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An Effective Approach for solving MHD Viscous Flow Due to A Shrinking Sheet

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Abstract: In this paper, we present an effective technique combined between homotopy analysis method and traditional Padé approximation so-called (HAM Padé), the technique to obtain the analytic approximation solution of a certain type of nonlinear boundary value problem with one boundary condition at infinity. The analytic series solution obtained from the homotopy analysis method and the Padé diagonal approximation to handle the boundary condition at infinity. This technique apply to the boundary value problem resulting from the magnetohydrodynamic (MHD) viscous flow due to a shrinking sheet. The proposed technique success to obtain the two branches of solutions for important parameter. Comparison of the present solution is made with the existing solution and excellent agreement is noted.

Keywords: Homotopy analysis method, Padé approximation, $\hbar$-curves, MHD viscous flow, Nonlinear boundary value problems.

1 Introduction

Nonlinear phenomena, that appear in many areas of scientific fields such as solid state physics, plasma physics, fuid mechanics, population models and chemical kinetics, can be modelled by nonlinear ordinary differential equations. In particular, the nonlinear ordinary differential equations with one boundary condition at infinity, that will be examined here, is of much interest. In most cases, analytic solutions of these differential equations are very difficult to achieve, so these equations may be approximated using semi-analytical techniques and its modifications such as differential transform method (DTM)[1,2], homotopy perturbation method (HPM)[3], Adomian's decomposition method(ADM)[4, 5], variational iteration method (VIM)[6]. However, these methods cannot provide us with a simple way to adjust and control the convergence region and rate of giving approximate series. Therefore, in this work , we used the homotopy analysis method, first developed by Liao [7] for general nonlinear problems, In recent years, this method has been successfully employed to solve many types of nonlinear problems in science and engineering such as [8]-[14]. The proposed technique obtained the series solution by the homotopy analysis method, but the series solution containing the unknown parameter α, therefore, applying the final value theorem for Laplace transform on the boundary condition at infinity and applying this condition to the corresponding Padé approximate $[t/t]$ on the series solution to find the unknown parameter. It is worth mentioning that the proposed scheme is an elegant recipe of HAM and Padé approximation [15]. The advantage of proposed idea is its capability of combining two powerful techniques for obtaining fast convergent series for nonlinear equations.

Boundary layer behaviour over a moving continuous solid surface is a relevant type of flow which is present in many industrial processes such as manufacture and drawing of plastics and rubber sheets, processing of sheet-like materials in paper production, cooling of metallic sheets and manufacture of metal and polymer solid cylinders [16] and crystal growing just to name a few. In this paper, the proposed technique is used to solve the magnetohydrodynamic (MHD) viscous flow due to a shrinking sheet [17], Wang [18] was the first to study the unsteady viscous flow induced by a shrinking film. The proof of the existence and (non) uniqueness, the exact solutions, both numerical and in closed form, are given by Miklavcic and Wang [19] for the steady viscous hydrodynamic flow due to a shrinking sheet for a specific value of the suction parameter. Miklavcic and Wang [19] concluded that the solution for shrinking sheets may not

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be unique at certain suction rates for both two-dimensional and axisymmetric flows. Sajid and Hayat [17] studied the magnetohydrodynamic (MHD) viscous flow due to shrinking sheet for the cases of two-dimensional and axisymmetric shrinking by traditional homotopy analysis method (HAM), but in this study solving the problem by present technique (HAM Padé) for these cases, shows the results obtained by HAM Padé good agreement compared to exact solution [20] and other methods. Also the proposed technique has succeeded to pridect and obtain the two branches of solutions for important parameters.

