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More Results on the Upper Solution Bounds of the Continuous ARE

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Abstract: In this study, by constructing different equivalent forms of the continuous algebraic Riccati matrix equation (CARE) and using some linear algebraic techniques, we present the upper matrix bounds which depend on any positive definite matrix for the unique positive semidefinite solution of the CARE. Based on these bounds, we develope iterative algorithms to obtain more sharper solution bounds. Furthermore, we give numerical examples to demonstrate that the new bounds are tighter than previous results in some cases.

Keywords: Continuous algebraic Riccati matrix equation, iterative algorithm, matrix bound, matrix inequality.

1 Introduction

The algebraic Riccati and Lyapunov matrix equations are widely used and they play an important role in various of engineering such as control system design and analysis [12,13,19,23], and signal processing [32]. The continuous algebraic Riccati and Lyapunov matrix equations that we generally encounter in the literature are defined as below:

The continuous algebraic Riccati matrix equation (CARE) is

$$PA + A^T P - PBB^T P = -Q \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices, $Q \in \mathbb{R}^{n\times n}$ is a given positive semidefinite matrix, and the matrix $P \in \mathbb{R}^{n\times n}$ is the unique positive semidefinite solution of the CARE (1). When $B = 0$ and A is stable matrix, the CARE (1) becomes the continuous algebraic Lyapunov matrix equation (CALE)

$$A^T P + PA = -Q.$$

It is well known that the unique positive semidefinite solution P to the CARE (1) exists if the pair (A,B) is controllable (stabilizable) and the pair $\left(A, Q^{1/2}\right)$ is observable (detectable).

The characteristics and structures of these equations have considerable role in many areas of modern engineering such as optimal control [2,13], robust control [2,15], filter design [25], stability analysis [1,27] in control theory [6,13,22,23] including optimization stability theory. For example, consider the following linear system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ x(0) = 0 \end{cases} \tag{2}$$

For the continuous-time linear quadratic regulator (LQR) problem, suppose the pair (A,B) is stabilizable and the pair $\left(A, Q^{1/2}\right)$ is detectable, then there exists a unique optimal control $u_0(t)$ which minimizes [5]

$$J_C(x) = \int_0^\infty \left[x^T(t)\, Qx(t) + u^T(t)\, u(t)\right] dt.$$

The vector $u_0(t)$ is given by

$$u_0(t) = -Kx(t)$$

where $K = B^T P$ and P is the unique positive semidefinite solution of the CARE (1).

Moreover, as known by [12], it can be seen that in the optimal regulator problem, the optimal cost can be written as

$$J^* = x_0^T P x_0$$

where $x_0 \in \mathbb{R}^n$ is the initial state of the considered system (2) and P is positive definite solution of the CARE (1).

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Then as it is denoted by [21] an interpretation of $tr(P)$ is that $tr(P)/n$ is the average value of the cost given by J^* as x_0 varies over the surface of a unit sphere.

Also, we should denote that many of numerical algorithms for obtaining the solutions of these equations have been reported in the literature [5,6,7,8,9,10,24,32]. The computing of these equations's analytic solutions are rather complicated in applications when the dimensions of system matrices are high. The exact solutions of these equations require a lot of heavy computational burdens and have time consuming. Therefore, in order to save time and decrease the burden of computation, instead of the exact solution, only the bounds as an approximation of the exact solution are sometimes needed. For example, for some applications such as stability analysis [27], without the burden of hard calculations, bounds are needed only for solution matrices. Furthermore, the solution bounds of the CARE (1) can be used to treat many control problems.

Therefore, during the past decades, numerous researchers have been devoted to obtain the bounds for the solution of the continuous algebraic Riccati matrix equations and a number of results have been reported in the literature. These results include matrix bounds for the solution matrix [4,8,16,17,20,21,26,30,32] and some characteristics of the solution matrix, specially eigenvalues [11,14,18,22], trace [14,22] and determinant [14,22] bounds are derived. However, matrix bounds are more general and usefull since they can be used to derive the other quantities.

This work is organized as following: Firstly, by generating the equivalent forms of the continuous algebraic Riccati matrix equation, using matrix inequalities, matrix identities, and some linear algebraic techniques, we propose new upper matrix bounds for the solution of the CARE (1). Also, we improve the algorithms which similar to previous studies to obtain tighter solution bounds. Lastly, we give illustrative examples to show that our results are more effective and less restrictive having compared with the some previous results in some cases.

