function of X and for $f(Y_i) \ge 1, i = 1, ..., n$, we have,*

(a)

$$S \le_{cx} S_N$$

(b)

$$\tilde{S} \le_{cx} \tilde{S}_N$$

(c)

$$\sum_{i=1}^n \mathbb{E}[X_i \mid \Lambda] \le_{cx} S_N$$

(d)

$$\sum_{i=1}^n \mathbb{E}\left[\tilde{X}_i \mid \Lambda\right] \le_{cx} \tilde{S}_N$$

Proof.(**a**) We have $f(Y_i) \ge 1, i = 1, ..n$ and we used property 10 and 6, we obtain

$$X_1 + X_2 + ... + X_n \le_{cx} X_1 f(Y_1) + X_2 f(Y_2) + ... + X_n f(Y_n)$$

thus

$$S \le_{cx} S_N$$

(**b**) We will omit the proof here because the idea is very similar to the proof in (**a**).
(**c**) According to Dhaene et al. [2] we have, $\sum_{i=1}^n \mathbb{E}[X_i \mid \Lambda] \le_{cx} S$ and (a), we deduce that

$$\sum_{i=1}^n \mathbb{E}[X_i \mid \Lambda] \le_{cx} S_N$$

(**d**) According to Zeghdoudi and Remita [13] we have $\sum_{i=1}^n \mathbb{E}\left[\tilde{X}_i \mid \Lambda\right] \le_{cx} \tilde{S}$, using property 12 and (**b**), we obtain

$$\sum_{i=1}^n \mathbb{E}\left[\tilde{X}_i \mid \Lambda\right] \le_{cx} \tilde{S}_N.$$

In addition, if $f(Y_i) \le 1, i = 1, ..n$, we can check easily that

$$S_N \le_{cx} \tilde{S}_N \le_{cx} S \le_{cx} \tilde{S}.$$

4 Some examples and application

In this section we will describe several examples that show how distribution function of the sum of random variables can be approximated by convex order of random variable (see *Rüschendorf* [8]) for lower convex order of random variables and comparison of two families of copulas.

4.1 Individual and collective risk model

The classical individual and collective model of risk theory has the form $X_{Ind} = \sum_{i=1}^{n} b_i I_i,\ X_{Coll} = \sum_{i=1}^{n} b_i N_i$, where

$$I_i \backsim Bernoulli(p_i) \text{ and } N_i \backsim poisson(\lambda_i).$$

With probability p_i contract i will yield a claim of size $b_i \geq 0$ for any of the n policies. As an application of stochastic and stop loss ordering we get that the collective risk model X_{Coll} leads to an overestimate of the risks and, therefore, also to an increase of the corresponding risk premiums for the whole portfolio $X_{Ind} \leq_{sl(cx)} X_{Coll}$.

4.2 Reinsurance contracts

We consider reinsurance contracts $I(X)$ for a risk X, where $0 \leq I(X) \leq X$ is the reinsured part of the risk X and $X - I(X)$ is the retained risk of the insurer. Consider the stop loss reinsurance contract $I_a(X) = (X-a)_+$, where a is chosen such that $EI_a(X) = EI(X)$. Then for any reinsurance contract $I(X)$

$$X - I_a(X) \leq_{sl(cx)} X - I(X).$$

4.3 Dependent portfolios increase risk

Let $Y_i = \sum_{i=1}^{m} \alpha_i X_i$, where α_i and $X_i \backsim Bernoulli$ with $\sum_{i=1}^{m} \alpha_i = 1$,then $Y_i \backsim Bernoulli$. It is interesting to compare the total risk $T_n = \sum_{i=1}^{n} Y_i$ in the mixed model (X_i) with the total risk $S_n = \sum_{i=1}^{n} W_i$ in an independent portfolio model (W_i), where $W_i \backsim Bernoulli$ are distributed identical to X_i. Then we obtain

$$S_n \leq_{sl(cx)} T_n.$$

4.4 Comparison of two families of copulas

Definition 8(copulas). *$C(u_1,...,u_n)$ is distribution function whose marginal are all uniformly distributed (see Nelson [10]).*

Now we consider two risks X and Y with given survival functions $\bar{F}$ and $\bar{G}$. A sufficient condition of the stop-loss order is given by:

