Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100430

Existence of Nonoscillatory Solutions of Higher Order Nonlinear Neutral Nonhomogeneous Equations with Distributed Deviating Arguments

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Received: 12 Apr. 2016, Revised: 29 May 2016, Accepted: 30 May 2016
Published online: 1 Jul. 2016

Abstract: We obtain sufficient conditions for the existence of a nonoscillatory solution of higher order nonlinear neutral differential equations with distributed deviating arguments. For this purpose, we use the Banach contraction principle.

Keywords: Neutral equations, Fixed point, Higher-order, Nonoscillatory solution, Distributed delay.

1 Introduction

In recent years, the existence of nonoscillatory solution of the first, second and higher order neutral differential equations have been studied. We refer the reader to the papers [1–12] and the references cited therein.

In the present article, we consider the following higher-order nonlinear neutral differential equations

$$\left[r(t)\left[x(t)+p(t)x(t-\tau)\right]^{(n-1)}\right]' +(-1)^n\left[\int_{a_1}^{b_1} q_1(t,\xi)g_1(x(t-\xi))d\xi - \int_{a_2}^{b_2} q_2(t,\xi)g_2(x(t-\xi))d\xi - f(t)\right] = 0 \quad (1)$$

and

$$\left[r(t)\left[x(t)+\int_{a_3}^{b_3}\tilde{p}(t,\xi)x(t-\xi)d\xi\right]^{(n-1)}\right]' +(-1)^n\left[\int_{a_1}^{b_1} q_1(t,\xi)g_1(x(t-\xi))d\xi - \int_{a_2}^{b_2} q_2(t,\xi)g_2(x(t-\xi))d\xi - f(t)\right] = 0, \quad (2)$$

where $n \geqslant 2$ is a positive integer, $\tau > 0$, $b_i > a_i \geqslant 0, i = 1,2,3$, $p \in C([t_0,\infty),\mathbb{R})$, $\tilde{p} \in C([t_0,\infty) \times [a_3,b_3],\mathbb{R})$, $r \in C([t_0,\infty),(0,\infty))$, $q_i \in C([t_0,\infty) \times [a_i,b_i],[0,\infty))$, i=1,2, $f \in C([t_0,\infty),\mathbb{R})$ and $g_i \in C(\mathbb{R},\mathbb{R})$, $i = 1,2$. We assume that g_i, $i = 1,2$, satisfy local Lipschitz condition and $g_i(x)x > 0$, $i = 1,2$, for $x \neq 0$.

The aim this paper is to extend the results of [6] to the case of distributed deviating argument and give sufficient conditions for the existence of a bounded nonoscillatory solution of (1) and (2).

Let $m = \max\{b_1,b_2,\tau\}$. By a solution of (1) we mean a function $x \in C([t_1-m,\infty),\mathbb{R})$, for some $t_1 \geqslant t_0$, such that $x(t)+p(t)x(t-\tau)$ is $n-1$ times continuously differentiable and $r(t)(x(t)+p(t)x(t-\tau))^{(n-1)}$ is continuously differentiable on $[t_1,\infty)$ and such that (1) is satisfied for $t \geqslant t_1$. Similarly, Let $m_1 = \max\{b_1,b_2,b_3\}$. By a solution of (2) we mean a function $x \in C([t_1-m_1,\infty),\mathbb{R})$, for some $t_1 \geqslant t_0$, such that $x(t)+\int_{a_3}^{b_3}\tilde{p}(t,\xi)x(t-\xi)d\xi$ is $n-1$ times continuously differentiable and $r(t)(x(t)+\int_{a_3}^{b_3}\tilde{p}(t,\xi)x(t-\xi)d\xi)^{(n-1)}$ is continuously differentiable on $[t_1,\infty)$ and such that (2) is satisfied for $t \geqslant t_1$.

As it is customary, a solution of (1) (or (2)) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, the solution is called nonoscillatory.

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2 Main Results

In what follows, we use the notation $Q_1(s)=\int_{a_1}^{b_1} q_1(s,\xi)d\xi$ and $Q_2(s)=\int_{a_2}^{b_2} q_2(s,\xi)d\xi$.

