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Hermite-Hadamard-Fejér Type Inequalities for Strongly (s,m)-Convex Functions with Modulus c, in Second Sense

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Abstract: We introduce the class of strongly (s,m)-convex functions modulus $c>0$ in the second sense, and prove inequalities of Hermite-Hadamard-Fejér type for such mappings. This strengthen results given for (s,m)-convex functions in [7].

Keywords: Inequalities of Hermite type, convexity generalized, inequalities of Fejér type

In recent years several generalizations and extensions of the classical notion of convex function have been introduced and the theory of inequalities has produce important contributions in that respect. This research deals with some inequalities related to the renowned works, on classical convexity, of Charles Hermite [11], Jaques Hadamard [12] and Lipót Fejér [9]. The inequalities of Hermite-Hadamard and Fejér have been object of intense investigation and have produce many applications. Proofs of them can be found in the literature (see e.g. [2,3,4,16,23,24,31] and references therein). In this paper we establish some results related with these inequalities for strongly (s,m)-convex functions.

Definition 1.*Let I be a real interval and let $f: I\subset\mathbb{R}\to\mathbb{R}$. If*

$$f(tx+(1-t)y)\leq t\,f(x)+(1-t)\,f(y),$$

for all $x,y\in I$ and $t\in[0,1]$ then f is said to be **convex on** *I*.

The Hermite-Hadamard inequality gives us an estimate of the (integral) mean value of a convex function; more precisely:

Theorem 1([1,18,23]). *Let f be a convex function on $[a,b]$, with $a<b$. Then*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{(b-a)}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}. \tag{1}$$

In [9], Fejér gives a generalization of (1) as follows:

Theorem 2.*Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a convex function and let $a,b\in I$ with $a<b$. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx\leq\int_a^b f(x)g(x)dx \leq\frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{2}$$

where $g:[a,b]\to\mathbb{R}$ is non negative, integrable and symmetric with respect to $(a+b)/2$, that is, $g(a+b-x)=g(x)$.

*Remark.*Clearly a $1-$convex function is a convex function in the ordinary sense. The $0-$convex functions are the "starshaped" functions; that is, those functions that satisfy the inequality $f(tx)\leq t\,f(x)$, for $t\in[0,1]$.

In 1984, G. Toader [29] introduces the concept of function m-convex. Several papers have been written on functions m-convex and we refer some of them below.

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Definition 2([4,5,29]). *A function $f:[0,b]\to\mathbb{R}$ is said to be m-convex, where $m\in(0,1]$, if for every $x,y\in[0,b]$ and $t\in[0,1]$, we have:*
$f(tx+m(1-t)y)\le tf(x)+m(1-t)f(y)$.

It is important to note that for $m\in(0,1)$ there are continuous and differentiable $m-$convex functions which are not convex in the classical sense (see [29]).

In [5], S.S. Dragomir and G. Toader demostrated the following Hermite-Hadamard type inequality:

Theorem 3. *Let $f:[0,+\infty)\to\mathbb{R}$ be an $m-$convex function, with $m\in(0,1]$. If $0\le a<b<+\infty$ and $f\in L^1[a,b]$ then*

$$\frac{1}{b-a}\int_a^b f(x)dx \tag{3}$$

$$\le \min\left\{\frac{f(a)+mf(\frac{b}{m})}{2},\frac{f(b)+mf(\frac{a}{m})}{2}\right\}. \tag{4}$$

In [15] the reader may find some others generalizations of this inequality.

Another result of this type which holds for convex functions is embodied in the following theorem, in [6],

Theorem 4. *Let $f:[0,\infty)\to\mathbb{R}$ be a $m-$convex function with $m\in(0,1]$ and that $0\le a<b$. If $f\in L_1[a,b]$, then one has the inequalities*

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{2(b-a)}\int_a^b\left(f(t)+mf\left(\frac{t}{m}\right)\right)dt$$
$$\le\frac{m+1}{4}\left[\frac{f(a)+f(b)}{2}+m\frac{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)}{2}\right]. \tag{5}$$

Remark. Notice that if we make $m=1$ in (5) we get the left hand side of inequality (1); that is:

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx.$$

In the year 1966, B.T. Polyak in [26] studied the concept of strongly convex function modulus $c>0$, which is defined as follows:

Definition 3([26]). *Let C be a nonempty convex subset of the normed space $(X,\|\cdot\|)$. A real valued function f is said to be strongly convex, modulus c, on C if for each $x,y\in C$ and $t\in[0,1]$*

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y)-ct(1-t)\|x-y\|^2. \tag{6}$$

Clearly for $c=0$ in (6), f is just a convex function.