2 The homotopy analysis method coupled with Padé approximations(HAM Padé)

Consider the nonlinear ordinary differential equation in unbounded domain:

$$f^{(r)}(\eta)+g\left(f,f',\ldots,f^{(r)}\right)-y(\eta)=0 \qquad (1)$$

With boundary conditions

$$\left.\frac{d^i f(\eta)}{dx^i}\right|_{x=0}=A_i,\ f'(\eta)|_\infty=B, i=0,1,2,\ldots,(r-2) \quad (2)$$

$g\left(f,f',\ldots,f^{(r)}\right)$ is the nonlinear function, and $y(\eta)$ is the non-homogeneous term. The equation (1) can be write in the form

$$\check{N}[f(\eta)]=0, \qquad (3)$$

where $\check{N}$ is a nonlinear operator, η denote independent variable, and $f(\eta)$ is an unknown function. The first step in this technique adds the new condition $f^{(r-1)}(0)=\alpha$, where α is unknown parameter and will determine later, we apply the traditional homotopy analysis method on the problem

$$\check{N}[f(\eta)]=0, \qquad (4)$$

$$\left.\frac{d^i f(\eta)}{dx^i}\right|_{x=0}=A_i, f^{(r-1)}(0)=\alpha \qquad (5)$$

the general zero-order deformation equation and the corresponding boundary conditions are as follows

$$(1-q)L[\phi(\eta,\alpha,q)-f_0(\eta,\alpha)]=$$
$$q\hbar H(\eta)\left(\check{N}[\phi(\eta,\alpha,q)]\right), \quad (6)$$

where $q\in[0,1]$ denote the so-called embedding parameter. $\hbar\neq 0$ is an auxiliary parameter, L denotes is an auxiliary linear operator, $\phi(\eta,\alpha,q)$ is an unknown function, $f_0(\eta,\alpha)$ is an initial guess of $f(\eta,\alpha)$ and $H(\eta)$ denotes a non-zero auxiliary function. It is obvious that when the embedding parameter $q=0$ and $q=1$, equation (6) becomes

$$\phi(\eta,\alpha,0)=f_0(\eta,\alpha),\ \phi(\eta,\alpha,1)=f(\eta,\alpha) \qquad (7)$$

respectively. Thus as q increases from 0 to 1, the solution $\phi(\eta,\alpha,q)$ varies from the initial guess $f_0(\eta,\alpha)$ to the solution $f(\eta,\alpha)$. Expanding $\phi(\eta,\alpha,q)$ in the Taylor series with respect to q, one has

$$\phi(\eta,\alpha,q)=f_0(\eta,\alpha)+\sum_{n=1}^{+\infty}f_n(\eta,\alpha)q^n, \qquad (8)$$

where

$$f_n(\eta,\alpha)=\left.\frac{1}{n!}\frac{\partial^n\phi(\eta,\alpha,q)}{\partial q^n}\right|_{q=0} \qquad (9)$$

The initial guess $f_0(\eta,\alpha)$ of the solution $f(\eta,\alpha)$ can be determined by the rule of solution expression as follows. expressed by a set of base functions

$$\left\{(\eta)^k \middle| k=0,1,2,3\ldots\right\}, \qquad (10)$$

in the form

$$f(\eta,\alpha)=\sum_{k=0}^{+\infty}f_k(\alpha)\left(\eta\right)^k \qquad (11)$$

The initial guess $f_0(\eta,\alpha)$ can be chosen from the equation (11) so that it achieves the boundary condition(5). The second goal is to determine the higher order terms $f_n(\eta,\alpha)(n,1,2,\ldots.)$. Define the vector

$$\vec{u}_i(\eta)=\{u_0(\eta,\alpha),u_1(\eta,\alpha),\ldots,u_i(\eta,\alpha)\} \qquad (12)$$

Differentiating equation (6) n times with respect to the embedding parameter q and then setting $q=0$ and finally dividing them by $n!$, we have the so-called nth-order deformation equation:

$$L[f_n(\eta,\alpha)-\chi_n f_{n-1}(\eta,\alpha)]=\hbar H(\eta)R\left(\vec{f}_{n-1}(\eta,\alpha)\right)(13)$$

where

$$R\left(\vec{f}_{n-1}(\eta,\alpha)\right)=\frac{1}{n-1!}\frac{\partial^{n-1}\left(\check{N}[\phi(\eta,\alpha,q)]\right)}{\partial q^{n-1}}|_{q=0}, \quad (14)$$

and

$$\chi_n=\begin{cases}0 & \text{when } n\leq 1\\ 1 & \text{otherwise}\end{cases} \qquad (15)$$