Theorem 1. *Assume that* $0\leqslant p(t)\leqslant p<1$ *and*

$$\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{s^{n-2}}{r(s)}Q_i(u)duds<\infty, \quad i=1,2$$

and (3)

$$\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{s^{n-2}}{r(s)}|f(u)|duds<\infty.$$

Then (1) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0,\infty)$ with the sup norm. Set

$$A=\{x\in\Lambda : M_1\leqslant x(t)\leqslant M_2, \quad t\geqslant t_0\},$$

where M_1 and M_2 are positive constants such that

$$pM_2+M_1<M_2.$$

Let $\alpha\in(pM_2+M_1,M_2)$, L_i, $i=1,2$, denote Lipschitz constants of functions g_i, $i=1,2$, on the set A, respectively and $L=\max\{L_1,L_2\}$, $\beta_i=\max_{x\in A}\{g_i(x)\}$, $i=1,2$, respectively. From (3), we can choose a $t_1>t_0$,

$$t_1\geqslant t_0+\max\{b_1,b_2,\tau\} \tag{4}$$

sufficiently large such that

$$\frac{1}{(n-2)!}\int_t^{\infty}\frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^{s}[Q_1(u)\beta_1+|f(u)|]duds \leqslant M_2-\alpha, \quad t\geqslant t_1, \tag{5}$$

$$\frac{1}{(n-2)!}\int_t^{\infty}\frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^{s}[Q_2(u)\beta_2+|f(u)|]duds \leqslant \alpha-M_1-pM_2, \quad t\geqslant t_1 \tag{6}$$

and

$$p+\frac{L}{(n-2)!}\int_t^{\infty}\frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^{s}(Q_1(u)+Q_2(u))duds \leqslant \theta_1<1, \quad t\geqslant t_1, \tag{7}$$

where θ_1 is a constant. Define a mapping $T: A\longrightarrow\Lambda$ as follows

$$(Tx)(t) = \begin{cases} \alpha-p(t)x(t-\tau)+\frac{1}{(n-2)!} \\ \times\int_t^{\infty}\frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^{s}\Big[\int_{a_1}^{b_1}q_1(u,\xi)g_1(x(u-\xi))d\xi \\ -\int_{a_2}^{b_2}q_2(u,\xi)g_2(x(u-\xi))d\xi-f(u)\Big]duds, \\ \quad t\geqslant t_1 \\ (Tx)(t_1), \quad t_0\leqslant t\leqslant t_1. \end{cases}$$

Obviously, Tx is continuous. For $t\geqslant t_1$ and $x\in A$, using (5) and (6), respectively, we obtain

$$\begin{aligned}(Tx)(t) &\leqslant \alpha+\frac{1}{(n-2)!}\int_t^{\infty}\frac{(s-t)^{n-2}}{r(s)} \\ &\times\int_{t_1}^{s}\left[\int_{a_1}^{b_1}q_1(u,\xi)g_1(x(u-\xi))d\xi-f(u)\right]duds \\ &\leqslant \alpha+\frac{1}{(n-2)!}\int_t^{\infty}\frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^{s}[Q_1(u)\beta_1+|f(u)|]duds \\ &\leqslant M_2\end{aligned}$$

and

$$\begin{aligned}(Tx)(t) &\geqslant \alpha-p(t)x(t-\tau)-\frac{1}{(n-2)!}\times \\ &\int_t^{\infty}\frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^{s}\left[\int_{a_2}^{b_2}q_2(u,\xi)g_2(x(u-\xi))d\xi+f(u)\right]duds \\ &\geqslant \alpha-pM_2-\frac{1}{(n-2)!}\int_t^{\infty}\frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^{s}[Q_2(u)\beta_2+|f(u)|]duds \\ &\geqslant M_1.\end{aligned}$$