If the inequality in (6) is reversed, then f is said to be strongly concave, modulus c.

The notion of strongly convex function has many applications in optimization theory and economics (See for example [14,19,20,23,25,27,30]).

More recently N. Eftekhari, in [8], establishes several inequalities for functions whose first derivative in absolute value are (s,m)-convex function. Some estimate to the left hand side of the Hermite-Hadamard type inequality for $(s,m)-$convex functions in the second sense are given.

Definition 4. *[7,8] A function $f:[0,b]\to\mathbb{R}$, $b>0$, is said to be (s,m)-convex, where $(s,m)\in(0,1]^2$, if*
$f(tx+m(1-t)y)\le t^sf(x)+m(1-t)^sf(y)$ for all $x,y\in[0,b]$ and $t\in[0,1]$.

It is readily seen that for $(s,m)\in\{(0,0),(1,1),(1,m)\}$ respectively, one obtains the following classes of functions: increasing, convex and m-convex functions respectively.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$, $[a,b]\subseteq[0,+\infty)$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$. If $|f'|$ is $(s,m)-$convex in the second sense on $[a,b]$ for $(s,m)\in(0,1]^2$, then the following inequality holds:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le\frac{b-a}{4(s+2)}\left[\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{m}{s+1}\left(\left|f'\left(\frac{a}{m}\right)\right|+(\left|f'\left(\frac{b}{m}\right)\right|\right)\right],$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le\frac{b-a}{2^{s+2}(s+2)}\left[|f'(a)|+|f'(b)|\right.$$
$$\left.+\frac{m(2^{s+2}-s+3)}{s+1}\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)\right],$$

1 Some basic properties

For the reader's convenience, we recall here the definitions of the Gamma function $\Gamma(x):=\int_0^{+\infty}e^{-t}t^{x-1}dt$ and the Beta function $B(x,y):=\int_0^1 t^{x-1}(1-t)^{y-1}dt$.

1. $B(x,y)=\dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$,
2. $\Gamma(x+1)=x\Gamma(x)$, and
3. $\Gamma(n)=(n-1)!,\quad n\in\mathbb{N}$.

In this paper we combine the notions of (s,m)-convex function and strongly convex function to define the concept of strongly (s,m)-convex functions with modulus c, in second sense.

Definition 5. *A function $f:[0,+\infty)\to\mathbb{R}$ is said to be strongly (s,m)-convex functions with modulus c in second sense, where $(s,m)\in(0,1]^2$, if*

$$f(tx+m(1-t)y)\le t^sf(x)+m(1-t)^sf(y)-ct(1-t)|x-y|^2$$

holds for all $x,y\in[0,+\infty)$ and $t\in[0,1]$.

Remark. 1. If $s=1$, f is strongly m-convex with modulus c (see [17]).

2.If $m=1$ and $h(t)=t^s$, f strongly h-convex with modulus c, have been introduced by [10].

In this article we prove some Hermite-Hadamard-Fejér type inequalities for strongly (s,m)-convex functions, modulus c in the second sense, using similar technics as those used in [28]. For the sake of brevity we will omit the the word "*in the second sense*" throughout the rest of this paper. Likewise we will assume that c is a positive real number.

Proposition 1. *Let $f:[0,+\infty)\to\mathbb{R}$ be a strongly (s,m)-convex function of modulus c , where $s,m\in(0,1]$, and let $a,b\in[0,+\infty)$ be, with $a<b$. Then for any $x\in[a,b]$ there is $t\in[0,1]$ such that*

$$\begin{aligned}&f(a+b-x)\\ \le\ & t^s(f(b)+f(a))+m(1-t)^s\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)-f(x)\\ &-ct(1-t)\left(\left|\frac{a}{m}-b\right|^2+\left|a-\frac{b}{m}\right|^2\right).\end{aligned}$$