Now the solution of the nth-order deformation equation(13) for $n\geq 1$ when $H(\eta)=1$ becomes

$$f_n(\eta,\alpha)=\chi_n f_{n-1}(\eta,\alpha)+L^{-1}\left[\hbar R\left(\vec{f}_{n-1}(\eta,\alpha)\right)\right] \quad (16)$$

and its boundary conditions(5) become

$$\left.\frac{d^i f(\eta)}{dx^i}\right|_{x=0}=0,\ f^{(r-1)}(0)=0, \qquad (17)$$

can be easily solve the equations (16) and (17) by using some symbolic software programs such as Mathematica or Maple. In this way, starting by $f_0(\eta,\alpha)$, we obtain the functions $f_n(\eta,\alpha)$ for $n,1,2,3,\ldots.$ form equations (16,17) successively. Accordingly, the $N-th$ order of approximate solution of the problem (4) and (5) is given by

$$f(\eta,\alpha) \cong F_N(\eta,\alpha) = \sum_{n=0}^{N} f_n(\eta,\alpha) \tag{18}$$

The auxiliary parameter $\hbar$ can be employed to adjust the convergence region $(R_{\hbar})$ of the series in the traditional homotopy analysis solution. By means of the so-called $\hbar$-curve, it is straightforward to choose an appropriate range for $\hbar$ which ensures the convergence of the solution series. As pointed out by Liao[21], the appropriate region for $\hbar$ is a horizontal line segment, but the series solution(18) obtained by HAM Padé technique containing the unknown parameter α, to find the convergence region $(R_{\hbar\alpha})$ of this series, we can write $f^{(\beta)}(0,\hbar,\alpha)$ from equation (18), as follows

$$\begin{aligned} f^{(\beta)}(0,\hbar,\alpha) &\cong F_N^{(\beta)}(0,\hbar,\alpha) \\ &= \sum_{i=0}^{\gamma} g_{i\beta}(\hbar)\alpha^i,\ \beta = r, r+1, r+2, \ldots \end{aligned} \tag{19}$$

The auxiliary parameter $\hbar$ can be employed to adjust the convergence region of the series in the HAM Padé technique solution(18), by plot $g_{ir}(\hbar), i(0,1,..,\gamma)$ against to the convergence controlling auxiliary parameter $\hbar$, the appropriate region for $\hbar$ is a horizontal line segment the convergence regions of each function $g_{ir}(\hbar)$, but the convergence region $(R_{\hbar\alpha})$ of the series(18) discover by the intersection between the convergence regions to $g_{ir}(\hbar)$ functions, in general, the convergence region $(R_{\hbar\alpha})$ is the intersection between the convergence regions to $g_{i\beta}(\hbar)$ functions. Some problems don't need to do so, if $f^{(r)}(0,\hbar,\alpha) = g(h)f(\alpha)$ then the convergence region not depend on the unknown parameter α, by plotting g(h), we get the convergence region$(R_{\hbar\alpha})$. The advantage of the proposed scheme to find the convergence region of the series(18) not depend on the unknown parameter α.

Theorem 2.1.[22]. If the function $f(\eta)$ is bounded for all $\eta > 0 : |f(\eta)| < M$ and the $\lim_{\eta\to\infty} f(\eta) = f(\infty)$ exists, then $\lim_{S\to 0} SF(S) = f(\infty)$, where $F(S)$ Laplace transform of $f(\eta)$.
Obviously in equation (18) the function $f(\eta,\alpha)$ Containing the unknown parameter α. Using Theorem 2.1, the condition $f'(\infty) = B$ leads to the condition $\lim_{S\to 0} S\tilde{F}(S) = B$, where $\tilde{F}(S)$ Laplace transform of $f'(\eta)$. Applying this condition to the corresponding Padé approximate, we can use some symbolic software programs such as Mathematica or Maple to perform this task

3 Application

3.1 Problem Statement

In Cartesian coordinates the continuity and momentum equations for MHD viscous flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{20}$$

