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Annular Regions Containing All the Zeros of a Polynomial via Special Numbers

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Abstract: In this paper, we obtain some results concerning annular regions containing all the zeros of a given polynomial. These annular regions have radii in terms of the Bell numbers, Pell numbers, Stirling numbers, Fibonacci numbers, Motzkin numbers, Catalan numbers, and/or the Schröder numbers. Also, we show, by means of examples, that for some polynomials our results sharpen some of the known results in this direction.

Keywords: zeros, polynomial, annular region.

1 Introduction

Let $p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n$ be a polynomial of degree n. By the Fundamental Theorem of Algebra (historically, the first important result concerning the roots of an algebraic equation), $p(z)$ has exactly n zeros in the complex plane, counting multiplicity. But this theorem does not say anything regarding the location of zeros of the polynomial, that is, the region which contains some or all of the zeros of a polynomial. Problems involving location of the zeros of a polynomial, besides being of theoretical interest, find applications in many areas of applied mathematics such as coding theory, cryptography, combinatorics, number theory, mathematical biology and engineering [2,6,21,25,28,30]. In particular, problems dealing with location of zeros of the polynomial play an important role, for example, in solving digital audio signal processing problems [35], control engineering problems [5], and eigenvalue problems in mathematical physics [34].

Since Abel and Ruffini proved that there is no general algebraic solution to polynomial equations of degree five or higher, the problem of finding a region containing all the zeros of a polynomial became much more interesting, and over a period a large number of results have been provided in this direction. It may be remarked that there are methods, for example Ehrlich-Aberth's type (see [1,17,27]) for the simultaneous determination of the zeros of algebraic polynomials, and there are studies to accelerate convergence and increase computational efficiency of these methods (for example, see [24,29]). These methods which are of course very useful, because they give approximations to the zeros of a polynomial can possibly become more efficient when combined with the results dealing with the region containing all the zeros of a polynomial, because an accurate estimate of the annulus containing all the zeros of a polynomial can considerably reduce the amount of work needed to find exact zeros, and so there is always a need for better estimates for the region containing all the zeros of a polynomial. Several monographs have been written on this subject and related subject of approximation theory (for example, see [9,23,25,26,32]).

To see how the study of the location of zeros of a polynomial can be useful in control theory, let us consider a transfer function $H(s)$ in a dynamical system. If we have an input function, say, $X(s)$, and an output function $Y(s)$, we define $H(s) = \frac{Y(s)}{X(s)}$. In discrete time systems, the function can also be written as $H(z) = \frac{Y(z)}{X(z)}$ and is often referred to as the pulse transfer function. The zeros z_i of the system satisfy $Y(z_i) = 0$, and poles z_j of the system satisfy $X(z_j) = 0$. Poles and zeros of a transfer function are the frequencies for which the value of the transfer function becomes infinity or zero, respectively. The values of the poles and the zeros determine whether the system is stable, and how well the system performs. Control systems, in the simplest sense, can be designed

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by assigning simple values to the poles and zeros of the system. Physically reliable control systems must have a number of poles greater than or equal to the number of zeros. Systems that satisfy this relationship are called proper. So, the problem of finding the roots of either $Y(z_i)=0$ or $X(z_j)=0$, and the location of these roots are very important from a stability point of view. As a matter of fact, the closer the zeros are to the imaginary axis, the greater the stabilizing effect. This, for example, somewhat illustrates how the problem of finding the location of zeros can be of great importance.

The paper is organized as follows. In Section 2 we give a brief overview of the subject, as of when it started till date. Our results are formulated and proved in Section 3 and thereafter, some examples in Section 5.

2 Preliminaries

We start by presenting the earliest known result in this subject.