In the following, let $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times m}$ denote the sets of $n\times n$ and $n\times m$ real matrices. Let $X\in\mathbb{R}^{n\times n}$ be an arbitrary symmetric matrix, then the eigenvalues of X are arranged so that $\lambda_1(X)\geq\lambda_2(X)\geq\cdots\geq\lambda_n(X)$. For any $X\in\mathbb{R}^{m\times n}$, the singular values of X are arranged so that $s_1(X)\geq s_2(X)\geq\cdots\geq s_{\min\{m,n\}}(X)$. If $X\in\mathbb{R}^{n\times n}$, let X^T, X^{-1}, $tr(X)$, and $\det(X)$ denote the transpose, the inverse, the trace, and the determinant of X, respectively. Write $X\geq(>)0$, if X is a positive semidefinite (positive definite) matrix. For X and Y are symmetric matrices of the same size, if $X-Y\geq 0$, then we write $X\geq Y$. If $X\geq Y$, then we have $\lambda_i(X)\geq\lambda_i(Y)$, $i=1,2,\ldots,n$. This expression is called Weyl's monotonicity principle. If $X\geq 0$, then $X^{1/2}$ denotes the unique positive semidefinite square root of X. The identity matrix in $\mathbb{R}^{n\times n}$ is shown by I.

The following Lemmas are used to prove the main results of this study.

Lemma 1.1. [3] If $X\in\mathbb{R}^{n\times n}$ is a symmetric matrix, then the following inequality holds:

$$\lambda_n(X)I\leq X\leq\lambda_1(X)I.$$

Lemma 1.2. [28] For any matrix $A\in\mathbb{R}^{n\times m}$ and positive semidefinite matrices $X,Y\in\mathbb{R}^{n\times n}$ such that $X\geq Y\geq(>)0$, it holds that $A^TXA\geq A^TYA$, with strict inequality if X and Y are positive definite and A is of full rank.

Lemma 1.3. [29] Let $X,Y\in\mathbb{R}^{n\times n}$ be two symmetric matrices and there exist an integer k such that $1\leq k\leq n$. Then for any index sequences $1\leq i_1\leq i_2\leq\cdots\leq i_k\leq n$, we have

$$\sum_{t=1}^{k}\lambda_{i_t}(X)\lambda_{n-t+1}(Y)\leq\sum_{t=1}^{k}\lambda_{i_t}(XY)\leq\sum_{t=1}^{k}\lambda_{i_t}(X)\lambda_t(Y).$$

Lemma 1.4. [29] Let $X,Y\in\mathbb{R}^{n\times n}$ be two symmetric matrices and there exist an integer k such that $1\leq k\leq n$. Then for any index sequences $1\leq i_1\leq i_2\leq\cdots\leq i_k\leq n$, we have

$$\sum_{t=1}^{k}\lambda_{i_t}(X)+\sum_{t=1}^{k}\lambda_{n-t+1}(Y)\leq\sum_{t=1}^{k}\lambda_{i_t}(X+Y)\leq\sum_{t=1}^{k}\lambda_{i_t}(X)+\sum_{t=1}^{k}\lambda_t(Y).$$

Lemma 1.5. [28] Let $A\in\mathbb{R}^{n\times n}$ be a positive definite matrix and let B be an $n\times m$ matrix. Then for any positive semidefinite $X\in\mathbb{R}^{m\times m}$,

$$\begin{pmatrix}A & B\\ B^T & X\end{pmatrix}\geq 0\Leftrightarrow X\geq B^TA^{-1}B.$$

Lemma 1.6. [29] Let A,B, and X be $n-$ square matrices such that $\begin{pmatrix}A & B\\ B^* & X\end{pmatrix}\geq 0$. Then,

$$\pm(B+B^*)\leq A+X.$$

Lemma 1.7. [11] The positive semidefinite solution P of the CARE (1) has the following upper bound on its maximal eigenvalue:

$$\lambda_1(P)\leq\lambda_1\left(D^TD\right)\frac{\lambda_1\left[\left(Q+K^TK\right)D^TD\right]}{\lambda_n\left(MD^TD\right)}\equiv\eta \qquad (3)$$

where K is any matrix stabilizing $A+BK$ (i.e., $Re(\lambda_i(A+BK))<0$ for all i) and the nonsingular matrix D and positive definite matrix M are chosen to yield the LMI (linear matrix inequality)

$$(A+BK)^TD^TD+D^TD(A+BK)\leq -M.$$

This eigenvalue upper bound is always computed if there exists a unique positive semidefinite solution of the CARE (1).

2 Main Results

In this section, we propose the upper solution matrix bounds to the CARE (1) and the algorithms for obtained bounds.