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\sigma {B_0}^2}{\rho}u + \\ \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \end{aligned} \tag{21}$$

$$\begin{aligned} u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{\sigma {B_0}^2}{\rho}v + \\ \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \end{aligned} \tag{22}$$

$$\begin{aligned} u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \\ \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \end{aligned} \tag{23}$$

where $\nu = \mu/\rho$ is the kinematic viscosity σ is the electrical conductivity. We have applied the magnetic field B_0 in the z-direction and the induced magnetic fields neglected. The above equations are derived by considering the zero electric field and incorporating the small magnetic Reynold number assumption. The boundary conditions applicable to the present flow are

$$\begin{aligned} u = -ax,\ v = -a(m-1)y,\ w = -W,\ at\ y = 0, \\ u \to 0\ as\ y \to \infty \end{aligned} \tag{24}$$

In which $a > 0$ is shrinking constant,W is the suction velocity. $m = 1$ when the sheet shrinks in x-direction only and $m = 2$ when the sheet shrinks axisymmetrically. Introducing the following similarity transformations

$$\begin{aligned} u = axf'(\eta,\ v = a(m-1)yf'(\eta), \\ w = -\sqrt{a\nu}mf(\eta),\ \eta = \sqrt{\frac{a}{\nu}}z \end{aligned} \tag{25}$$

equation (20) is identically satisfied and equation (23) can be integrated to given

$$\frac{p}{\rho} = \nu\frac{\partial w}{\partial z} - \frac{w^2}{2} + constant \tag{26}$$

and equations (21), (22) and (24) reduces to the following boundary value problem[17]

$$f'''(\eta) - M^2 f'(\eta) - f'^2(\eta) + mf(\eta)f''(\eta) = 0, \tag{27}$$

$$f(0) = s, f'(0) = -1, f'(\infty) = 0, \tag{28}$$

where $s = W/m\sqrt{\nu a}$ and $M^2 = \sigma\, {B_0}^2/\rho a$.

3.2 The Method Implementation and Results

The new boundary condition according (5)

$$f(0) = s,\ f'(0) = -1,\ f''(0) = \alpha, \tag{29}$$

where $\alpha > 0$. We choose only initial guess $f_0(\eta,\alpha)$ according to initial condition (29) and equation (11)

$$f_0(\eta,\alpha) = s - \eta + \alpha\frac{\eta^2}{2}, \tag{30}$$

and choose the auxiliary linear operator

$$L[\phi(\eta,\alpha,p)] = \frac{\partial^3\phi(\eta,\alpha,p)}{\partial\eta^3}, \tag{31}$$

with the property

$$L\left[c_0 + c_1\eta + c_2\eta^2\right] = 0. \tag{32}$$

The solution of the nth-order deformation equation (16) for $n \geq 1$

$$\begin{aligned} f_n(\eta,\alpha) = \chi_n f_{n-1}(\eta,\alpha)\ &+\hbar\int\int\int f'''_{n-1} + \\ &\sum_{j=0}^{n-1}\left(m f_{n-1-j} f''_j - f'_{n-1-j} f'_j\right) - \\ &M^2 f'_{n-1} \mathrm{d}\eta\mathrm{d}\eta\mathrm{d}\eta + c_0 + c_1\eta + c_2\eta^2 \end{aligned} \tag{33}$$

c_0, c_1, c_2 are determined by the boundary condition equation

$$f(0) = 0,\ f'(0) = 0,\ f''(0) = 0 \tag{34}$$

Now given the solution equation (33) at $n = 1$

$$\begin{aligned} f_1(\eta,\alpha) = \ &\frac{1}{120}\hbar\left(-20 + 20M^2 + 20ms\alpha\right)\eta^3 \\ &+\frac{1}{120}\hbar\left(10\alpha - 5m\alpha - 5M^2\alpha\right)\eta^4 \\ &+\frac{1}{120}\hbar\left(-2\alpha^2 + m\alpha^2\right)\eta^5 \end{aligned} \tag{35}$$

$f_n(\eta)(n = 2,3,4,\ldots)$ can be calculated similarly, the approximate analytic solution is

$$f(\eta,\alpha) \cong F_N(\eta,\alpha) = \sum_{n=0}^{N} f_n(\eta,\alpha), \tag{36}$$