Theorem 1(Gauss). *Let $p(z)=a_0+a_1z+a_2z^2+a_3z^3+\cdots+a_nz^n$ be a real polynomial. Then $p(z)$ has no zeros outside the circle $|z|=R$, where $R=\max\limits_{1\le j\le n}(n2^{1/2}|a_j|)^{1/j}$.*

However, in the case of arbitrary real or complex a_j, Gauss [18] in 1849 showed that R may be taken as the positive root of the equation:

$$z^n-2^{1/2}(|a_1|z^{n-1}+\cdots+|a_n|)=0.$$

Around 1829, Cauchy [7] (also, see the book of Marden [23, Theorem 27.1, p. 122]) derived more exact bounds for the moduli of the zeros of a polynomial than those given by Gauss, by proving the following

Theorem 2(Cauchy). *Let $p(z)=z^n+\sum\limits_{j=0}^{n-1}a_jz^j$, be a complex polynomial. Then all the zeros of $p(z)$ lie in the disc*

$$\{z:|z|\le\eta\}\subset\{z:|z|<1+A\},$$

where $A=\max\limits_{0\le j\le n-1}|a_j|$, and η is the unique positive root of the real coefficient equation

$$z^n-|a_{n-1}|z^{n-1}-|a_{n-2}|z^{n-2}-\cdots-|a_1|z-|a_0|=0 \quad (1)$$

The result is best possible and the bound is attained when $p(z)$ is the polynomial on the left hand side of (1).

If one applies the above Theorem 2 of Cauchy to the polynomial $P(z)=z^np(1/z)$ and combine it with Theorem 2, one easily gets

Theorem 3. *All the zeros of the polynomial $p(z)=a_0+a_1z+\cdot+a_nz^n$, $a_n\ne 0$, lie in the annulus $r_1\le|z|\le r_2$, where r_1 is the unique positive root of the equation*

$$|a_n|z^n+|a_{n-1}|z^{n-1}+\cdot+|a_1|z-|a_0|=0,$$

and r_2 is the unique positive root of the equation

$$|a_0|+|a_1|z+\cdot+|a_{n-1}|z^{n-1}-|a_n|z^n=0.$$

Although the above result gives an annulus containing all the zeros of a polynomial, it is implicit, in the sense, that in order to find the annulus containing all the zeros of a polynomial, one needs to compute the zeros of two other polynomials.

In a bid to get an explicit bound, Datt and Govil [10] (see also Dewan [13]) proved

Theorem 4. *Let $p(z)=z^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$, be a polynomial of degree n and $A=\max\limits_{0\le j\le n-1}|a_j|$, as defined in Theorem 2. Then p(z) has all its zeros in the ring shaped region*

$$\frac{|a_0|}{2(1+A)^{n-1}(An+1)}\le|z|\le 1+\lambda_0A, \qquad (2)$$

where λ_0 is the unique positive root of the equation $x=1-1/(1+Ax)^n$ in the interval $(0,1)$. The upper bound $1+\lambda_0A$ in the above given region (2) is best possible and is attained for the polynomial $p(z)=z^n-A\left(z^{n-1}+\ldots+z+1\right)$.

In case one does not wish to solve the equation $x=1-1/(1+Ax)^n$, then in order to apply the above result of Datt and Govil [10], one can apply the following result also due to Datt and Govil [10], which in every case clearly gives an improvement over Theorem 2 of Cauchy [7].

Theorem 5. *Let $p(z)=z^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$, be a polynomial of degree n and*

$$A=\max_{0\le j\le n-1}|a_j|.$$

Then p(z) has all its zeros in the ring shaped region

$$\frac{|a_0|}{2(1+A)^{n-1}(An+1)}\le|z|\le 1+\left(1-\frac{1}{(1+A)^n}\right)A.$$

Since, always $\left(1-\frac{1}{(1+A)^n}\right)<1$, the above Theorem 5 in every situation sharpens Theorem 2 due to Cauchy.

Although, since the beginning, binomial coefficients defined by $C(n,k)=\frac{n!}{k!(n-k)!}$, $0!=1$ (in the sequel, we will interchange between $C(n,j)$ and C_j^n as it deems convenient) have appeared in the derivation or as a part of closed expressions of bounds, the Fibonacci's numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_j = F_{j-1} + F_{j-2}$ for $j \geq 2$ have not appeared either in implicit bounds or explicit bounds for the moduli of the zeros. Diaz-Barrero [14] proved the following result, which gives circular domains containing all the zeros of a polynomial where binomial coefficients and Fibonacci's numbers appear.