Theorem 2.1. Let P be the positive semidefinite solution of the CARE (1) and the positive semidefinite matrix M_1 is defined by

$$M_1 \equiv BB^T - AX_1A^T + \lambda_n(X_1)AA^T \tag{4}$$

where the positive constant matrix X_1 is chosen as

$$BB^T - AX_1A^T > 0. \tag{5}$$

Then P has the following upper bound

$$P \le \left\{ \frac{1}{\lambda_n(M_1)} \left[\begin{array}{l} Q + X_1^{-1} - \lambda_n(X_1)X_1^{-2} \\ +\lambda_n(X_1)\rho_1\left(I + X_1^{-1}A^TAX_1^{-1}\right) \end{array} \right] \right\}^{1/2} \equiv P_{u1} \tag{6}$$

where the nonnegative constant ρ_1 for $\mu_1 = \lambda_n(X_1)\left[1 + s_1^2\left(AX_1^{-1}\right)\right]$ is defined by

$$\rho_1 \equiv \frac{1}{2\lambda_n(M_1)} \times \left\{ \mu_1 + \left[\begin{array}{l} \mu_1^2 + 4\lambda_n(M_1) \\ \times\lambda_1\left(Q + X_1^{-1} - \lambda_n(X_1)X_1^{-2}\right) \end{array} \right]^{1/2} \right\}. \tag{7}$$

Proof. By multiplying (-1) the CARE (1) and adding $X_1^{-1} + PAX_1A^TP$ to both sides of the CARE (1), we have

$$P\left[BB^T - AX_1A^T\right]P = Q + X_1^{-1} - \left(PA - X_1^{-1}\right)X_1\left(PA - X_1^{-1}\right)^T$$

where X_1 is a positive constant matrix. Using Lemma 1.1 and the definition (4) of the matrix M_1, from the above equation, we write the following inequality:

$$PM_1P \le Q + X_1^{-1} + \lambda_n(X_1)\left(PAX_1^{-1} + X_1^{-1}A^TP - X_1^{-2}\right) \tag{8}$$

For $P \ge 0$, from Lemma 1.5, we get the following positive semidefinite block matrix

$$\begin{pmatrix} P^{1/2} & 0 \\ P^{1/2} & 0 \end{pmatrix}\begin{pmatrix} P^{1/2} & P^{1/2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} P & P \\ P & P \end{pmatrix} \ge 0. \tag{9}$$

By pre-and post- multiplying the first row by $X_1^{-1}A^T$ and the first column by AX_1^{-1}, from the matrix (9), we obtain

$$\begin{pmatrix} X_1^{-1}A^TPAX_1^{-1} & X_1^{-1}A^TP \\ PAX_1^{-1} & P \end{pmatrix} \ge 0 \tag{10}$$

and then applying Lemma 1.6 to (10) shows that

$$X_1^{-1}A^TP + PAX_1^{-1} \le X_1^{-1}A^TPAX_1^{-1} + P. \tag{11}$$

Therefore, combining the inequalities (8) and (11) yields

$$PM_1P \le Q + X_1^{-1} - \lambda_n(X_1)X_1^{-2} + \lambda_n(X_1)\left(X_1^{-1}A^TPAX_1^{-1} + P\right), \tag{12}$$

and from Lemma 1.1 and Lemma 1.2, (12) becomes

$$\lambda_n(M_1)P^2 \le Q + X_1^{-1} - \lambda_n(X_1)X_1^{-2} + \lambda_n(X_1)\left(X_1^{-1}A^TPAX_1^{-1} + P\right). \tag{13}$$

Furthermore, applying Lemma 1.1 and Lemma 1.2 to the right side of (13) gives

$$\lambda_n(M_1)P^2 \le Q + X_1^{-1} - \lambda_n(X_1)X_1^{-2} + \lambda_n(X_1)\lambda_1(P)\left(X_1^{-1}A^TAX_1^{-1} + I\right). \tag{14}$$

Utilizing Weyl's monotonicity principle for (14) implies

$$\lambda_n(M_1)\lambda_1^2(P) \le \lambda_1\left[\begin{array}{l} Q + X_1^{-1} - \lambda_n(X_1)X_1^{-2} \\ +\lambda_n(X_1)\lambda_1(P)\left(X_1^{-1}A^TAX_1^{-1} + I\right) \end{array} \right]$$

and by Lemma 1.4, we obtain

$$\lambda_n(M_1)\lambda_1^2(P) \le \lambda_1\left[Q + X_1^{-1} - \lambda_n(X_1)X_1^{-2}\right] + \lambda_n(X_1)\lambda_1(P)\lambda_1\left(X_1^{-1}A^TAX_1^{-1} + I\right) \tag{15}$$

By solving (15) with respect to $\lambda_1(P)$, we get

$$\lambda_1(P) \le \frac{1}{2\lambda_n(M_1)} \times \left\{ \mu_1 + \left[\begin{array}{l} \mu_1^2 + 4\lambda_n(M_1) \\ \times\lambda_1\left(Q + X_1^{-1} - \lambda_n(X_1)X_1^{-2}\right) \end{array} \right]^{1/2} \right\} \equiv \rho_1. \tag{16}$$

Substituting (16) into (14) and then solving the obtained inequality according to P gives the upper bound (6).