Hence $f^{(3)}(0,\hbar,\alpha)$ is

$$\begin{aligned} &f^{(3)}(0,\hbar,\alpha) \cong F^{(3)}{}_{10}(0,\hbar,\alpha) = g_{03}(\hbar) + g_{13}(\hbar)\alpha = \\ &\hbar\left(-1+M^2\right)\left(10 + 45\hbar + 120\hbar^2 + 210\hbar^3 + 252\hbar^4 + \right. \\ &\qquad\left. 210\hbar^5 + 120\hbar^6 + 45\hbar^7 + 10\hbar^8 + \hbar^9\right) + \\ &\hbar\mathrm{ms}\alpha\left(10 + 45\hbar + 120\hbar^2 + 210\hbar^3 + 252\hbar^4 + \right. \\ &\qquad\left. 210\hbar^5 + 120\hbar^6 + 45\hbar^7 + 10\hbar^8 + \hbar^9\right) \end{aligned} \tag{37}$$

Fig.1, $\hbar$-curves of $g_{03}(\hbar)$ and $g_{13}(\hbar)$ when $m = 2, s = 1$, $M = 2$,the convergence region ($R_{\hbar\alpha}$) of series (36) is the intersection between the convergence regions to $g_{03}(\hbar)$ and $g_{13}(\hbar)$, but from the equation (37) and Fig.1, find that the functions $g_{03}(\hbar)$ and $g_{13}(\hbar)$ in the same behavior but movable only on the vertical axis therefore this problem is a special case, can write

$$\begin{aligned} f^{(3)}(0,\hbar,\alpha) &= F^{(3)}{}_{10}(0,\hbar,\alpha) = g(\hbar)\left(-1 + M^2 + ms\alpha\right) \\ &= \hbar\left(10 + 45\hbar + 120\hbar^2 + 210\hbar^3 + 252\hbar^4 + 210\hbar^5 + \right. \\ &\quad\left. 120\hbar^6 + 45\hbar^7 + 10\hbar^8 + \hbar^9\right)\left(-1 + M^2 + ms\alpha\right) \end{aligned} \tag{38}$$

From equation(38), the function $f^{(3)}(0,\hbar,\alpha)$ is product of two functions, one only in $\hbar$ ($g(\hbar)$) and other in the parameters α, M, s and m, this means the convergence region of the series solution not depend on the parameters α, M, s and m, also the convergence region of the series solution constant region for all different values of parameters therefore, $f'''(0,\hbar,\alpha) = g(\hbar)$, we can plotting $g(\hbar)$, against to the convergence controlling auxiliary parameter $\hbar$, to find the convergence region of the solution series fig.2, the $\hbar$-curve of $g(\hbar)$ for $N-th$ order of approximation, it is easy to discover the valid region of $\hbar$ that corresponds to the line segment nearly parallel to the horizontal axis (constant $g(\hbar)$ value), from fig.2. the region of convergence($R_{\hbar\alpha}$) is $[-0.5, -1.2]$.

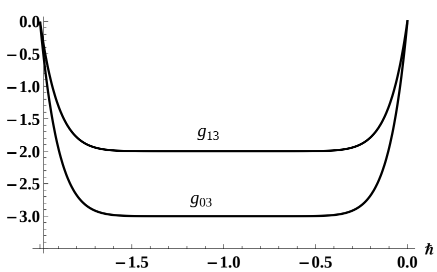

Fig. 1: $\hbar$-curves of $g_{03}(\hbar)$ and $g_{13}(\hbar)$ when $m = 2$, $s = 1$, $M = 2$ at $10-th$ order

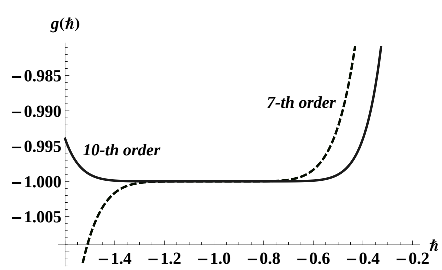

Fig. 2: $\hbar$-curve of $g(\hbar)$ in equation (38) for different $N-th$ order of approximation