Thus, we proved that $TA\subset A$. We observe that A is a bounded, closed, convex subset of Λ. We now show that T is a contraction mapping on A. For $x_1,x_2\in A$ and $t\geqslant t_1$,

$$\begin{aligned}(Tx_1)(t)-(Tx_2)(t)| &\leqslant p(t)|x_1(t-\tau)-x_2(t-\tau)|+\tfrac{1}{(n-2)!}\times \\ &\int_t^{\infty}\tfrac{(s-t)^{n-2}}{r(s)}\int_{t_1}^{s}\Big(\int_{a_1}^{b_1}q_1(u,\xi)|g_1(x_1(u-\xi))-g_1(x_2(u-\xi))|d\xi \\ &+\int_{a_2}^{b_2}q_2(u,\xi)|g_2(x_1(u-\xi))-g_2(x_2(u-\xi))|d\xi\Big)duds\end{aligned}$$

or using (7)

$$\begin{aligned}|(Tx_1)(t)-(Tx_2)(t)| &\leqslant ||x_1-x_2|| \\ &\times\left(p+\frac{L}{(n-2)!}\int_t^{\infty}\frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^{s}(Q_1(u)+Q_2(u))duds\right) \\ &\leqslant \theta_1||x_1-x_2||.\end{aligned}$$

This implies with the sup norm that

$$||Tx_1-Tx_2||\leqslant\theta_1||x_1-x_2||,$$

where in view of (7), $\theta_1<1$, which shows that T is a contraction mapping on A. As a result, T has a fixed point $x\in A$, and x is a positive solution of (1). This completes the proof.

Theorem 2. *Assume that* $1<p\leqslant p(t)\leqslant p_0<\infty$ *and (3) holds. Then (1) has a bounded nonoscillatory solution.*

Proof. Let Λ be the same set as in the proof of Theorem 1. Set

$$A=\{x\in\Lambda : M_3\leqslant x(t)\leqslant M_4, \quad t\geqslant t_0\},$$

where M_3 and M_4 are positive constants such that

$$p_0M_3+M_4<pM_4.$$

Let $\alpha\in(p_0M_3+M_4,pM_4)$, L_i, $i=1,2$, denote Lipschitz constants of functions g_i, $i=1,2$, on the set A, respectively and $L=\max\{L_1,L_2\}$, $\beta_i=\max_{x\in A}\{g_i(x)\}$, $i=1,2$, respectively. In view of (3), we can choose a $t_1>t_0$,

$$t_1+\tau \geq t_0+\max\{b_1,b_2\} \tag{8}$$

sufficiently large such that

$$\frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s [Q_1(u)\beta_1+|f(u)|]\,duds \leqslant pM_4-\alpha, \quad t\geqslant t_1, \tag{9}$$

$$\frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s [Q_2(u)\beta_2+|f(u)|]\,duds \leqslant \alpha-M_4-p_0M_3, \quad t\geqslant t_1 \tag{10}$$

and

$$\frac{1}{p}\left(1+\frac{L}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s (Q_1(u)+Q_2(u))\,duds\right) \leqslant \theta_2<1, \quad t\geqslant t_1, \tag{11}$$

where θ_2 is a constant. Define a mapping $T:A\longrightarrow \Lambda$ as follows

$$(Tx)(t) = \begin{cases} \frac{1}{p(t+\tau)}\Big\{\alpha-x(t+\tau)+\frac{1}{(n-2)!}\int_{t+\tau}^\infty \frac{(s-t-\tau)^{n-2}}{r(s)} \\ \times\int_{t_1+\tau}^s\Big[\int_{a_1}^{b_1} q_1(u,\xi)g_1(x(u-\xi))d\xi \\ -\int_{a_2}^{b_2} q_2(u,\xi)g_2(x(u-\xi))d\xi-f(u)\Big]duds\Big\}, \\ t\geqslant t_1 \\ (Tx)(t_1), \quad t_0\leqslant t\leqslant t_1. \end{cases}$$