Now, we can develop the following iterative algorithm to obtain sharper solution estimates depending on the upper bound P_{u1} for the CARE (1):

Algorithm 2.1.

Step 1. Set $S_{1_0} \equiv P_{u1}$, where P_{u1} is defined by (6).

Step 2. Calculate

$$S_{1_i} = \left\{ \frac{1}{\lambda_n(M_1)} \left[\begin{array}{l} Q + X_1^{-1} - \lambda_n(X_1)X_1^{-2} \\ +\lambda_n(X_1)\left(X_1^{-1}A^TS_{1_{i-1}}AX_1^{-1} + S_{1_{i-1}}\right) \end{array} \right] \right\}^{1/2} \tag{17}$$

for $i = 1, 2, \ldots$. Then S_{1_i}'s are also upper bounds for the solution of the CARE (1).

Proof. Setting $i = 1$ in (17), we have

$$S_{1_1} = \left\{ \frac{1}{\lambda_n(M_1)} \left[\begin{array}{l} Q + X_1^{-1} - \lambda_n(X_1)X_1^{-2} \\ +\lambda_n(X_1)\left(X_1^{-1}A^TS_{1_0}AX_1^{-1} + S_{1_0}\right) \end{array} \right] \right\}^{1/2} \tag{18}$$

Applying Lemma 1.1 to (18) gives

$$S_{1_1} \le \left\{ \frac{1}{\lambda_n(M_1)} \left[\begin{array}{l} Q + X_1^{-1} - \lambda_n(X_1)X_1^{-2} \\ +\lambda_n(X_1)\lambda_1\left(S_{1_0}\right)\left(X_1^{-1}A^TAX_1^{-1} + I\right) \end{array} \right] \right\}^{1/2}. \tag{19}$$

Since $S_{1_0} \equiv P_{u1}$, by using Lemma 1.1 and Lemma 1.2, and the definition (7) of ρ_1, we obtain

$$S_{1_1} \le \left\{ \frac{1}{\lambda_n(M_1)} \left[\begin{array}{l} Q + X_1^{-1} - \lambda_n(X_1)X_1^{-2} \\ +\lambda_n(X_1)\rho_1\left(X_1^{-1}A^TAX_1^{-1} + I\right) \end{array} \right] \right\}^{1/2} = S_{1_0}. \tag{20}$$

Assume that $S_{1_{i-1}} \le S_{1_{i-2}}$. Then by Lemma 1.2, we write

$$S_{1_i} = \left\{ \frac{1}{\lambda_n(M_1)} \begin{bmatrix} Q + X_1^{-1} - \lambda_n(X_1) X_1^{-2} \\ + \lambda_n(X_1)\left(X_1^{-1} A^T S_{1_{i-1}} A X_1^{-1} + S_{1_{i-1}}\right) \end{bmatrix} \right\}^{1/2} \le \left\{ \frac{1}{\lambda_n(M_1)} \begin{bmatrix} Q + X_1^{-1} - \lambda_n(X_1) X_1^{-2} \\ + \lambda_n(X_1)\left(X_1^{-1} A^T S_{1_{i-2}} A X_1^{-1} + S_{1_{i-2}}\right) \end{bmatrix} \right\}^{1/2} = S_{1_{i-1}}. \quad (21)$$

By the mathematical induction method, it can be concluded that $S_{1_i} \le S_{1_{i-1}} \le \cdots \le S_{1_1} \le S_{1_0}$.

Corollary 2.1. Let the positive semidefinite matrix P satisfy (1). Then

$$P \le \left\{ \frac{1}{s_n^2(B)} \left[Q + \frac{1}{\alpha} \rho_1^* \left(\alpha^2 I + A^T A\right) \right] \right\}^{1/2} \equiv P_{u1}^* \quad (22)$$

where the positive constant α is chosen such that $BB^T > \alpha AA^T$ and the positive constant

$$\rho_1^* \equiv \frac{\alpha^2 + s_1^2(A) + \sqrt{\left(\alpha^2 + s_1^2(A)\right)^2 + 4\alpha^2 s_n^2(B) \lambda_1(Q)}}{2\alpha s_n^2(B)}$$

such as $s_n(B) \ne 0$.