From equation (19), $f^{(4)}(0,\hbar,\alpha)$ is

$$f^{(4)}(0,\hbar,\alpha) = {F^{(4)}}_{10}(0,\hbar,\alpha) = m\left(-1+M^2\right)s\,\hbar^2\left(45+240\hbar+630\hbar^2+1008\hbar^3+1050\hbar^4+720\hbar^5+315\hbar^6+80\hbar^7+9\hbar^8\right)+\left(m^2s^2\,\hbar^2\left(45+240\hbar+630\hbar^2+1008\hbar^3+1050\hbar^4+720\hbar^5+315\hbar^6+80\hbar^7+9\hbar^8\right)+\left(-m-2+M^2\right)\hbar(10+45\hbar+120\hbar^2+210\hbar^3+252\hbar^4+210\hbar^5+120\hbar^6+45\hbar^7+10\hbar^8+\hbar^9)\right)\alpha \qquad (39)$$

In spite of $f^{(3)}(0,\hbar,\alpha)$ in this problem is a special case, it is possible to clarify the idea of the proposed scheme to find a convergence region $((R_{\hbar\alpha})$, fig.3. $\hbar$-curves of $g_{04}(\hbar)$ and $g_{14}(\hbar)$ in equation (39) when $m=2, s=1$, $M=2$ at $10-th$ order, the convergence region the intersection between $g_{04}(\hbar)$ and $g_{14}(\hbar)$ almost the same region intersection between $g_{03}(\hbar)$ and $g_{13}(\hbar)$, with the assertion that the functions $g_{04}(\hbar)$ and $g_{14}(\hbar)$ do not have the same behavior.

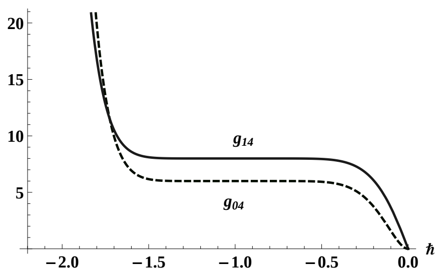

Fig. 3: $\hbar$-curves of $g_{04}(\hbar)$ and $g_{14}(\hbar)$ when $m=2,\ s=1,\ M=2$ at $10-th$ order

To find the unknown parameter $\alpha = f''(0)$. Using Theorem 2.1, the condition $f'(\infty) = 0$leads to the condition $\lim_{S\to 0} S\tilde{F}(S) = 0$, where $\tilde{F}(S)$ Laplace transform of $f'(\eta)$. Applying this condition to the corresponding Padé approximate $[t/t]$ for small values of t, we get the value of unknown parameter $\alpha = f''(0)$. Table 1, Shows comparison of the value of $\alpha = f''(0)$ obtained by the present method and the numerical solution [23], traditional homotopy analysis method(HAM)[17], homotopy Perturbation method coupled with Padé approximations(HPM- Padé)[24] and Adomian decomposition method coupled with Padé approximations(ADM-Padé)[25] when $m=2$, $s=1$, $M=2$ at $\hbar=-1$ and different $N-th$ order of approximation, table 1, also shows the good agreement between results obtained by the present method with other applied methods, in spite of the present method need only small values of t but other methods like[24,25] used large values of t and the results of the present method accurate than [24,25] to the numerical solution [23], table 1, also shows the value of α has not changed to increase the number of order approximation from $N=10$ to 15 order. Table 2. shows that the value of $\alpha = f''(0)$ for the diagonal approximate $[6/6]$ for different values of α when $m=2, s=1, M=2$. Table 2, shows when $\hbar \in [-0.7,-1.1]$ given the value of α six decimal places uncertain compared by the numerical solution[23], this means the solution obtained by the present method is more general as compared to homotopy perturbation method coupled with Padé approximations(HPM-Padé) [24] and Adomian decomposition nethod coupled with Padé approximations (ADM-Padé) [25].

