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A Large Diffusion Expansion for the Transition Function of Lévy Ornstein-Uhlenbeck Processes

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Abstract: We consider the Lévy Ornstein- Uhlenbeck process X_t described by the equation $dX_t = -\lambda X_t\, dt + dL_t$, $\lambda > 0$ and L_t a Lévy white noise. The corresponding semigroup is expressed by an expectation with respect to a pure jump Ornstein- Uhlenbeck process. A large diffusion expansion is then obtained.
The expansion is organized by using suitable generalized Feynman graphs and rules. Applications on information sciences will be given.

Keywords: Lévy Ornstein- Uhlenbeck processes, large diffusion expansion, Feynman graphs and rules.

1 Introduction

We consider the Lévy Ornstein- Uhlenbeck (OU) equation for an $\mathbb{R}^d-$valued process X_t, $t \in [0,\infty[$:

$$\begin{cases} dX_t = -\lambda X_t\, dt + dL(t) & , \lambda > 0 \\ X_0 = x_0, \;\; (t,x_0) \in]0,\infty[\times\mathbb{R}^d, & \end{cases} \tag{1}$$

where $L_t = L(t)$ is a Lévy process in $\mathbb{R}^d$. See below for more details about L_t.
In their paper [21], Ornstein and Uhlenbeck studied a free particle in Brownian motion, moving in a rarefied gas and affected by a friction force proportional to the pressure. In order to understand the displacement process X_t of the particle, they investigated the velocity process $\upsilon(t) = \partial_t X_t$, which is known as the Ornstein-Uhlenbeck process. The work of Ornstein and Uhlenbeck continues Einstein's fundamental work (in 1905), see. [13] on Brownian motion itself and also the work of Smoluchowski (in 1906), see. [26] who, derived the Fokker-Planck equation for the OU process and also determined the transition density.
After Paul Lévy's characterisation in the 1930s of all processes with stationary independent increments, many researchers were interested in detailed properties of the distributions of these processes. Moreover, important classes of stochastic processes were obtained as generalizations of the class of Lévy processes. An important one, that attract researchers due to its large applications, especially in Mathematical Finance, is the Lévy Ornstein-Uhlenbeck process. The latter is used for example as volatility process in stochastic volatility models, see, e.g, [10].
The aim of this work is to provide expansions for the transition semigroup for the Lévy OU-process.
In the Gaussian case one has the well known "Mehler's Formula", for the transition semigroup, which we denote by $P_t^M(x_0, dx)$, $t > 0$, $x, x_0 \in \mathbb{R}^d$:

$$P_t^M(x_0,dx) = (\det\mathbf{D})^{-\frac{1}{2}}\left(\frac{2\pi}{\lambda}(1-e^{-2\lambda t})\right)^{-\frac{d}{2}} \times \exp\left\{\alpha_t \langle (x-e^{-\lambda t}x_0), \mathbf{D}^{-1}(x-e^{-\lambda t}x_0)\rangle\right\}dx, \quad (E1)$$

where $\mathbf{D} = (D_{ij})_{i,j=1,\ldots,d}$ is a symmetric strictly positive definite matrix and $\alpha_t = -\frac{1}{2}\frac{\lambda}{1-e^{-2\lambda t}}$, $t > 0$.
Using the terminology in [23], the Lévy process $\{L_t, t > 0\}$ with both Gaussian and non-Gaussian component, is given by its characteristic (or generating) functional:

$$\mathbb{E}(e^{i\langle \xi, L_t\rangle}) = e^{t\,\psi(\xi)}, \forall t \geq 0, \xi \in \mathbb{R}^d, \tag{2}$$

with ψ a function from $\mathbb{R}^d$ into $\mathbb{R}$ of the form

$$\psi(\xi) = i\langle a,\xi\rangle - \langle \xi, \mathbf{D}\,\xi\rangle + \int_{\mathbb{R}^d\setminus\{0\}} (e^{i\langle x,\xi\rangle} - 1)\,\nu(dx). \tag{3}$$

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Here $a \in \mathbb{R}^d$ and $\mathbf{D}$ is as above, ν is a positive Lévy measure satisfying:

$$\int_{|x|\leq 1} |x|\, \nu(dx) < \infty, \quad \int_{|x|\geq 1} |x|^2\, \nu(dx) < \infty. \tag{4}$$