Proof. Since any $x\in[a,b]$ can be represented as $x=ta+(1-t)b$, $t\in[0,1]$, then

$$\begin{aligned}&f(a+b-x)\\ =\ & f(a+b-(ta+(1-t)b))=f((1-t)a+tb)\\ =\ & f\left(tb+m(1-t)\frac{a}{m}\right)\\ \le\ & t^sf(b)+m(1-t)^sf\left(\frac{a}{m}\right)-ct(1-t)\left|\frac{a}{m}-b\right|^2\\ =\ & t^sf(b)+m(1-t)^sf\left(\frac{a}{m}\right)-t^sf(a)+t^sf(a)+m(1-t)^sf\left(\frac{b}{m}\right)\\ &-m(1-t)^sf\left(\frac{b}{m}\right)-ct(1-t)\left|a-\frac{b}{m}\right|^2-t(1-t)c\left|\frac{a}{m}-b\right|^2\\ &+ct(1-t)\left|a-\frac{b}{m}\right|^2\\ =\ & t^s(f(b)+f(a))+m(1-t)^s\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)-t^sf(a)\\ &-m(1-t)^sf\left(\frac{b}{m}\right)-ct(1-t)\left|a-\frac{b}{m}\right|^2-t(1-t)c\left|\frac{a}{m}-b\right|^2\\ &+ct(1-t)\left|a-\frac{b}{m}\right|^2\\ \le\ & t^s(f(b)+f(a))+m(1-t)^s\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)\\ &-f\left(ta+m(1-t)\frac{b}{m}\right)-t(1-t)c\left(\left|\frac{a}{m}-b\right|^2-\left|a-\frac{b}{m}\right|^2\right)\\ =\ & t^s(f(b)+f(a))+m(1-t)^s\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)-f(x)\\ &-ct(1-t)\left(\left|\frac{a}{m}-b\right|^2-\left|a-\frac{b}{m}\right|^2\right).\end{aligned}$$

Proposition 2. *If $f_i:[0,+\infty)\to\mathbb{R}$, $i=1,\dots,n$ are strongly (s,m)-convex functions, modulus $c_i>0$, where $(s,m)\in(0,1]^2$, then the function given by $f:=\max\limits_{i=1,2,\dots,n}\{f_i\}$ is also strongly (s,m)-convex functions, modulus $c:=\min\limits_{i=1,2,\dots,n}\{c_i\}>0$.*

Proof. If $x,y\in[0,+\infty)$ and $t\in[0,1]$,

$$\begin{aligned}&f_i(tx+m(1-t)y)\\ \le\ & t^sf_i(x)+m(1-t)^sf_i(y)-ct(1-t)|x-y|^2\\ \le\ & t^s\max_{i=1,2,\dots,n}\{f_i(x)\}+m(1-t)^s\max_{i=1,2,\dots,n}\{f_i(y)\}\\ &-\min_{i=1,2,\dots,n}\{c_i\}t(1-t)|x-y|^2\\ =\ & t^sf(x)+m(1-t)^sf(y)-ct(1-t)|x-y|^2.\end{aligned}$$

Proposition 3. *Let $f_n:[0,+\infty)\to\mathbb{R}$ be is a sequence of functions. If f_n is strongly (s,m)-convex function, modulus $c_n>0$, for all $n\ge k$, $f_n(x)\to f(x)$ (on $[0,+\infty)$) and $c_n\to c$, then f is strongly (s,m)-convex functions, modulus c .*

Proof. If $x,y\in[0,+\infty)$ and $t\in[0,1]$, we have, for $n\ge k$.

$$\begin{aligned}&f_n(tx+m(1-t)y)\\ \le\ & t^sf_n(x)+m(1-t)^sf_n(y)-c_nt(1-t)|x-y|^2,\end{aligned}$$

If we have $n\to\infty$ on both sides of the inequality, we obtain that

$$f(tx+m(1-t)y)\le t^sf(x)+m(1-t)^sf(y)-ct(1-t)|x-y|^2,$$

which completes the proof.

Proposition 4. *Let $f:[0,+\infty)\to[0,+\infty)$ be a strongly $(s_1,m)-$convex function, modulus $c_1>0$ and let $g:[0,+\infty)\to[0,+\infty)$ be a strongly (s_2,m)-convex function, modulus $c_2>0$, where $s_1,s_2,m\in(0,1]$. Then $f+g$ is strongly (s,m)-convex function, modulus $c_1+c_2>0$, where $s=\min\{s_1,s_2\}$.*

Proof.