Cut-criterion (Karlin and Novikoff [7]): Let X and Y be two risks with $\mathbb{E}[X] \leq \mathbb{E}[Y]$. If there exists a constant c such that

$$\begin{cases} \bar{F}(x) \geq \bar{G}(x) \text{ for all } x < c, \\ \bar{F}(x) \leq \bar{G}(x) \text{ for all } x \geq c, \end{cases}$$

then

$$X \preceq_{st} Y$$

Definition 9(Bivariate orthant convex order). *Given non-negative random vectors $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$. We say that $\mathbf{X}$ is smaller than $\mathbf{Y}$ in the orthant convex order denoted as $\mathbf{X} \preceq_{uo-cx} \mathbf{Y}$ if the inequalities*

$$\mathbb{E}[v_1(X_1)v_2(X_2)] \leq \mathbb{E}[v_1(Y_1)v_2(Y_2)]$$

holds for all non-decreasing convex function $v_1$and v_2 .

Characterization: $\mathbf{X} \preceq_{uo-cx} \mathbf{Y}$ if and only, if

1.$\mathbb{E}[(X_i-d_i)_+] \leq \mathbb{E}[(Y_i-d_i)_+]$ for all $d_i > 0,\ i = 1,2$
2.$\mathbb{E}[(X_1-d_1)_+(X_2-d_2)_+] \leq \mathbb{E}[(Y_1-d_1)_+(Y_2-d_2)_+]$ for all $d_1, d_2 > 0$.

Consequently:

$$\mathbf{X} \preceq_{uo-cx} \mathbf{Y} \Rightarrow X_i \preceq_{st} Y_i, i = 1,2$$

This shows that $\preceq_{uo-cx}$can be viewed as bivariate extension of stop-loss order.

Crossing condition for the bivariate orthant convex order:

Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be non-negative random vectors with survival functions $\bar{F}$ and $\bar{G}$. Let h be a level curve defined by

$$\bar{F}(x, h(x)) - \bar{G}(x, h(x)) = 0,\ x \geq 0.$$

Let

$$C = \left\{(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+ : y \leq h(x)\right\}$$

we denote by $\bar{C}$ the complement of C in $\mathbb{R}^+ \times \mathbb{R}^+$.

*Remark.*The concordance order is used to compare members of a given copula family C_θ when the dependence parameter varies:

$$\theta_1 \leq \theta_2 \Rightarrow C_{\theta_1} \preceq_C C_{\theta_2}$$

In general, there is no comparison between a copulas from different families with $\preceq_C$:

$$C_{\theta_1} \npreceq_C C_{\theta_2} \text{ and } C_{\theta_2} \npreceq_C C_{\theta_1}$$

*Example 1.*Let C_{θ_1} be a Clayton copula with parameter $\theta_1 = 1$ and C_{θ_2} be a Frank copula with parameter $\theta_2 = 2$. Since $\preceq_{uo-cx}$ is weaker than $\preceq_C$. Thus one can expect to rank the copulas C_{θ_1} and C_{θ_2} with respect to $\preceq_{uo-cx}$ instead of $\preceq_C$. Therefore, one can use our cut-criterion to establish a such comparison with respect $\preceq_{uo-cx}$. To this end, we can see that $C_{\theta_1} \preceq_{uo-cx} C_{\theta_2}$.This means that the upper orthant convex order can be more convenient for compare the concordance between two different families of copulas.

5 Conclusions

In this work, we give an extensive bibliographic overview of the developments of the theory of stochastic orderings, comonotonicity and their applications. Also, we present the problems of optimal allocation of policy limits and deductibles are studied. By applying the bivariate characterizations of stochastic ordering relations, we reconsider the general model and derive some new refined results on orderings of optimal allocations of policy limits and deductibles from the viewpoint of the policyholder. In addition, when the severity and the chance of the loss are both larger, a larger policy limit and a smaller policy deductible will be allocated to that risk by a risk-averse policyholder. Moreover, we obtain an convex upper and lower bound in terms of comonotonic portfolios for $S_N = X_1 f(Y_1) + X_2 f(Y_2) + ... + X_n f(Y_n)$ (our model). For future studies, we may try to explore the following directions. First, we can relax the condition imposed on $f(Y_i)$ and introduce financial risks to the model. Second, we can remakes same work for obtain the optimal allocation of policy limits and deductibles in a model with mixture and discount factors.

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