Obviously, Tx is continuous. For $t\geqslant t_1$ and $x\in A$, using (9) and (10), respectively, we have

$$\begin{aligned}(Tx)(t) &\leqslant \frac{1}{p(t+\tau)}\left[\alpha+\frac{1}{(n-2)!}\int_{t+\tau}^\infty \frac{(s-t-\tau)^{n-2}}{r(s)}\right. \\ &\times \left.\int_{t_1+\tau}^s\left[\int_{a_1}^{b_1} q_1(u,\xi)g_1(x(u-\xi))d\xi-f(u)\right]duds\right] \\ &\leqslant \frac{1}{p}\left[\alpha\right. \\ &+\left.\frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s[Q_1(u)\beta_1+|f(u)|]\,duds\right] \\ &\leqslant M_4\end{aligned}$$

and

$$\begin{aligned}(Tx)(t) &\geqslant \frac{1}{p(t+\tau)}\left[\alpha-x(t+\tau)-\frac{1}{(n-2)!}\int_{t+\tau}^\infty \frac{(s-t-\tau)^{n-2}}{r(s)}\right. \\ &\times\left.\int_{t_1+\tau}^s\left[\int_{a_2}^{b_2} q_2(u,\xi)g_2(x(u-\xi))d\xi+f(u)\right]duds\right] \\ &\geqslant \frac{1}{p_0}\left[\alpha-M_4\right. \\ &-\left.\frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s[Q_2(u)\beta_2+|f(u)|]\,duds\right] \\ &\geqslant M_3.\end{aligned}$$

Thus, we showed that $TA\subset A$. We observe that A is a bounded, closed, convex subset of Λ. We now show that T is a contraction mapping on A. For $x_1,x_2\in A$ and $t\geqslant t_1$, from (11)

$$\begin{aligned}&|(Tx_1)(t)-(Tx_2)(t)| \\ &\leqslant \frac{||x_1-x_2||}{p} \\ &\times\left(1+\frac{L}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s (Q_1(u)+Q_2(u))\,duds\right) \\ &\leqslant \theta_2||x_1-x_2||.\end{aligned}$$

This implies with the sup norm that

$$||Tx_1-Tx_2||\leqslant \theta_2||x_1-x_2||,$$

where in view of (11), $\theta_2<1$, which proves that T is a contraction mapping on A. Consequently, T has a fixed point $x\in A$, and x is a positive solution of (1). This completes the proof of Theorem 2.

Theorem 3. *Assume that $-1<p\leqslant p(t)\leqslant 0$ and (3) holds. Then (1) has a bounded nonoscillatory solution.*

Proof. Let Λ be the same set as in the proof of Theorem 1. Set

$$A=\{x\in\Lambda: M_5\leqslant x(t)\leqslant M_6, \quad t\geqslant t_0\},$$

where M_5 and M_6 are positive constants such that

$$M_5<(1+p)M_6.$$

Let $\alpha\in(M_5,(1+p)M_6)$, L_i, $i=1,2$, denote Lipschitz constants of functions g_i, $i=1,2$, on the set A, respectively and $L=\max\{L_1,L_2\}$, $\beta_i=\max_{x\in A}\{g_i(x)\}$, $i=1,2$, respectively. By making use of (3), we can choose a $t_1>t_0$ sufficiently large satisfying (4) such that

$$\frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s [Q_1(u)\beta_1+|f(u)|]\,duds \leqslant (1+p)M_6-\alpha, \quad t\geqslant t_1,$$

$$\frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s [Q_2(u)\beta_2+|f(u)|]\,duds \leqslant \alpha-M_5, \quad t\geqslant t_1$$

and

$$-p+\frac{L}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s (Q_1(u)+Q_2(u))\,duds \leqslant \theta_3<1, \quad t\geqslant t_1,$$

where θ_3 is a constant. Consider the operator $T:A\longrightarrow\Lambda$ defined by

$$(Tx)(t) = \begin{cases} \alpha-p(t)x(t-\tau) \\ +\frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s\Big[\int_{a_1}^{b_1} q_1(u,\xi)g_1(x(u-\xi))d\xi \\ -\int_{a_2}^{b_2} q_2(u,\xi)g_2(x(u-\xi))d\xi-f(u)\Big]duds, \quad t\geqslant t_1 \\ (Tx)(t_1), \quad t_0\leqslant t\leqslant t_1. \end{cases}$$

Clearly, Tx is continuous. Since the rest of the proof is similar to that of Theorem 1, it is omitted.