Later on we shall also assume that the Fourier transform $\hat{\nu}$ of ν is a smooth function on $\mathbb{R}^d$.
The transition semigroup $P_t(x_0, dx)$ of L_t is then the kernel of the semigroup P_t with infinitesimal generator G, s.t. $P_t = e^{tG}$ on $C_0(\mathbb{R}^d)$ (the space of continuous functions vanishing at infinity equipped with sup-norm) and given by

$$(Gf)(x) = -\sum_{j=1}^{d} a_j \frac{\partial}{\partial x_j} f(x) + \sum_{j,l=1}^{d} D_{jl} \frac{\partial^2}{\partial x_j \partial x_l} f(x) + \int_{\mathbb{R}^d\setminus\{0\}} [f(x+y) - f(x)]\, \nu(dy),\ x \in \mathbb{R}^d, \tag{5}$$

for $f \in C_0^2(\mathbb{R}^d)$, see, eg., [[23], p. 208]. P_t satisfies then the corresponding jump-diffusion Kolmogorov equation

$$\frac{\partial}{\partial t} P_t f = G P_t f,\ t > 0 \tag{6}$$

The large diffusion expansion of the transition density will be given by a power series in a parameter β (proportional to the inverse of the determinant of $\mathbf{D}$), the latter series expansion is not convergent and has complicated terms, however it is well interpreted using the recently discovered generalized Feynman graphs and Feynman rules, see, e.g. [14,15,16,25],the later will help in clarifying the problems related with this expansion.
The graphical representation for the large diffusion expansion of the transition probabilities of the Lévy OU process, as done in this work, seems to be new even for classical OU processes. It simplifies the analytic expressions and could lead to a lot of applications , as in the simulation of Lévy OU processes, see, e.g. [28], in applications to mathematical finance, see, e.g, [10], neurobiology, see, e.g. [4], quantum field theory and statistical mechanics, see, e.g, [2,3].
Before we go over to describe the contents of the present paper, let us mention that our study of SDE's of type (1) can be extended to other classes of non linear S(P)DEs, such that KPZ equations, see, e.g, [25], and the beam epitaxy equations, see, e.g, [19]. Also let us mention that, to the best of our knowledge, graphical representations and linked cluster theorems for Lévy OU processes have not been considered before.
The remainder of this paper is organized as follow:
In section 2 we present some results on Lévy noise, that help the reader to understand the other sections of this work. Section 3 is devoted to the study of the transition density of the Lévy OU process, the latter will be be given by a series which is not convergent but has a meaning as an asymptotic series. We recall then a basic result known as "Linked Cluster" theorem.
The aim of section 4 is to introduce generalized Feynman graphs and rules, we achieve then our main result by giving a graphical representation of the large diffusion expansion of the transition density of the Lévy OU process.
Section 5 is devoted to some applications on information systems.

2 Lévy noise

In this section we recall some properties of the Lévy noise $L_t, t \geq 0$, that are useful for the current work.
We define the compensated jump measure $\tilde{N}$, also called the compensated Poisson random measure by:

$$\tilde{N}(dt,dz) := N(dt,dz) - \nu(dz)\,dt, \tag{7}$$

where $N(t,B) := \sum_{0\leq s\leq t} \chi_B(\Delta L(s))$, B a Borel set in $\mathbb{R}^d$, and $\Delta L(s) := L(s) - L(s^-)$ is the jump of L at time s, we have $\nu(B) = \mathbb{E}(N(1,B))$.
By a result of Itô and Lévy, see, e.g. [23], [[8], p. 108-109], the Lévy process $L_t = L(t), t \geq 0$, admits the following integral representation:

$$L(t) = at + DW(t) + \int_0^t \int_{|z|<1} z N(ds,dz) + \int_0^t \int_{|z|\geq 1} z N(ds,dz), \tag{8}$$

for some constant $a \in \mathbb{R}^d$ and with D as in section1 . Here $W = W(t), t \geq 0$ is a standard Wiener process on $\mathbb{R}^d$.
By assuming $\mathbb{E}[|\ L(t)\ |^2] < \infty, t \geq 0$, then $\int_{|z|\geq 1} |\ z\ |^2\ \nu(dz) < \infty$ and the representation (8) becomes

$$L(t) = a_1 t + DW(t) + \int_0^t \int_{|z|<1} z \tilde{N}(ds,dz) + \int_0^t \int_{|z|\geq 1} z N(ds,dz), \tag{9}$$

where $a_1 = a + \int_{|z|\geq 1} z\, \nu(dz)$.
A Lévy process L_t which satisfies the representation (9) with $a_1 = D = 0$ is called a pure jump Lévy process(it is without Gaussian and deterministic components). If $\tilde{L}(t)$ is a pure jump Lévy process on $\mathbb{R}^d$, then its characteristic function is given by:

$$E\left(e^{i\langle u,\tilde{L}(t)\rangle}\right) = e^{t\int_{\mathbb{R}^d\setminus\{0\}}\left(e^{i\langle u,y\rangle}-1\right)\nu(\mathrm{d}y)}, \tag{10}$$