Theorem 6. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ $(a_j \neq 0,\ 0 \leq j \leq n)$ be a complex monic polynomial. Then all its zeros lie in the disk $C_1 = \{z \in \mathbb{C} : |z| \leq r_1\}$ or $C_2 = \{z \in \mathbb{C} : |z| \leq r_2\}$, where*

$$r_1 = \max_{1\leq k\leq n} \left\{ \sqrt[k]{\frac{2^{n-1}C_2^{n+1}}{k^2 C_k^n}|a_{n-k}|} \right\},$$

$$r_2 = \max_{1\leq k\leq n} \left\{ \sqrt[k]{\frac{F_{3n}}{C_k^n 2^k F_k}|a_{n-k}|} \right\},$$

where F_j are the Fibonacci's numbers, and C_k^n the binomial coefficients.

Diaz-Barrero [15] also proved the following result

Theorem 7. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ $(a_j \neq 0,\ \ 0 \leq j \leq n)$ be a nonconstant complex polynomial. Then all its zeros lie in the annulus $C = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$, where*

$$r_1 = \frac{3}{2} \min_{1\leq j\leq n} \left\{ \frac{2^n F_j C_j^n}{F_{4n}} \left| \frac{a_0}{a_j} \right| \right\}^{1/j},$$

$$r_2 = \frac{2}{3} \max_{1\leq j\leq n} \left\{ \frac{F_{4n}}{2^n F_j C_j^n} \left| \frac{a_{n-j}}{a_n} \right| \right\}^{1/j}.$$

Here F_j being the Fibonacci's numbers, and C_j^n the binomial coefficients.

The following result of Kim [22] also provides an annulus containing all the zeros of a polynomial.

Theorem 8. *Let $p(z) = \sum_{k=0}^{n} a_k z^k$ $(a_k \neq 0,\ \ 0 \leq k \leq n)$ be a nonconstant polynomial with complex coefficients. Then all the zeros of $p(z)$ lie in the annulus $A = \{z : r_1 \leq |z| \leq r_2\}$, where*

$$r_1 = \min_{1\leq k\leq n} \left\{ \frac{C_k^n}{2^n - 1} \left| \frac{a_0}{a_k} \right| \right\}^{1/k}, \quad r_2 = \max_{1\leq k\leq n} \left\{ \frac{2^n - 1}{C_k^n} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}.$$

Here again, as usual, C_k^n denote the binomial coefficients.

The following two results by Diaz-Barrero and Egozcue [16], also provide annuli containing all the zeros of a polynomial.

Theorem 9. *Let $p(z) = \sum_{k=0}^{n} a_k z^k$ $(a_k \neq 0, 1 \leq k \leq n)$ be a non-constant complex polynomial. Then for $j \geq 2$, all the zeros of $p(z)$ lie in the annulus $C = \{z : r_1 \leq |z| \leq r_2\}$, where*

$$r_1 = \min_{1\leq k\leq n} \left\{ \frac{C(n,k) A_k B_j^k (bB_{j-1})^{n-k}}{A_{jn}} \left| \frac{a_0}{a_k} \right| \right\}^{1/k}$$

and

$$r_2 = \max_{1\leq k\leq n} \left\{ \frac{A_{jn}}{C(n,k) A_k B_j^k (bB_{j-1})^{n-k}} A_{jn} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}.$$

Here, $B_n = \sum_{k=0}^{n-1} r^k s^{n-1-k}$ and $A_n = cr^n + ds^n$, where c, d are real constants and r,s are the roots of the equation $x^2 - ax - b = 0$ in which a,b are strictly positive real numbers. For $j \geq 2$, $\sum_{k=0}^{n} C(n,k)(bB_{j-1})^{n-k} B_j^k A_k = A_{jn}$. Furthermore, $C(n,k)$ is the binomial coefficient.

Theorem 10. *Let $p(z) = \sum_{k=0}^{n} a_k z^k$ $(a_k \neq 0, 1 \leq k \leq n)$ be a non-constant polynomial with complex coefficients. Then, all its zeros lie in the ring shaped region $C = \{z : r_1 \leq |z| \leq r_2\}$, where*

$$r_1 = \min_{1\leq k\leq n} \left\{ \frac{2^k P_k C(n,k)}{P_{2n}} \left| \frac{a_0}{a_k} \right| \right\}^{1/k}$$

and

$$r_2 = \max_{1\leq k\leq n} \left\{ \frac{P_{2n}}{2^k P_k C(n,k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}.$$

Here P_k is the k^{th} Pell number, namely, $P_0 = 0$, $P_1 = 1$ and for $k \geq 2$, $P_k = 2P_{k-1} + P_{k-2}$. Furthermore, $C(n,k) = \frac{n!}{k!(n-k)!}$ are the binomial coefficients.