Table 1: Comparison of the value of $\alpha = f''(0)$ obtained by the present method and the other methods when $m=2$, $s=1$, $M=2$ at $\hbar=-1$ and different $N-th$ order of approximation

	$N-th$ order	10	15
Present Method $\alpha = f''(0)$	[4/4]	2.89161032	2.89161032
	[5/5]	2.8916048	2.8916048
	[6/6]	2.8916046	2.8916046
	[7/7]	2.8916046	2.8916046
HPM-Padé[24]	[10/10]	2.89161	
ADM Padé[25]	[25/25]	2.89160	
HAM[17]		2.89160	
Numerical[23]		2.8916045	

Table 2: The value of α obtained by the present method for different values of $\hbar$ at $15-th$ order, when $s=1,\ m=2,\ M=2$

$\hbar$	−0.5	−0.7	−0.8	−1	−1.1	−1.2
α[6/6]	2.891559	2.8916049	2.8916046	2.8916046	2.8916048	2.891607
Numerical[23]	2.8916045					

For the special case of $m=1$, the exact solution of equation(27) is [20]

$$f(\eta) = s - \frac{2}{s\pm\sqrt{s^2-(4-4M^2)}} + \frac{2}{s\pm\sqrt{s^2-(4-4M^2)}}e^{\frac{s\pm\sqrt{s^2-\left(4-4M^2\right)}}{-2}\eta}, \qquad (40)$$

$$f''(0) = \frac{s\pm\sqrt{s^2-(4-4M^2)}}{2} \qquad (41)$$

In order to have a physical solution, $f''(0) > 0$ is required[20] so the solution exists only for s is positive with $s^2 \geq 4\left(1-M^2\right)$; also, when $M \geq 1$ we find only one solution for different values of s, however, for $0 < M < 1$, there are two solution branches, for example $M=0.5$ of equation (41) we find that the two solutions are at $s > \sqrt{3}$, at $s=\sqrt{3}$ find only one solution and $s < \sqrt{3}$ no solution. Table 3, shows a comparison of the present method multiple solutions for $\alpha = f''(0)$ against the exact solution and numerical solution when $m=1, M=0.5$ for different values of $s \geq \sqrt{3}$ at $\hbar = -1$, table [3], shows when $s > \sqrt{3}$ the results of the present method are the same of the exact solution but the numerical method [23] found only the upper solution, also from table3, the value of α obtained by the present method when $s = \sqrt{3}$ only one solution, the lower and upper solution to have the same value according to the exact solution(41), therefore the results clearly demonstrate the efficiency, accuracy and simplicity of the present method. We don't compare with [24,25] because the authors of [24,25] have not studied this case. The figs. 4 and 5 show the form of velocity

Table 3: Comparison of the present method multiple solutions for $\alpha = f''(0)$ against the exact solution[20] and numerical solution[23] when $m = 1$, $M = 0.5$ for different values of s at $\hbar = -1$

	Lower Solution		Upper Solution		
s	HAM Padé[6/6]	[20]	HAM Padé[6/6]	[23]	[20]
1.9	0.559488	0.559488	1.3405125	1.3405083	1.3405125
1.85	0.600000	0.600000	1.250000	1.250000	1.250000
1.8	0.655051	0.655051	1.1449489	1.1449427	1.1449489
1.75	0.750000	0.750000	1.0000000	1.0000000	1.0000000
1.73	0.8660254	0.8660254	0.8660254	-	0.8660254

profiles $f'(\eta)$ for different values of the parameters s and M. In these figures (a) corresponds to the two-dimensional shrinking($m = 1$) and (b) corresponds to axisymmetric shrinking ($m = 2$). It is shown in fig. 4 that the velocity increases by increases the suction parameter s for both two dimensional and axisymmetric shrinking. The effect of the Hartman number M on the velocity are similar to that of suction parameter and are shown in fig. 5.