$u \in \mathbb{R}^d$ and ν is the intensity measure, also called Lévy measure, satisfying $\int_{\mathbb{R}^d\setminus\{0\}}(|\ y\ |^2 \wedge 1)\, \nu(dy) < \infty$.
For information on Lévy processes and related equations, see, e.g., [8],[11], [18] , [23] .
Moreover, the Lévy-Itô decomposition (8) for $\tilde{L}_t$ takes then the form:

$$\tilde{L}_t = \int_B x N(t,\mathrm{d}x) + \int_{\mathbb{R}^d\setminus B} x N(t,\mathrm{d}x),\ t \geq 0, \tag{11}$$

where, in this case, N is a Poisson random measure on $\mathbb{R}_+ \times (\mathbb{R}^d - \{0\})$ (the Poisson random measure associated with the jumps $\Delta Z_t := \tilde{L}_t - \tilde{L}_{t_-}$, i.e. $N([0,t)\times A) = \{0 \le s < t | \Delta Z_s \in A\}$, for each $t \ge 0$, $A \in \mathscr{B}(\mathbb{R}^d \backslash \{0\})$, $\tilde{N}(t,A) := N(t,A) - t\nu(A)$, for all $A \in \mathscr{B}(\mathbb{R}^d, \{0\})$, $0 \in \overline{A}, \overline{A}$ the closure of A.) We have $\nu(A) = E(N(1,A))$; for each $t > 0$, $\omega \in \Omega$, $\tilde{N}(t,\cdot)(\omega)$ is the compensated Poisson random measure (to $N(t,\cdot)(\omega)$) on the Borel $\sigma-$algebra $\mathscr{B}(\mathbb{R}^d \backslash \{0\})$; $\tilde{N}(t,A)$, $t \ge 0$ is, in particular, a martingale-valued measure.

3 Large diffusion expansion for the transition probabilities

In this section the transition probability density of the Lévy OU process X_t will be given by a series which is not convergent in general but rather has only the meaning of an asymptotic series, see, e.g, [14,27]. Under the use of Linked Cluster Theorem we prove a large diffusion expansion for the transition density of the Lévy OU process.
From known results, see, e.g., [8], [23], under the above assumptions on L_t, equation (1) has a unique strong solution $X_t, t \ge 0$ which can be assumed to be a *càdlàg* process and is given by

$$X_t = e^{-\lambda t} x_0 + \int_0^t e^{-\lambda (t-s)} dL_s, x_0 \in \mathbb{R}^d, \qquad (12)$$

the integral being a stochastic one. We shall call X_t Lévy OU process.
Let $(\Omega, \mathscr{B}, P)$ the probability space underlying X_t. Then X_t is a Markov process, with transition semigroup $P_t(x_0, dx)$, $x_0, x \in \mathbb{R}^d$. One has thus

$$P_t(x_0, A) = \int_A P_t(x_0, dy) = P(X_t \in A \mid X_0 = x_0), \qquad (13)$$

for any Borel subset A of $\mathbb{R}^d$.
It is proven, e.g., in [[23], p. 106], that if L_t is given as above by the characteristic functional (2) then

$$\int_{\mathbb{R}^d} e^{i\langle \xi, y\rangle} P_t(x,dy) = \exp\Big[i e^{-\lambda t}\langle \xi, x\rangle + \int_0^t \psi(e^{-\lambda s}\xi) ds\Big], \xi, x \in \mathbb{R}^d, \qquad (14)$$

with ψ given by (3).
If P_t^J is the transition semigroup of a pure jump Lévy O-U process Y_t, satisfying (12) with L replaced by $\tilde{L}$ as given in (10), then P_t^J satisfies:

$$\int_{\mathbb{R}^d} e^{i\langle \xi, y\rangle} P_t^J(x,dy) = \exp\Big[i e^{-\lambda t}\langle \xi, x_0\rangle + \int_0^t \tilde{\psi}(e^{-\lambda s}\xi) ds\Big], \xi \in \mathbb{R}^d, \qquad (15)$$

where

$$\tilde{\psi}(\xi) = \int_{\mathbb{R}^d \backslash \{0\}} (e^{i\langle x, \xi\rangle} - 1)\nu(dx). \qquad (16)$$

Let P_t^M be the Gaussian transition density of the Lévy OU process given by equation (E1), if we denote this process by X_t then X_t satisfies (1)with $L := L_t^d$ given by the generating triplet $(2D, 0, 0)_0$.
For simplicity the matrix $\mathbf{D}$, in equation (3), will henceforth be taken to be diagonal, i.e: $\mathbf{D} = D \cdot \mathbf{1}$, where D is constant and $\mathbf{1}$ the unit matrix on $\mathbb{R}^d$.
In the following we set: $\beta = \frac{1}{2D}(\frac{\lambda}{1-e^{-2\lambda t}}), t \ge 0$.