Recently, Dalal and Govil [8] unified the above results by proving the following

Theorem 11. *Let $A_k > 0$ for $1 \leq k \leq n$, and be such that $\sum_{k=1}^{n} A_k = 1$. If $p(z) = \sum_{k=0}^{n} a_k z^k$ $(a_k \neq 0, 1 \leq k \leq n)$ is a non-constant polynomial with complex coefficients, then all the zeros of $p(z)$ lie in the annulus $C = \{z : r_1 \leq |z| \leq r_2\}$, where*

$$r_1 = \min_{1\leq k\leq n} \left\{ A_k \left| \frac{a_0}{a_k} \right| \right\}^{1/k}$$

and

$$r_2 = \max_{1\leq k\leq n} \left\{ \frac{1}{A_k} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}.$$

The above theorem, by appropriate choice of the numbers $A_k > 0$ for $1 \leq k \leq n$, includes as special case Theorems 6, 7, 8, 9 and 10, and this has been shown in the Table 1 in the paper of Dalal and Govil [8, p. 9612]. Recently, Govil and Kumar [19] used Theorem 11 to obtain annular regions involving the Motzkin, Catalan and Narayana numbers. Motivated by their paper, [4] and [33], we obtain more results in this direction.

3 Main Results

Our first result connects the $n^{th}-$Bell number, B_n, which counts the partitions of a set with n elements and the Stirling number (of the second kind) with parameters n and k, denoted by $S(n,k)$, that enumerates the number of partitions of a set with n elements consisting k disjoint, nonempty sets. Here, B_n is defined recursively as: $B_0 = 1$, $B_{n+1} = \sum_{k=0}^{n} C(n,k)B_k$, for $n \geq 0$ and $S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j C(k,j)(k-j)^n$.

Theorem 12. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be a non-constant complex polynomial of degree* n, *with* $a_k \neq 0$, $1 \leq k \leq n$. *Then all the zeros of* $p(z)$ *lie in the annulus* $C = \{z : r_1 \leq |z| \leq r_2\}$, *where*

$$r_1 = \min_{1\leq k\leq n}\left\{\frac{S(n,k)}{B_n}\left|\frac{a_0}{a_k}\right|\right\}^{1/k}$$

and

$$r_2 = \max_{1\leq k\leq n}\left\{\frac{B_n}{S(n,k)}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}.$$

Theorem 13. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be a non-constant complex polynomial of degree* n, *with* $a_k \neq 0$, $1 \leq k \leq n$. *Then all the zeros of* $p(z)$ *lie in the annulus* $C = \{z : r_1 \leq |z| \leq r_2\}$, *where*

$$r_1 = \min_{1\leq k\leq n}\left\{\frac{C(2n-k,k)C_{n-k}}{S_n - C_n}\left|\frac{a_0}{a_k}\right|\right\}^{1/k}$$

and

$$r_2 = \max_{1\leq k\leq n}\left\{\frac{S_n - C_n}{C(2n-k,k)C_{n-k}}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k},$$

where $C_n = \frac{C(2n,n)}{n+1}$ *is the* $n^{th}-$*Catalan number and* S_n *the* $n^{th}-$*Schröder number given recursively by* $S_0 = 1$, $S_n = S_{n-1} + \sum_{j=0}^{n-1} S_j S_{n-1-j}$, *for* $n \geq 1$.