Theorem 4. *Assume that* $-\infty < p_0 \leqslant p(t) \leqslant p < -1$ *and (3) holds. Then (1) has a bounded nonoscillatory solution.*

Proof. Let Λ be the same set as in the proof of Theorem 1. Set

$$A = \{x \in \Lambda : M_7 \leqslant x(t) \leqslant M_8, \quad t \geqslant t_0\},$$

where M_7 and M_8 are positive constants such that

$$-p_0 M_7 < (-p-1)M_8.$$

Let $\alpha \in (-p_0M_7, (-p-1)M_8)$, L_i, $i = 1,2$, denote Lipschitz constants of functions g_i, $i = 1,2$, on the set A, respectively and $L = \max\{L_1, L_2\}$, $\beta_i = \max_{x\in A}\{g_i(x)\}$, $i = 1,2$, respectively. By using (3), one can choose a $t_1 > t_0$ sufficiently large satisfying (8) such that

$$\frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s [Q_2(u)\beta_2 + |f(u)|]\,duds \leqslant (-p-1)M_8 - \alpha, \quad t \geqslant t_1$$

$$\frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s [Q_1(u)\beta_1 + |f(u)|]\,duds \leqslant \alpha + p_0)M_7, \quad t \geqslant t_1$$

and

$$\frac{-1}{p}\left(1 + \frac{L}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s (Q_1(u)+Q_2(u))\,duds\right) \leqslant \theta_4 < 1, \quad t \geqslant t_1,$$

where θ_4 is a constant. Define a mapping $T : A \longrightarrow \Lambda$ as follows

$$(Tx)(t) = \begin{cases} \frac{1}{p(t+\tau)}\{-\alpha - x(t+\tau) + \frac{1}{(n-2)!}\int_{t+\tau}^\infty \frac{(s-t-\tau)^{n-2}}{r(s)} \\ \times \int_{t_1+\tau}^s \left[\int_{a_1}^{b_1} q_1(u,\xi)g_1(x(u-\xi))d\xi \right. \\ \left. - \int_{a_2}^{b_2} q_2(u,\xi)g_2(x(u-\xi))d\xi - f(u))\right] duds\}, \\ t \geqslant t_1 \\ (Tx)(t_1), \quad t_0 \leqslant t \leqslant t_1. \end{cases}$$

Clearly Tx is continuous. Since the rest of the proof is similar to that of Theorem 2, it is omitted.

Theorem 5. *Assume that* $0 \leqslant \int_{a_3}^{b_3} \tilde{p}(t,\xi)d\xi \leqslant p < 1$ *and (3) holds. Then (2) has a bounded nonoscillatory solution.*

Proof. Let Λ be the same set as in the proof of Theorem 1. Set

$$A = \{x \in \Lambda : N_1 \leqslant x(t) \leqslant N_2, \quad t \geqslant t_0\},$$

where N_1 and N_2 are positive constants such that

$$pN_2 + N_1 < N_2.$$

Let $\alpha \in (pN_2 + N_1, N_2)$, L_i, $i = 1,2$, denote Lipschitz constants of functions g_i, $i = 1,2$, on the set A, respectively and $L = \max\{L_1, L_2\}$, $\beta_i = \max_{x\in A}\{g_i(x)\}$, $i = 1,2$, respectively. From (3), one can choose a $t_1 > t_0$,

$$t_1 \geqslant t_0 + \max\{b_1, b_2, b_3\} \tag{12}$$

sufficiently large such that

$$\frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s [Q_1(u)\beta_1 + |f(u)|]\,duds \leqslant N_2 - \alpha, \quad t \geqslant t_1,$$

$$\frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s [Q_2(u)\beta_2 + |f(u)|]\,duds \leqslant \alpha - N_1 - pN_2, \quad t \geqslant t_1$$

and

$$p + \frac{L}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s (Q_1(u)+Q_2(u))\,duds \leqslant \theta_5 < 1, \quad t \geqslant t_1,$$

where θ_5 is a constant. Consider the operator $T : A \longrightarrow \Lambda$ defined by

$$(Tx)(t) = \begin{cases} \alpha - \int_{a_3}^{b_3} \tilde{p}(t,\xi)x(t-\xi)d\xi + \\ \frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s \left[\int_{a_1}^{b_1} q_1(u,\xi)g_1(x(u-\xi))d\xi \right. \\ \left. - \int_{a_2}^{b_2} q_2(u,\xi)g_2(x(u-\xi))d\xi - f(u)\right] duds, t \geqslant t_1 \\ (Tx)(t_1), \quad t_0 \leqslant t \leqslant t_1. \end{cases}$$