Proposition 1 *Let* $C_t = (\frac{\pi}{\beta})^{-\frac{d}{2}}, t \ge 0$, $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ *and* $V(x) := | x |^2$.
The transition semigroup of the Lévy OU process X_t *solution of (1) satisfies:*

$$P_t(x_0, A) = C_t\, \mathbb{E}_Y\Big[\int_A e^{-\beta V(y - Y_t)} dy\Big], t > 0 \qquad (17)$$

for any Borel subset A *of* $\mathbb{R}^d$, *where* $\mathbb{E}_Y$ *is the expectation with respect to the probability measure for the pure jump type Lévy OU process* Y_t *satisfying* $dY_t = -\lambda\, dY_t + d\tilde{L}_t$, $Y_0 = x_0$, *with* $\tilde{L}_t$ *given by (10).*

Proof. Following [[24], Lemma 2.2], X_t can be decomposed into two processes Y_t, R_t s.t. $X_t = Y_t + R_t$, and Y_t, R_t are independent of each other and

1. Y_t is a solution of the equation (1) with L_t replaced by $\tilde{L}_t^J$, and $\tilde{L}_t^J$ of pure jump type, with characteristic functional $\tilde{\psi}$ given by equation (16), the initial condition being $Y_0 = x_0$;
2. R_t is a solution of the equation (1) with L_t replaced by $L_t^d := L_t - \tilde{L}_t$, and initial condition $R_0 = 0$, i.e., R_t satisfies $dR_t = -\lambda R_t\, dt + dL_t^d$.

We have, for any Borel subset A in $\mathbb{R}^d$:
(i)- $P_t(x_0, A) = \mathbb{E}\Big(\chi_A(X_t)\Big)$ where $\mathbb{E}$ is the expectation with respect to the probability measure underlying X_t (started at $X_0 = x_0$)(see (13)).
(ii)-Using the decomposition $X_t = Y_t + R_t$ we have

$$P_t(x_0, A) = P(X_t \in A) = \mathbb{E}\Big(\chi_A(Y_t + R_t)\Big). \qquad (18)$$

But

$$\begin{aligned}\mathbb{E}\Big(\chi_A(Y_t + R_t)\Big) &= \mathbb{E}\Big[\mathbb{E}\Big(\chi_A(Y_t + R_t) \mid \sigma(Y_t)\Big)\Big] \\ &= \mathbb{E}\Big[\mathbb{E}_Y\Big(\chi_{A-Y_t}(R_t) \mid \sigma(Y_t)\Big)\Big] \\ &= \mathbb{E}\Big[\mathbb{E}_Y\Big(\chi_{A-Y_t}(R_t)\Big)\Big], \qquad (19)\end{aligned}$$

since the $\sigma-$algebra $\sigma(Y_t)$ generated by Y_t is independent of R_t, and where in the last equality we have used Fubini theorem. $\mathbb{E}_Y$ stands for expectation with respect to Y.
Using Fubini theorem to interchange $\mathbb{E}_Y$ and $\mathbb{E}$, the definition of P_t^M, and recalling the definitions of β, C_t, V then yields the result. ■

Remark 1 *1.The expression given by equation (17) looks like a sum over states in statistical mechanics, where* V *is the potential energy and* $e^{-\beta V}$ *is a Boltzmann weight, see, e.g. [22],* 2.*In the Gaussian case studied by Einstein the coefficient D is determined in terms of molecular quantities.*

3.*The case where $\psi(t)$ in (3) is purely Poisson, i.e, $\nu \neq 0$ and $a = \sigma^2 = 0$, the coordinate process L_t is then a marked Poisson process with intensity z, if we write $\nu = z\mu$ and it can be interpreted as a non-interacting classical continuous particle in the configurational grand canonical ensemble with activity z, see, e.g., [22], where each particle carries a distributed random charge. The case, where both a Gaussian and a Poisson process contribute to the random process L, i.e, $z > 0$ and $\sigma^2 > 0$, can be interpreted as a grand canonical ensemble mesoscopic charge particle (Poisson contribution) and a white noise fluctuation of the charge density to microscopic particles(Gaussian contribution). Moreover the Gaussian contribution can be seen as a scaling limit of a Poisson contribution, see, e.g. [3].*

The idea is now to perform a "high temperature" T_β-expansion of P_t corresponding to a small β, ($\beta = \frac{1}{kT_\beta}, k > 0, T_\beta > 0$), which is in our case, since β is proportional to $\frac{1}{D}$, as given by Prop. (1) is a *large diffusion expansion*, looking upon D as a diffusion coefficient.

Proposition 2 *The large diffusion expansion of the transition semigroup of the Lévy OU process $X_t, t \geq 0$ in Prop. (1), is given by a formal expansion in power of β, i.e:*

$$P_t(x,A) = C_t \sum_{m=0}^{\infty} \frac{(-\beta)^m}{m!} \sum_{k=0}^{m} \sum_{l=0}^{m-k} (-2)^{m-k-l} \binom{m}{k} \binom{m-k}{l} \times \int_A y^{2k} \mathbb{E}[Y_t^{2l} \langle y, Y_t \rangle^{m-k-l}]\, dy, \tag{20}$$

Y_t is the pure jump process described in Prop. (1) and $\mathbb{E}$ is the corresponding expectation.