Theorem 14. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be a non-constant complex polynomial of degree* n, *with* $a_k \neq 0$, $1 \leq k \leq n$. *Then all the zeros of* $p(z)$ *lie in the annulus* $C = \{z : r_1 \leq |z| \leq r_2\}$, *where*

$$r_1 = \min_{1\leq k\leq n}\left\{\frac{C(n,k)M_k}{C_{n+1}-1}\left|\frac{a_0}{a_k}\right|\right\}^{1/k}$$

and

$$r_2 = \max_{1\leq k\leq n}\left\{\frac{C_{n+1}-1}{C(n,k)M_k}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k},$$

where C_n *is the same as in Theorem 13 and* M_k *is the* $k^{th}-$*Motzkin number defined recursively as*

$$M_0 = M_1 = M_{-1} = 1;\ M_{k+1} = \frac{2k+3}{k+3}M_k + \frac{3k}{k+3}M_{k-1},\ k \geq 1.$$

Theorem 15. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be a non-constant complex polynomial of degree* n, *with* $a_k \neq 0$, $1 \leq k \leq n$. *Then all the zeros of* $p(z)$ *lie in the annulus* $C = \{z : r_1 \leq |z| \leq r_2\}$, *where*

$$r_1 = \min_{1\leq k\leq n}\left\{\frac{C(n,k)^2}{C(2n,n)-1}\left|\frac{a_0}{a_k}\right|\right\}^{1/k}$$

and

$$r_2 = \max_{1\leq k\leq n}\left\{\frac{C(2n,n)-1}{C(n,k)^2}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}.$$

Theorem 16. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be a non-constant complex polynomial of degree* n, *with* $a_k \neq 0$, $1 \leq k \leq n$. *Then all the zeros of* $p(z)$ *lie in the annulus* $C = \{z : r_1 \leq |z| \leq r_2\}$, *where*

$$r_1 = \min_{1\leq k\leq n}\left\{\frac{F_k}{F_{n+2}-1}\left|\frac{a_0}{a_k}\right|\right\}^{1/k}$$

and

$$r_2 = \max_{1\leq k\leq n}\left\{\frac{F_{n+2}-1}{F_k}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k},$$

where F_n *denotes the* $n^{th}-$*Fibonacci number.*

Theorem 17. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be a non-constant complex polynomial of degree* n, *with* $a_k \neq 0$, $1 \leq k \leq n$. *Then all the zeros of* $p(z)$ *lie in the annulus* $C = \{z : r_1 \leq |z| \leq r_2\}$, *where*

$$r_1 = \min_{1\leq k\leq n}\left\{\frac{k\,C(n,k)}{n2^{n-1}}\left|\frac{a_0}{a_k}\right|\right\}^{1/k}$$

and

$$r_2 = \max_{1\leq k\leq n}\left\{\frac{n2^{n-1}}{k\,C(n,k)}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k},$$

For the proof of our results, we will need the following lemmas.

Lemma 1(see [20] for proof). *In combinatorics,it is known that for any* $n \in \mathbb{N}$, B_n *and* $S(n,k)$ *are connected as follows:*

$$\sum_{k=1}^{n} S(n,k) = B_n.$$

Lemma 2. *If M_n is the $n^{th}-$ Motzkin number and C_n the $n^{th}-$Catalan number, then for $n \geq 0$,*

$$C_0 = 1; \quad \sum_{k=0}^{n} C(n,k)M_k = C_{n+1}.$$

For the proof of Lemma 2 see [3, p. 99] and [12].

Lemma 3. *If S_n is the $n^{th}-$Schröder number, then for $n \geq 0$,*

$$\sum_{k=0}^{n} C(2n-k,k)C_{n-k} = S_n.$$

See [11, p. 2782] for the proof Lemma 3.

Lemma 4. *For $n \geq 0$,*

$$\sum_{k=0}^{n} C(n-k,k) = F_{n+1},$$

where F_n is the $n^{th}-$Fibonacci number.