Theorem 2.2. The positive semidefinite matrix M_2 is defined by

$$M_2 \equiv BB^T - X_2 + \lambda_n(X_2) I \quad (23)$$

where the positive constant X_2 is selected by

$$BB^T - X_2 > 0. \quad (24)$$

Then the positive semidefinite solution P to the CARE (1) has the upper bound

$$P \le \left\{ \frac{1}{\lambda_n(M_2)} \begin{bmatrix} Q + A^T\left(X_2^{-1} - \lambda_n(X_2) X_2^{-2}\right) A \\ + \lambda_n(X_2) \rho_2 \left(I + X_2^{-1} A^T A X_2^{-1}\right) \end{bmatrix} \right\}^{1/2} \equiv P_{u2} \quad (25)$$

where the nonnegative constant ρ_2 for $\mu_2 = \lambda_n(X_2)\left[1 + s_1^2\left(AX_2^{-1}\right)\right]$ is defined by

$$\rho_2 \equiv \frac{1}{2\lambda_n(M_2)} \times \left\{ \mu_2 + \begin{bmatrix} \mu_2^2 + 4\lambda_n(M_2) \\ \times \lambda_1\left[Q + A^T\left(X_1^{-1} - \lambda_n(X_1) X_1^{-2}\right) A\right] \end{bmatrix}^{1/2} \right\}. \quad (26)$$

Proof. By multiplying (-1) the CARE (1) and adding $PX_2P + A^T X_2^{-1} A$ to both sides of the CARE (1), we have

$$P\left(BB^T - X_2\right) P = Q + A^T X_2^{-1} A - \left(P - A^T X_2^{-1}\right) X_2 \left(P - A^T X_2^{-1}\right)^T.$$

When the above inequality is rearranged by using Lemma 1.1, it is written

$$PM_2P = Q + A^T\left(X_2^{-1} - \lambda_n(X_2) X_2^{-2}\right) A + \lambda_n(X_2)\left(PX_2^{-1}A + A^T X_2^{-1} P\right) \quad (27)$$

where M_2 is given by (23). From the matrix (9), by pre- and post- multiplying the first row by $A^T X_2^{-1}$ and the first column by $X_2^{-1}A$ gives

$$\begin{pmatrix} A^T X_2^{-1} P X_2^{-1} A & A^T X_2^{-1} P \\ P X_2^{-1} A & P \end{pmatrix} \ge 0$$

and by application of Lemma 1.6 to the above block matrix shows that

$$A^T X_2^{-1} P + P X_2^{-1} A \le A^T X_2^{-1} P X_2^{-1} A + P. \quad (28)$$

Therefore from the inequalities (27) and (28), it is obtained

$$PM_2P = Q + A^T\left(X_2^{-1} - \lambda_n(X_2) X_2^{-2}\right) A + \lambda_n(X_2)\left(A^T X_2^{-1} P X_2^{-1} A + P\right).$$

Consequently, following the above procedures, along the lines of Theorem 2.1's proof, it can be obtained the upper bound P_{u2} for the CARE (1).

Algorithm 2.2.

Step 1. Set $S_{2_0} \equiv P_{u2}$, where P_{u2} is defined by (25).

Step 2. Compute

$$S_{2_i} = \left\{ \frac{1}{\lambda_n(M_2)} \begin{bmatrix} Q + A^T\left(X_2^{-1} - \lambda_n(X_2) X_2^{-2}\right) A \\ + \lambda_n(X_2)\left(A^T X_2^{-1} S_{2_{i-1}} X_2^{-1} A + S_{2_{i-1}}\right) \end{bmatrix} \right\}^{1/2} \quad (29)$$

for $i = 1, 2, \ldots$. Then S_{2_i}'s are also upper bounds for the solution of the CARE(1).

Corollary 2.2. Let the positive semidefinite matrix P satisfy (1). Then

$$P \le \left\{ \frac{1}{s_n^2(B)} \left[Q + \frac{1}{\beta} \rho_2^* \left(\beta^2 I + A^T A\right) \right] \right\}^{1/2} \equiv P_{u2}^* \quad (30)$$

where the positive constant β is chosen such that $BB^T > \beta I$ and the positive constant

$$\rho_2^* \equiv \frac{\beta^2 + s_1^2(A) + \sqrt{\left(\beta^2 + s_1^2(A)\right)^2 + 4\beta^2 s_n^2(B) \lambda_1(Q)}}{2\beta s_n^2(B)} \quad (31)$$

for $s_n(B) \ne 0$.

Theorem 2.3. Define the matrices M_3 and U

$$M_3 \equiv BB^T + AX_3A^T \quad (32)$$

and

$$U \equiv s_1^2(A)\, \eta / 2I + AX_3^{-1} \quad (33)$$

where X_3 is a positive definite matrix. Then the positive solution P to the CARE (1) satisfies

$$P \le \left\{ \frac{1}{\lambda_n(M_3)} \begin{bmatrix} Q - X_3^{-1} \\ + \lambda_1(X_3)\left(X_3^{-2} + \rho_3\left(I + U^T U\right)\right) \end{bmatrix} \right\}^{1/2} \equiv P_{u3} \quad (34)$$

where the nonnegative constant ρ_3 for $\mu_3 = \left[1+s_1^2(U)\right]\lambda_1(X_3)$ is defined by

$$\rho_3 \equiv \frac{1}{2\lambda_n(M_3)} \times \left\{\mu_3 + \begin{bmatrix}\mu_3^2+4\lambda_n(M_3)\\ \times\lambda_1\left(Q-X_3^{-1}+\lambda_1(X_3)X_3^{-2}\right)\end{bmatrix}^{1/2}\right\} \quad (35)$$

so that the positive constant η is given by (3).