Proof. Expanding the exponential function under the expectation in (17), we get

$$P_t(x_0,A) = C_t \mathbb{E}[\int_A e^{-\beta V(y-Y_t)}]\, dy = C_t \mathbb{E}\left(\int_A \sum_{m=0}^{\infty} \frac{(-\beta)^m}{m!} V^m(y-Y_t) \right) dy. \tag{21}$$

If we now exchange, formally, the sum with the integral and use Fubini theorem we get

$$P_t(x_0,A) = C_t \sum_{m=0}^{\infty} \frac{(-\beta)^m}{m!} \int_A \mathbb{E}[V^m(y-Y_t)]\, dy. \tag{22}$$

Since by definition $V(x) = \mid x \mid^2$, we have

$$V^m(y-Y_t) = \sum_{k=0}^{m} \binom{m}{k} y^{2k} \sum_{l=0}^{m-k} (-2)^{m-k-l} \binom{m-k}{l} Y_t^{2l} \times \langle y, Y_t \rangle^{m-k-l}. \tag{23}$$

Inserting now equation (23) into (21) yields the results. ■

The Fourier transform (or characteristic function) of the process Y_t is given, for $k \in \mathbb{R}^d$, see (15), by:

$$\begin{aligned}\mathscr{F}(Y_t)(k) &= \mathbb{E}[e^{i\langle Y_t,k\rangle}] \\ &= \int_\Omega e^{i<Y_t(w),k>} P(dw) \\ &= \int_{\mathbb{R}^d} e^{i<y,k>} P_{Y_t}(dy) \\ &= \exp\left[ie^{-\lambda t}\langle k, x_0\rangle + \int_0^t \tilde{\psi}(e^{-\lambda(t-s)}k)\, ds\right], \end{aligned} \tag{24}$$

here P_{Y_t} is the image measure of P under the map $\Omega \longrightarrow \mathbb{R}^d$ given by Y_t and $P_{Y_t}(dy) = P_t(0,dy)$ and $\tilde{\psi}$ is the characteristic function given by equation (16).

In the following we compute $\mathbb{E}[Y_t^r \langle y, Y_t\rangle^s]$, for $r,s \in \mathbb{N}$. For simplicity we write the formulae for $d = 1$, an easy adaptation yields the case $d > 1$. Let $J \subseteq \mathbb{N}$ be a finite set. The collection of all partitions of J is denoted by $\mathscr{P}(J)$. A partition is a decomposition of J into disjoint, non-empty subsets, i.e. $I \in \mathscr{P}(J) \Longleftrightarrow \exists k \in \mathbb{N}, I = \{I_1,\cdots,I_k\}, I_j \subseteq S, I_j \cap I_l = \emptyset \forall 1 \leq j < l \leq k, \cup_{l=1}^k I_l = J$.

Definition 1 *Let $X_1,\cdots,X_n, n \in \mathbb{N}$, be $\mathbb{R}-$valued random variables on some probability space $(\Omega,\mathscr{A},P)$. Denote $\langle X_1 \cdots X_n\rangle := \mathbb{E}(X_1 \cdots X_n)$. Let $I = \{I_1,\ldots,I_k\}$ a partition of the set $\{1,\ldots,n\}$, then $\langle X_1 \cdots X_n\rangle$ are the moments of P and the truncated moments functions $\langle X_1 \cdots X_n\rangle^T$ are recursively defined by*

$$\langle X_1 \cdots X_n \rangle = \sum_{\substack{I \in \mathscr{P}(\{1,\ldots,n\}) \\ I=\{I_1,\ldots,I_k\}}} \prod_{l=1}^{k} \left\langle \prod_{j \in I_k} X_j \right\rangle^T \tag{25}$$

where, for a finite set A, $\mathscr{P}(A)$ stands for the set of all partitions I of A into nonempty disjoint subsets $\{I_1,\ldots,I_k\}$.

Proposition 3 *Assume the Lévy measure ν has all moments and let K be a subset of $\{1,\cdots,2n\}$ then in the sense of formal power series:*

$$P_t(x_0,A) = C_t \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} \sum_{j_1,\ldots,j_{2n}=1}^{d} \delta_{j_1 j_{n+1}} \cdots \delta_{j_n j_{2n}} \times \sum_{K \subseteq \{1\cdots 2n\}} (-2)^{\sharp K} \int_A \prod_{l \in K} (-y_{j_l}) \mathbb{E}[\prod_{l \in K^c} Y_{t_{j_l}}]dy, \tag{26}$$

where $K^c = \{1\cdots 2n\} \backslash K$ and δ_{jk} being the Kronecker symbol and Y and y are as in Prop. 1.