Proof of Lemma 4: For $n = 0$ and $n = 1$, we have that $F_1 = 1$ and $F_2 = 1+0 = 1$, respectively. Now , for $n \geq 2$, assume that

$$\sum_{k=0}^{n-1} C(n-1-k,k) = F_n, \quad \text{and} \quad \sum_{k=0}^{n-2} C(n-2-k,k) = F_{n-1}.$$

So by the Pascal recursion,

$$C(n-k,k) = C(n-k-1,k-1) + C(n-k-1,k),$$

we have therefore (by the induction hypothesis, Fibonacci recursion, and $C(n,k) = 0$, when either $k > n$ or $k < 0$.)

$$\begin{aligned}
\sum_{k=0}^{n} C(n-k,k) &= \sum_{k=0}^{n} C(n-k-1,k-1) + \sum_{k=0}^{n} C(n-k-1,k) \\
&= \sum_{k=1}^{n-1} C(n-k-1,k-1) + \sum_{k=0}^{n-1} C(n-k-1,k) \\
&= \sum_{k=0}^{n-2} C(n-k-2,k) + \sum_{k=0}^{n-1} C(n-k-1,k) \\
&= F_{n-1} + F_n \\
&= F_{n+1}.
\end{aligned}$$

Lemma 5. *Let $n,k \in \mathbb{N}$, with $n \geq k$. Then* $k\,C(n,k) = n\,C(n-1,k-1)$.

Proof of Lemma 5*:*

$$\begin{aligned}
k\,C(n,k) &= k\frac{n!}{(n-k)!k!} \\
&= k\frac{n(n-1)!}{k(n-k)!(k-1)!} \\
&= n\frac{(n-1)!}{(n-k)!(k-1)!} \\
&= n\,C(n-1,k-1).
\end{aligned}$$

Lemma 6. *For $n \geq 0$,*

$$\sum_{k=1}^{n} k\,C(n,k) = n2^{n-1}.$$

Proof of Lemma 6: From Lemma 5 we obtain that

$$\begin{aligned}
\sum_{k=1}^{n} k\,C(n,k) &= \sum_{k=1}^{n} n\,C(n-1,k-1) \\
&= n\sum_{k=1}^{n} C(n-1,k-1) \\
&= n\sum_{k=0}^{n-1} C(n-1,k) \\
&= n2^{n-1}.
\end{aligned}$$

Lemma 7. *Let n,m and r be nonnegative integers. Then*

$$\sum_{k=0}^{r} C(m,k)\,C(n,r-k) = C(n+m,r).$$

Proof of Lemma 7: In general, the product of two polynomials with degrees m and n, respectively, is given by

$$\left(\sum_{i=0}^{m} a_i x^i\right)\left(\sum_{j=0}^{n} b_j x^j\right) = \sum_{r=0}^{m+n}\left(\sum_{k=0}^{r} a_k b_{r-k}\right)x^r;$$

where we use the convention that $a_i = 0$ for all integers $i > m$ and $b_j = 0$ for all integers $j > n$. Note by the binomial theorem,

$$(1+x)^{m+n} = \sum_{r=0}^{m+n} C(m+n,r)x^r.$$

Using the binomial theorem also for the exponents m and n, and then the above formula for the product of polynomials, we obtain

$$\begin{aligned}
\sum_{r=0}^{m+n} C(m+n,r)x^r &= (1+x)^{m+n} \\
&= (1+x)^m(1+x)^n \\
&= \left(\sum_{i=0}^{m} C(m,i)x^i\right)\left(\sum_{j=0}^{n} C(n,j)x^j\right) \\
&= \sum_{r=0}^{m+n}\left(\sum_{k=0}^{r} C(m,k)C(n,r-k)\right)x^r,
\end{aligned}$$

where the above convention for the coefficients of the polynomials agrees with the definition of the binomial coefficients, because both give zero for all $i > m$ and $j > n$, respectively.

By comparing coefficients of x^r, the identity follows for all integers with $0 \leq r \leq m+n$. For larger integer r, both sides of the identity are zeros due to the definition of the binomial coefficients.

Lemma 8. *Let $n \geq 0$. Then*

$$\sum_{k=0}^{n} C(n,k)^2 = C(2n,n).$$

The proof of Lemma 8 follows easily by setting $m = r = n$ in Lemma 7.

Lemma 9. *Let $n \geq 1$. Then*

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1.$$

The proof of the above lemma follows by mathematical induction.

4 Proofs of Theorems

Proof of Theorem 12: From Lemma 1, we have that

$$\sum_{k=1}^{n} \frac{S(n,k)}{B_n} = 1.$$

If we take $A_k = \dfrac{S(n,k)}{B_n}$, then $A_k > 0$ and $\sum_{k=1}^{n} A_k = 1$, and hence by applying Theorem 11 for this set of values of A_k we get our desired result.