Proof. By adding $X_3^{-1}+PAX_3A^TP$ to both sides of the CARE (1) , we get

$$PM_3P = Q-X_3^{-1}+\left(PA+X_3^{-1}\right)X_3\left(PA+X_3^{-1}\right)^T \quad (36)$$

where $X_3>0$ and by use of Lemma 1.1, from (36), we write

$$P\ M_3\ P \le Q-X_3^{-1} + \lambda_1(X_3)\left(PAA^TP+PAX_3^{-1}+X_3^{-1}A^TP+X_3^{-2}\right). \quad (37)$$

Furthermore, for the term PAA^TP of (37), using Lemma 1, it can be written

$$PAA^TP \le s_1^2(A)P^2 \quad (38)$$

and

$$P^2 \le \left[P\lambda_1(P)IP\right]^{1/2} = \lambda_1(P)P \le \eta P \quad (39)$$

respectively. Thus from the inequalities (38) and (39), it can be seen that

$$PAA^TP \le s_1^2(A)\eta P. \quad (40)$$

Substituting (40) into (37) and organizing to the matrix U (from (33)) leads to

$$\begin{aligned} PM_3P &\le Q-X_3^{-1}+\lambda_1(X_3)X_3^{-2} \\ &+\lambda_1(X_3)\begin{bmatrix}P\left(s_1^2(A)\eta/2I+AX_3^{-1}\right)\\ +\left(s_1^2(A)\eta/2I+AX_3^{-1}\right)^TP\end{bmatrix} \\ &= Q-X_3^{-1}+\lambda_1(X_3)\left(X_3^{-2}+PU+U^TP\right). \end{aligned} \quad (41)$$

Now by pre-and post- multiplying the first row of the matrix (9) by U^T and the first column of the matrix (9) by U, and by Lemma 1.6, we can write

$$\begin{pmatrix}U^TPU & U^TP\\ PU & P\end{pmatrix} \ge 0 \Rightarrow PU+U^TP \le U^TPU+P. \quad (42)$$

Therefore, from the inequalities (41) and (42), we have

$$PM_3P \le Q-X_3^{-1}+\lambda_1(X_3)\left(X_3^{-2}+U^TPU+P\right) \quad (43)$$

As Lemma 1.1 and Lemma 1.2 are considered, from (43), we can write

$$\begin{aligned} \lambda_n\ (M_3)\ P^2 &\le Q-X_3^{-1}+\lambda_1(X_3)\left(X_3^{-2}+U^TPU+P\right) \\ &\le\ Q-X_3^{-1}+\lambda_1(X_3)\lambda_1(P)\left(X_3^{-2}+U^TU+I\right), \end{aligned} \quad (44)$$

and utilizing Weyl's monotonicity principle and Lemma 1.4 imply that

$$\begin{aligned} \lambda_n(M_3)\lambda_1^2(P) &\le \lambda_1\left[Q-X_3^{-1}+\lambda_1(X_3)X_3^{-2}\right] \\ &+\lambda_1(X_3)\lambda_1(P)\left(1+s_1^2(U)\right). \end{aligned} \quad (45)$$

Solving (45) in according to $\lambda_1(P)$ shows that

$$\begin{aligned} \lambda_1(P) &\equiv \frac{1}{2\lambda_n(M_3)} \\ &\times\left\{\mu_3+\begin{bmatrix}\mu_3^2+4\lambda_n(M_3)\\ \times\lambda_1\left(Q-X_3^{-1}+\lambda_1(X_3)X_3^{-2}\right)\end{bmatrix}^{1/2}\right\} \\ &\equiv \rho_3. \end{aligned}$$

Substituting ρ_3 into (44) and then solving the obtained inequality regarding to P gives the upper bound P_{u3}.

Algorithm 2.3.

Step 1. Set $S_{3_0} \equiv P_{u3}$, where P_{u3} is defined by (34).
Step 2. Work out

$$S_{3_i} = \left\{\frac{1}{\lambda_n(M_3)}\begin{bmatrix}Q-X_3^{-1}+\lambda_1(X_3)X_3^{-2}\\ +\lambda_1(X_3)\left(U^TS_{3_{i-1}}U+S_{3_{i-1}}\right)\end{bmatrix}\right\}^{1/2} \quad (46)$$

for $i=1,2,\ldots$. Then S_{3_i}'s are also upper bounds for the solution of the CARE(1).