Proof. From (21) and the definition of V we have:

$$P_t(x_0,A) = C_t\sum_{n=0}^{\infty}\frac{(-\beta)^n}{n!}\int_A \mathbb{E}[V^n(y-Y_t)]dy$$
$$= C_t\sum_{n=0}^{\infty}\frac{(-\beta)^n}{n!}\sum_{j_1,\dots,j_n=1}^{d}\int_A \mathbb{E}[\prod_{l=1}^{n}(Y_{t_{j_l}}-y_{j_l})^2]dy$$
$$= C_t\sum_{n=0}^{\infty}\frac{(-\beta)^n}{n!}\sum_{j_1,\dots,j_{2n}=1}^{d}\delta_{j_1 j_{n+1}}\cdots\delta_{j_n j_{2n}}$$
$$\times\int_A \mathbb{E}[\prod_{l=1}^{2n}(Y_{t_{j_l}}-y_{j_l})]dy$$
$$= C_t\sum_{n=0}^{\infty}\frac{(-\beta)^n}{n!}\sum_{j_1,\dots,j_{2n}=1}^{d}\delta_{j_1 j_{n+1}}\cdots\delta_{j_n j_{2n}}$$
$$\times\sum_{K\subseteq\{1\cdots 2n\}}(-2)^{\sharp K}\prod_{l\in K}(-y_{j_l})\int_A \mathbb{E}[\prod_{l\in K^c}Y_{t_{j_l}}]dy$$

■

Note that the moments of a product of random variables $Y_{t_{j_l}}$ can be calculated by differentiation of the Fourier transform $\hat{P}_{Y_{t_{j_l}}}(k)$ of the laws $P_{Y_{t_{j_l}}}(k)$ at $k=0$:

$$\mathbb{E}[\prod_{l\in K^c}Y_{t_{j_l}}] = (-i)^{\sharp K^c}(\prod_{l\in K^c}\frac{\partial}{\partial k_{j_l}})\hat{P}_{t_{j_l}}(k)\mid_{k=0}$$
$$= (-i)^{\sharp K^c}(\prod_{l\in K^c}\frac{\partial}{\partial k_{j_l}})\exp\{ie^{-\lambda t}\langle k,x_0\rangle$$
$$+\int_0^t\tilde{\psi}(e^{-\lambda(t-s)}k)ds\}\mid_{k=0}. \tag{27}$$

The last equation is based on the fact that if the process Y_t is given by $Y_t = e^{-\lambda t}x_0 + (K_\lambda\tilde{L})(t)$, where $(K_\lambda\tilde{L})(t) = \int_{\mathbb{R}}K_\lambda(t,s)\tilde{L}(ds)$, with $K_\lambda(t,s) = \chi_{[0,t]}(s)e^{-\lambda(t-s)}$, and if $f_1,\dots,f_n$ are given functions then

$$\mathbb{E}[\prod_{l\in K^c}\langle f_{j_l},Y_{t_{j_l}}\rangle] = \sum_{\substack{I\in P(K^c)\\ I=\{I_1\cdots I_k\}}}\prod_{l=1}^{k}c_l\int_{\mathbb{R}}K_\lambda f_{j_l}ds_l, \tag{28}$$

where $c_l = \int_{\mathbb{R}\setminus\{0\}}s^l\,\nu(ds)$.

In the following we set: $\partial_J = \frac{\partial}{\partial k_{j_1}\cdots\partial k_{j_q}}$ where $J=\{j_1,\cdots\cdots j_q\}$.

Lemma 1 *Let f and g be two functions defined on $\mathbb{R}$, differentiable k times, then the following holds:*

$$\partial_J f\circ g = \sum_{k=1}^{\sharp J}f^{(k)}\circ g\sum_{\substack{I\in P_k(J)\\ I=\{I_1\cdots I_k\}}}\prod_{l=1}^{k}\partial_{I_l}g \tag{29}$$

where $f^{(k)}(y) = \frac{d^k}{dy^k}f(y)$ and $P_k(J)$ is a $k-$partitions of J, i.e, $P_k(J) = \{I=\{I_1,\cdots I_k\}: I_l\neq\emptyset, I_l\cap I_{l'}=\emptyset, \bigcup I_l = J\}$.