Proof of Theorem 13: From Lemma 3, we have that

$$\sum_{k=1}^{n} \frac{C(2n-k,k)C_{n-k}}{S_n - C_n} = 1.$$

If we take $A_k = \frac{C(2n-k,k)C_{n-k}}{S_n - C_n}$, then $A_k > 0$ and $\sum_{k=1}^{n} A_k = 1$, and hence by applying Theorem 11 for this set of values of A_k we get the required annulus and thus the proof of Theorem 13 is complete.

Proof of Theorem 14: From Lemma 2, we have that

$$\sum_{k=1}^{n} \frac{C(n,k)M_k}{C_{n+1} - 1} = 1.$$

If we take $A_k = \frac{C(n,k)M_k}{C_{n+1}-1}$, then $A_k > 0$ and $\sum_{k=1}^{n} A_k = 1$, and hence by applying Theorem 11 for this set of values of A_k we get the desired annulus, and thus the proof of Theorem 14 is complete.

Proof of Theorem 15: From Lemma 8, we have that

$$\sum_{k=1}^{n} \frac{C(n,k)^2}{C(2n,n) - 1} = 1.$$

If we take $A_k = \frac{C(n,k)^2}{C(2n,n)-1}$, then $A_k > 0$ and $\sum_{k=1}^{n} A_k = 1$, and hence by applying Theorem 11 for this set of values of A_k we get the desired annulus given in Theorem 15.

Proof of Theorem 16: From Lemma 9, we have that

$$\sum_{k=1}^{n} \frac{F_k}{F_{n+2} - 1} = 1.$$

If we take $A_k = \frac{F_k}{F_{n+2}-1}$, then $A_k > 0$ and $\sum_{k=1}^{n} A_k = 1$, and hence by applying Theorem 11 for this set of values of A_k we get the desired annulus given be the radii in Theorem 16.

Proof of Theorem 17: From Lemma 6, we have that

$$\sum_{k=1}^{n} \frac{k\,C(n,k)}{n2^{n-1}} = 1.$$

If we take $A_k = \frac{k\,C(n,k)}{n2^{n-1}}$, then $A_k > 0$ and $\sum_{k=1}^{n} A_k = 1$, and hence by applying Theorem 11 for this set of values of A_k we get the desired annulus given be the radii in Theorem 17.

5 Computational Analysis

We now give examples of polynomials for which our results can compare favorably with the already known theorems as stated above.

Example 1. Consider the polynomial $p(z) = z^3 + 0.1z^2 + 0.1z + 0.7$.

Table 1: Computational Analysis I

Theorems	r_1	r_2	Area of the annulus
7	0.6402	1.2312	3.4730
8	0.4641	1.6984	8.382
12	0.519249	1.51829	6.39502
14	0.59943	1.31521	4.305399
16	0.7047	1.1187	2.37155
17	0.55934	1.4095	5.25812

As one can observe from Table 1, our Theorem 16 is giving a significantly better bound than obtainable from the known Theorems 7 and 8. In fact, the area of the annulus containing all the zeros of the polynomial $p(z)$ obtained by Theorem 16 is about 2.37155, which is about 68.29% of the area of the annulus obtained by Theorem 7 and about 28.29% of the area of the annulus obtained by Theorem 8.

Example 2. Consider the polynomial $p(z) = z^5 + 0.06z^4 + 0.29z^3 + 0.29z^2 + 0.29z + 0.001$.

It is clear from Table 2 that our Theorem 17 gives a better lower and upper bound for the polynomial $p(z)$, hence, a smaller area of the annulus containing all the zeros of the polynomial $p(z)$. Comparing the area

Table 2: Computational Analysis II

Theorems	r_1	r_2	Area of the annulus
7	0.00012233	1.6912	8.986
8	0.00055617	1.158	4.2125
12	0.51925	1.51829	6.3950
14	0.000132	1.5720	7.76345
15	0.000343	1.3063	5.36063
17	0.0010776	1.07703	3.6442

obtained by Theorem 17, one observe that this area is about 40.55% of the area obtained by Theorem 7 and 86.51% of the area of the annulus obtained by Theorem 8.

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