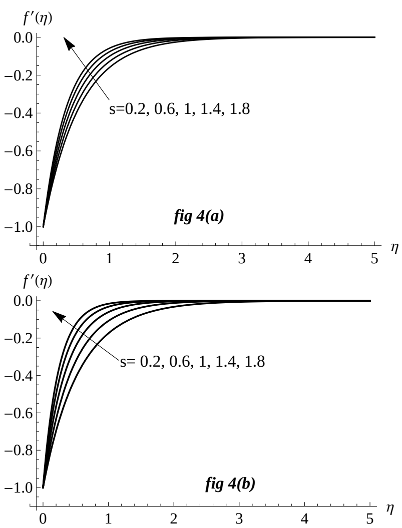

Fig. 4: The velocity profile obtained by present method when when $M = 2$ and several values of s (a) $m = 1$ (b) $m = 2$

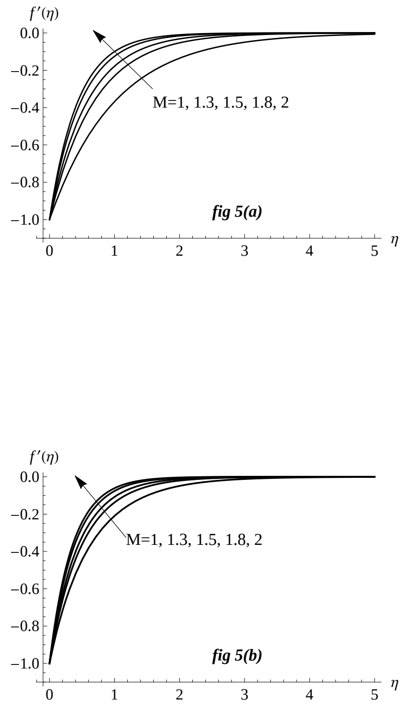

Fig. 5: The velocity profile obtained by present method when $s = 1$ and several values of M (a) $m = 1$ (b) $m = 2$

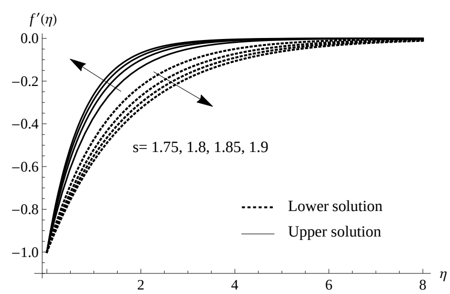

Fig. 6: The velocity profile (lower and upper) obtained by present method when $m = 1$, $M = 1/2$ and several values of s

Fig. 6 shows the multiple solutions obtained by present method for the two dimensional shrinking case when $M = 1/2$ and several values of s, from fig. 6, the lower solution profiles are generally further away from the wall compared with the upper solution branch even though the suction parameter s at the sheet is the same. Finally , as shown in fig.7. Comparison of $f(\eta)$ and $f'(\eta)$ obtained by the present method against the exact solution when $m = 1$, $M = 1/2$ and $s = \sqrt{3}$, it has been attempted to show the accuracy, capabilities of the present method.

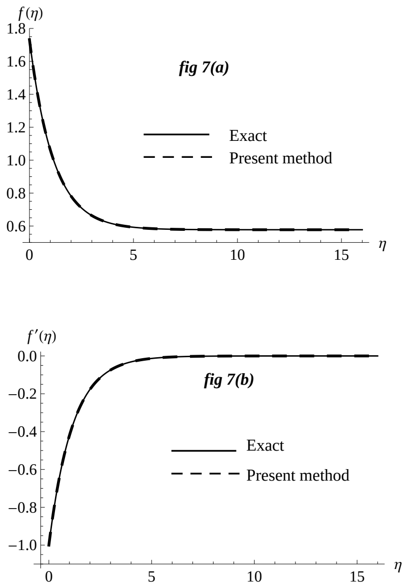

Fig. 7: Comparison of $f(\eta)$ and $f'(\eta)$ obtained by the present method against the exact solution when $m=1$, $M=1/2$ and $s=\sqrt{3}$.

4 Conclusion

The proposed technique combined between homotopy analysis method and traditional Padé approximation approach was presented for solving the nonlinear boundary value problem with one boundary condition at infinity. This technique succeeded in solving the MHD viscous flow due to a shrinking sheet. The figures 1, 2 and 3 and table 2, show that the proposed technique is more general as compared to some other methods such as HPM-Padé and ADM-Padé techniques. The results are in well agreement with the existing results and therefore elucidate the reliability and efficiency of the technique.

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