Proof. The proof is immediate by induction over $q=\sharp J$,. ■

The following analogue of the ”Linked Cluster” Theorem of statistical mechanics gives a connection between the ordinary expectations of products of Y_t and the truncated ones:

Theorem 1 *Let K be a subset of $\{1,\dots,2n\}$ and Y_t be the Lévy OU process described in Proposition 1, then:*

$$\mathbb{E}[\prod_{l\in K^c}Y_{t_{j_l}}] = \sum_{\substack{I\in P(K^c)\\ I=\{I_1\cdots I_k\}}}\prod_{l=1}^{k}\langle\prod_{q\in I_l}Y_{t,q}\rangle^T \tag{30}$$

Proof. The proof is based on lemma (1), by taking $f\circ g = \exp(g)$, we have then:

$$\mathbb{E}[\prod_{l\in K^c}Y_{t_{j_l}}] = \sum_{\substack{I\in P(K^c)\\ I=\{I_1\cdots I_k\}}}\partial_{I_l}\exp\{\int_0^t\tilde{\psi}(e^{-\lambda(t-t')}k)dt'\}_{|k=0}$$
$$= \sum_{\substack{I\in P(K^c)\\ I=\{I_1\cdots I_k\}}}\prod_{l=1}^{k}\langle\prod_{q\in I_l}Y_{t,q}\rangle^T$$

■

The following first main result holds:

Theorem 2 *The large diffusion expansion for the transition semigroup of the Lévy OU process X_t, defined by equation (1) is given, in the sense of formal power series, by:*

$$P_t(x_0,A) = C_t\sum_{n=0}^{\infty}\frac{(-\beta)^n}{n!}\sum_{j_1,\dots,j_{2n}=1}^{d}\delta_{j_1 j_{n+1}}\cdots\delta_{j_n j_{2n}}\sum_{K\subseteq\{1\cdots 2n\}}(-2)^{\sharp K}$$
$$\times\int_A\prod_{q\in K}(-y_{j_q})\sum_{\substack{I\in P(K^c)\\ I=\{I_1\cdots I_k\}}}\prod_{l=1}^{k}\langle\prod_{q\in I_l}Y_{t,q}\rangle^T dy \tag{31}$$

Proof. From proposition (3) and theorem (1) we have:

$$P_t(x_0,A) = C_t\sum_{n=0}^{\infty}\frac{(-\beta)^n}{n!}\sum_{j_1,\dots,j_{2n}=1}^{d}\delta_{j_1 j_{n+1}}\cdots\delta_{j_n j_{2n}}\sum_{K\subseteq\{1\cdots 2n\}}(-2)^{\sharp K}$$
$$\times\prod_{l\in K}(-y_{j_l})\mathbb{E}[\prod_{l\in K^c}Y_{t_{j_l}}]$$
$$= C_t\sum_{n=0}^{\infty}\frac{(-\beta)^n}{n!}\sum_{j_1,\dots,j_{2n}=1}^{d}\delta_{j_1 j_{n+1}}\cdots\delta_{j_n j_{2n}}\sum_{K\subseteq\{1\cdots 2n\}}(-2)^{\sharp K}$$
$$\times\int_A\prod_{l\in K}(-y_{j_l})\sum_{\substack{I\in P(K^c)\\ I=\{I_1\cdots I_k\}}}\prod_{l=1}^{k}\langle\prod_{q\in I_l}Y_{t,q}\rangle^T dy \tag{32}$$

■

Remark 2 *The disadvantages of the right hand side in the formula given by theorem (2) is that the corresponding N-truncated solution is a polynomial in x_0 and can not be used as a probability density. In addition the obtained formula is quite complicated!, We shall see, in the next section, that these disadvantages will be solved by the Feynman graphs techniques.*

Table 1: Different types of vertices.

	Empty
Inner	$\circ$
Outer	$\otimes$

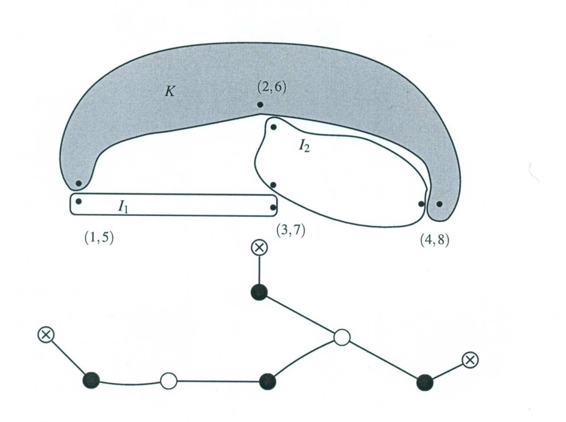

Figure 1: Construction of a generalized Feynman graph from the set K and the partition $I=\{I_1,I_2\}$

4 Feynman graph representation of the large diffusion expansion

The main objective of this section is to introduce the generalized Feynman graphs and Feynman rules to solve the disadvantages of the formula given by Theorem 2.
Let us consider the $n-$th terms in the large diffusion expansion given by Theorem 2, i.e :

$$\sum_{j_1,\ldots,j_{2n}=1}^{d}\delta_{j_1 j_{n+1}}\cdots\delta_{j_n j_{2n}}\sum_{K\subseteq\{1\cdots 2n\}}(-2)^{\sharp K}\int_A\prod_{q\in K}(-y_{j_q})$$
$$\times\sum_{\substack{I\in P(K^c)\\ I=\{I_1\cdots I_k\}\\ \sharp I_l=m_l}}\prod_{l=1}^{k}\langle\prod_{q\in I_l}Y_{t,q}\rangle^T dy. \qquad (33)$$

Definition 2 *A generalized Feynman graph is a graph with two types of vertices called inner empty and outer empty and directed, distinguishable edges. Outer empty vertices have only one leg, inner empty vertices have an arbitrary number of legs.*
The inner empty and the outer empty vertices are indistinguishable and have indistinguishable legs (i.e. are graphs that differ only by a relabling of such vertices, the legs are identified.)
The number of edges n, connected to the inner vertices, is also called the order of the Feynman graph.