Theorem 2.4. Define the matrices M_4 and V

$$M_4 \equiv BB^T+X_4, \quad (47)$$

$$V \equiv \frac{\eta}{2}I+X_4^{-1}A \quad (48)$$

where the positive definite matrix X_4. Then the positive solution P to the CARE (1) holds

$$\begin{aligned} P &\le \left\{\frac{1}{\lambda_n(M_4)}\begin{bmatrix}Q-A^T\left(X_4^{-1}+\lambda_1(X_4)X_4^{-2}\right)A\\ +\lambda_1(X_4)\rho_4\left(I+V^TV\right)\end{bmatrix}\right\}^{1/2} \\ &\equiv P_{u4} \end{aligned} \quad (49)$$

where the nonnegative constant ρ_4 for $\mu_4 = \left[1+s_1^2(V)\right]\lambda_1(X_4)$ is defined by

$$\rho_4 \equiv \frac{1}{2\lambda_n(M_4)} \quad (50)$$
$$\times\left\{\mu_4+\begin{bmatrix}\mu_4^2+4\lambda_n(M_4)\\ \times\lambda_1\left[Q-A^T\left(X_4^{-1}+\lambda_1(X_4)X_4^{-2}\right)A\right]\end{bmatrix}^{1/2}\right\}$$

such that the positive constant η is given by (3).

Proof. By adding $PX_4P+A^TX_4^{-1}A$ to both sides of the CARE (1) for $X_4>0$, by definition (47) of M_4, we get

$$\begin{aligned} PM_4P &= Q-A^TX_4^{-1}A \\ &+\left(P+A^TX_4^{-1}\right)X_4\left(P+A^TX_4^{-1}\right)^T. \end{aligned} \quad (51)$$

Applying Lemma 1.1 to (51) and using the inequality (39), from the definition (48) of V, we can easily write

$$\begin{aligned} PM_4P &\le Q+A^T\left(X_4^{-1}+\lambda_1(X_4)X_4^{-2}\right)A \\ &+\lambda_1(X_4)\left[P\left(\frac{\eta}{2}I+X_4^{-1}A\right)+\left(\frac{\eta}{2}I+X_4^{-1}A\right)^TP\right] \\ &= Q+A^T\left(X_4^{-1}+\lambda_1(X_4)X_4^{-2}\right)A \\ &+\lambda_1(X_4)\left(PV+V^TP\right). \end{aligned} \quad (52)$$

Therefore, continuing from the previous theorem's proof gives the bound P_{u4}.

Algorithm 2.4.

Step 1. Set $S_{4_0} \equiv P_{u4}$, where P_{u4} is defined by (49).

Step 2. Check out

$$S_{4_i} = \left\{ \frac{1}{\lambda_n(M_4)} \left[\begin{array}{l} Q + A^T \left(X_4^{-1} + \lambda_1(X_4) X_4^{-2}\right) A \\ +\lambda_1(X_4)\left(V^T S_{4_{i-1}} V + S_{4_{i-1}}\right) \end{array} \right] \right\}^{1/2} \quad (53)$$

for $i = 1, 2, \ldots$. Then S_{4_i}'s are also upper bounds for the solution of the CARE(1).

Remark 2.1. One of the most frequently used techniques in matrix theory is the continuity argument [28]. If a matrix Y is singular, consider $Y + \varepsilon I$. Choose $\delta > 0$ such that $Y + \varepsilon I$ is invertible for all ε, $0 < \varepsilon < \delta$. Then, we can say that it is obtainable a matrix $X = \varepsilon I$, $\varepsilon > 0$ such that $BB^T + X$ is nonsingular. Replace singular BB^T by nonsingular $BB^T + \varepsilon I$. Then upper bound given by Theorem 2.4 is always computable for the positive constant ε is chosen so that $BB^T + \varepsilon I > 0$. So far, all of the presented upper matrix bounds of the solution for the CARE (1) have the restriction that $s_n(B) \neq 0$. Therefore, we say that Theorem 2.4 is to improve this restriction.

Remark 2.2. As Chen and Lee in [4] and Zhang and Liu in [30] pointed out, to give a general comparison between any parallel upper bounds for the same measure is either difficult or actually impossible. Since many upper bounds in literature include different parameters depend on various assumptions, the mentioned comparisons also can find that it is hard to compare the sharpness of our upper bounds to the similar results. For this reason, we can give the following numerical examples to show the effectiveness of our results.