Clearly Tx is continuous. Since the remaining part of the proof is similar to that of Theorem 1, it is omitted.

Theorem 6. *Assume that* $-1 < p \leqslant \int_{a_3}^{b_3} \tilde{p}(t,\xi)d\xi \leqslant 0$ *and (3) holds. Then (2) has a bounded nonoscillatory solution.*

Proof. Let Λ be the same set as in the proof of Theorem 1. Set

$$A = \{x \in \Lambda : N_3 \leqslant x(t) \leqslant N_4, \quad t \geqslant t_0\},$$

where N_3 and N_4 are positive constants such that

$$N_3 < (1+p)N_4.$$

Let $\alpha \in (N_3, (1+p)N_4)$, L_i, $i = 1,2$, denote Lipschitz constants of functions g_i, $i = 1,2$, on the set A, respectively and $L = \max\{L_1, L_2\}$, $\beta_i = \max_{x\in A}\{g_i(x)\}$, $i = 1,2$, respectively. From (3), we can choose a $t_1 > t_0$ sufficiently large satisfying (12) such that

$$\frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s [Q_1(u)\beta_1 + |f(u)|]\,duds \leqslant (1+p)N_4 - \alpha, \quad t \geqslant t_1,$$

$$\frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s [Q_2(u)\beta_2 + |f(u)|]\,duds \leqslant \alpha - N_3, \quad t \geqslant t_1$$

and

$$-p+\frac{L}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s (Q_1(u)+Q_2(u))\,duds \leqslant \theta_6<1, \quad t\geqslant t_1,$$

where θ_6 is a constant. Consider the operator $T:A\longrightarrow \Lambda$ defined by

$$(Tx)(t) = \begin{cases} \alpha-\int_{a_3}^{b_3}\tilde{p}(t,\xi)x(t-\xi)d\xi+ \\ \frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_{t_1}^s \left[\int_{a_1}^{b_1} q_1(u,\xi)g_1(x(u-\xi))d\xi \right. \\ \left. -\int_{a_2}^{b_2} q_2(u,\xi)g_2(x(u-\xi))d\xi - f(u)\right] duds,\ t\geqslant t_1 \\ (Tx)(t_1), \quad t_0\leqslant t\leqslant t_1. \end{cases}$$

Clearly Tx is continuous. Since the rest of the proof is similar to that of Theorem 1, it is omitted.

Example 1. Consider the equation

$$\left[e^t\left[x(t)+(\frac{e^{-2t}+2}{e^3})x(t-3)\right]''\right]' - \left[\int_1^2 x(t-\xi)d\xi-\int_2^3 x(t-\xi)d\xi + e^{-t}(e^3-2e^2+e)-18e^{-2t}\right]=0, \qquad (13)$$

and note that $n=3$, $r(t)=e^t$, $p(t)=\frac{e^{-2t}+2}{e^3}$, $q_1(t,\xi)=q_2(t,\xi)=1$, $g_1(x)=g_2(x)=x$ and $f(t)=e^{-t}(e^3-2e^2+e)-18e^{-2t}$. The conditions of Theorem 1 are satisfied. In fact $x(t)=exp(-t)$ is a nonoscillatory solution of (13).

3 Conclusion

We considered the existence of bounded nonoscillatory solutions of the higher order nonlinear neutral nonhomogeneous equations with distributed deviating arguments. We presented four theorems for (1) and two theorems for (2) depending on the ranges of $p(t)$ and $\tilde{p}(t,\xi)$, and gave an example to support usability of our results.

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