We denote the set of all generalized Feynman graphs of order n by $\mathscr{F}_2(n)$.
The different types of vertices of a generalized Feynman graph are summarized in Table 1.

Lemma 2 *There exists a one to one correspondence between the set of pairs $\{(K,I): I=\{I_1,\cdots,I_k\}, K\subseteq\{1\cdots 2n\}\}$ and $\mathscr{F}_2(n)$.*

Proof. We have n pairs of indices $(1,n)\cdots(n,2n)$ and we have to choose a set K from $\{1\cdots 2n\}$ and then a partition of the remaining points. Now we represent the pairs by vertices with two legs, the legs in K by an outer empty vertex and we connect the legs in $I_1,\cdots,I_k$ to the inner empty vertices. ∎
Figure 1, gives an example of construction of a generalized Feynman graph from the set K and the partition $I=\{I_1,I_2\}$.
The following rule gives an analytic value to a generalized Feynman graph:

Definition 3 *We obtain an analytic value $\vartheta:\mathscr{F}_2(n)\longrightarrow\mathbb{R}$ for a generalized Feynman graph as follow:*

i). For every edge choose a number $l\in\{1,\cdots n\}$ and an index $j_l\in\{1\cdots d\}$.
ii). For every outer empty vertex connected to the $l-th$ edge multiply by $\int_A(-y_{j_l})dy$, $A\in\mathscr{B}(\mathbb{R}^d)$.
iii). For every inner empty vertex with l legs connected with the edges $q_1,\cdots,q_l$ multiply by $(-2)^l c_l$.
iv). Sum up over $j_1\cdots j_n$.

Remark 3 *Here we used "topological graphs", i.e, the direction of edges and their labeling is being neglected. Therefore we obtain multiplicity factors in front of each graph. We note the set of all topological graphs by $\mathscr{G}$, such that $\mathscr{G}=\{G'\in\mathscr{F}_2(n), G'$ differs from $\mathscr{G}$ only by labeling.$\}$*
The multiplicity of a graph $G\in\mathscr{F}_2(n)$ is equal to the cardinality of the topological graph G'.
Note that $\vartheta(G)$ only depends on the topological graphs!

From theorem (2), definition (2) and lemma (2), we are now able to state the second main result of this work:

Theorem 3 *The large diffusion expansion for the transition semigroup of the Lévy OU process X_t given by equation (1) is given by a sum over all generalized Feynman graphs $G\in\mathscr{F}_2(n)$ that are evaluated according to the rule fixed in definition (3), i.e,*

$$P_t(x_0,A)=C_t\sum_{n=0}^{\infty}\frac{(-\beta)^n}{n!}\sum_{G\in\mathscr{F}_2(n)}\vartheta(G)(x_0,A), \qquad (34)$$

where C_t is as given by Prop. (1).

Remark 4 *The series given by equation (34) in general diverge, but it can be given the meaning of an asymptotic series, which can be studied in a future work, we can refer for the moment to some results found in [4], [6].*

5 Applications on information systems

The current work can be used to simplify the complexity of the online social networks, which has been recently exploded in popularity and increasingly used from 2010 to 2015, as an example we can cite Google web, citations,... . This is really the case of many others online networks.
Our graph formalism can be used to represent such online networks by a Feynman graph where the edges of the graph are the relationships between peoples in a given office while the vertices are persons in such office.
Another example is to look to a given web as a Feynman graph where the vertices are individual sites or pages whereas the edges are the links between them, the previous section of the current work can be used to find the probability that a random web surfer, looked as a stochastic process, will be at a given page and by the same method developed in section 4 of this work, one can compute easily the transition probability densities of such process.
The current Feynman graphs are also considered one of the important modeling objects in many modern areas, such as online networks, they can analyse in a simple way the networks, complex networks from biological systems and image segmentation. Moreover the Feynman rules or algorithms as developed in section 4 of the current work simplify the graphs of any complex online networks, see.e.g, [9], with hundred million edges and several millions vertices ,and are simple to implement. In addition the approximations errors are bounded due to their stochastic nature.
Finally the main advantage of our graphical representations, as it s done in the current work, is that it allows researchers to easily handle online networks that are very large, with millions vertices and hundreds of millions of edges.

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