3 Numerical Examples

Example 3.1. [20] Consider the CARE (1) with

$$A = \begin{pmatrix} -3 & 0.5 \\ 0.1 & 0.2 \end{pmatrix}, BB^T = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, Q = \begin{pmatrix} 3 & 0.2 \\ 0.2 & 3 \end{pmatrix}.$$

Then, the unique positive definite solution P to the CARE (1) is

$$P_{exact} = \begin{pmatrix} 0.3967 & 0.0936 \\ 0.0936 & 1.9603 \end{pmatrix}, tr(P) = 2.3570, \det(P) = 0.7689$$

Our upper bounds P_{u2} and P_{u4}, and the algorithms (2 iterations) corresponding to these bounds are as the following:

$$P_{u2} \equiv \begin{pmatrix} 2.5997 & -0.0114 \\ -0.0114 & 2.0346 \end{pmatrix}, S_2 \equiv \begin{pmatrix} 2.5910 & -0.0260 \\ -0.0260 & 1.9681 \end{pmatrix}$$

with $X_2 = \begin{pmatrix} 3.2 & 0 \\ 0 & 0.2 \end{pmatrix}$,

$$P_{u4} \equiv \begin{pmatrix} 1.5087 & 0.0303 \\ 0.0303 & 2.0209 \end{pmatrix}, S_4 \equiv \begin{pmatrix} 1.3338 & 0.0403 \\ 0.0403 & 2.0122 \end{pmatrix}$$

with $X_4 = 3I$, $\eta = 2.01917$.

Also, some studies in recent years have resulted in the following:

The upper bound given by Corollary 2.1 in [4] gives

$$P_{u[4]} \equiv \begin{pmatrix} 3.2278 & -0.2618 \\ -0.2618 & 2.3232 \end{pmatrix}, tr(P) \le 5.5510, \det(P) \le 7.4302,$$

The upper bound presented by Theorem 1 in [8] shows

$$P_{u[8]} \equiv \begin{pmatrix} 2.4579 & 0.0387 \\ 0.0387 & 2.6609 \end{pmatrix}, tr(P) \le 5.1188, \det(P) \le 6.5387,$$

The upper matrix bound developed in Theorem 2 of [20] is

$$P_{u[20]} \equiv \begin{pmatrix} 2.7067 & -0.0445 \\ -0.0445 & 2.0238 \end{pmatrix}$$

with $R_1 = X_2^{-1}$ and we can also denote that this bound can not be computed for $R_1 \equiv X_4^{-1}$.

The proposed upper bound in [21] leads to

$$P_{u[21]} \equiv \begin{pmatrix} 6.8384 & 0.0002 \\ 0.0002 & 6.8385 \end{pmatrix}, tr(P) \le 13.677, \det(P) \le 46.7647,$$

for $\varepsilon = 0.014$.

In view of the above numerical experiments, it appears that our bounds give more precise solution estimates than the previously reported results.

Example 3.2. [7] Consider the CARE (1) with

$$A = \begin{pmatrix} 0.5 & 0 \\ 1 & -2.5 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

Then, the unique positive definite solution P to the CARE (1) is

$$P_{exact} = \begin{pmatrix} 0.6689 & 0.1228 \\ 0.1228 & 0.5879 \end{pmatrix}.$$

In this example $s_n(B) = 0$ and this means that BB^T is singular, so the upper matrix bound P_{u1} and P_{u2} cannot work for this case. However, the upper bounds P_{u3} and P_{u4} according to the arbitrary selections of matrices X_3 and X_4 and the algorithms (5 iterations) corresponding to these bounds are as following:

$$P_{u3} = \begin{pmatrix} 3.3988 & 0.0717 \\ 0.0717 & 1.2974 \end{pmatrix}, S_3 = \begin{pmatrix} 3.1916 & 0.0267 \\ 0.0267 & 0.9988 \end{pmatrix}$$

with $X_3 = I$,

$$P_{u3} = \begin{pmatrix} 3.0956 & -0.4006 \\ -0.4006 & 1.7915 \end{pmatrix}, S_3 = \begin{pmatrix} 2.5762 & -0.1640 \\ -0.1640 & 1.3689 \end{pmatrix}$$

with $X_3 = 0.5I$,

$$P_{u4} = \begin{pmatrix} 4.3299 & -1.3930 \\ -1.3930 & 6.2479 \end{pmatrix}, S_4 = \begin{pmatrix} 2.8038 & -1.2757 \\ -1.2757 & 5.6711 \end{pmatrix}$$

with $X_4 = I$,

$$P_{u4} = \begin{pmatrix} 1.2903 & 0.0081 \\ 0.0081 & 1.2704 \end{pmatrix}, S_4 = \begin{pmatrix} 1.2461 & 0.0137 \\ 0.0137 & 1.2187 \end{pmatrix}$$

with $X_4 = 15I$ for $\eta = 0.7974$.
$s_n(B) = 0$ means that BB^T is singular, so the upper matrix bounds presented in [4,21,30] cannot work for this case.

4 Perspective

Some upper matrix bounds for the solution of the CARE are improved by means of different equivalent forms of the CARE by using some linear algebraic techniques and matrix inequalities. Also, the algorithms for the bounds are given to obtain more precise bounds. Finally, by aid of several numerical examples, it is demonstrated that the solution upper bounds developed in this study are sharper than some results in the literature in certain circumstances .

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