$$\begin{aligned}&(f+g)(tx+m(1-t)y)\\ =\ & f(tx+m(1-t)y)+g(tx+m(1-t)y)\\ \le\ & t^{s_1}f(x)+m(1-t)^{s_1}f(y)-c_1t(1-t)|x-y|^2\\ &+t^{s_2}g(x)+m(1-t)^{s_2}g(y)-c_2t(1-t)|x-y|^2\\ \le\ & t^sf(x)+m(1-t)^sf(y)-c_1t(1-t)|x-y|^2\\ &+t^sg(x)+m(1-t)^sg(y)-c_2t(1-t)|x-y|^2\\ =\ & t^s(f(x)+g(x))+m(1-t)^s(f(x)+g(x))\\ &-(c_1+c_2)t(1-t)|x-y|^2\\ =\ & t^s(f+g)(x)+m(1-t)^s(f+g)(y)\\ &-(c_1+c_2)t(1-t)|x-y|^2.\end{aligned}$$

Proposition 5. *Let $f:[0,+\infty)\to\mathbb{R}$ be a strongly (s,m)-convex function, modulus $c>0$, where $(s,m)\in(0,1]^2$. If $\lambda>0$, then λf is strongly (s,m)-convex function, modulus $\lambda c>0$.*

Proof. For $x,y\in[0,+\infty)$ and $t\in[0,1]$, then

$$\begin{aligned}&\lambda f(tx+m(1-t)y)\\ \le\ & \lambda\left(t^sf(x)+m(1-t)^sf(y)-ct(1-t)|x-y|^2\right)\\ =\ & t^s(\lambda f)(x)+m(1-t)^s(\lambda f)(y)-c\lambda t(1-t)|x-y|^2.\end{aligned}$$

Theorem 6. *Let $f:[a,b]\subset[0,+\infty)\to\mathbb{R}$ be a function, $s,m\in(0,1]$ and $c\in(0,+\infty)$. If the function $g:[a,b]\to\mathbb{R}$, defined by $g(x)=f(x)-cx^2$ is $(s,m)-$convex then f is strongly $(s,m)-$convex with modulus c.*

Proof. Let $x,y\in[0,+\infty)$ and $t\in[0,1]$.

Suppose without loss of generality $x\le y$

$$
\begin{aligned}
& f(ty+m(1-t)x) \qquad (7)\\
&= g(ty+m(1-t)x)+c(ty+m(1-t)x)^2\\
&\le t^s g(y)+m(1-t)^s g(x)+c(ty+m(1-t)x)^2\\
&= t^s g(y)+m(1-t)^s g(x)+c\Big(t(1-1+t)y^2\\
&\quad +2mt(1-t)xy+m^2(1-t)(1-t)x^2\Big)\\
&= t^s g(y)+m(1-t)^s g(x)+c\Big(ty^2-t(1-t)y^2\\
&\quad +2mt(1-t)xy+m^2(1-t)x^2-m^2t(1-t)x^2\Big)\\
&= t^s g(y)+cty^2+m(1-t)^s g(x)+cm^2(1-t)x^2\\
&\quad -ct(1-t)\left(y^2-2mxy+m^2x^2\right)\\
&\le t^s g(y)+ct^s y^2+m(1-t)^s g(x)+cm(1-t)^s x^2\\
&\quad -ct(1-t)\left(y^2-2mxy+m^2x^2\right)\\
&= t^s\left[g(y)+cy^2\right]+m(1-t)^s\left[g(x)+cx^2\right]-ct(1-t)(y-mx)^2\\
&= t^s f(y)+m(1-t)^s f(x)-ct(1-t)(y-mx)^2. \qquad (8)
\end{aligned}
$$

It is easy to check that, if $x\le y$, then

$$(y-mx)^2\ge(y-x)^2. \qquad (9)$$

By replacing in (7), we obtain that

$$f(ty+m(1-t)x)\le t^s f(y)+m(1-t)^s f(x)-ct(1-t)(y-x)^2.$$

This complete is proof.

Example 1. (See [13]) Let $0<s<1$ and $a,b,d\in\mathbb{R}$. Defining, for $x\in[0,+\infty)$,

$$g(x)=\begin{cases} a, & \text{if } x=0\\ bx^s+d, & \text{if } x>0.\end{cases}$$

If $b\ge 0$ and $c\le a$ then f is $(s,1)-$convex.

Hence, by Theorem 6, we have that $g:[0,+\infty)\to\mathbb{R}$, defined by $f(x)=g(x)+cx^2$ is strongly $(s,1)-$convex function with modulus $c>0$.

Example 2. The function $g:[0,+\infty)\to\mathbb{R}$, given by

$$g(x)=\frac{1}{12}\left(x^4-5x^3+9x^2-5x\right)$$

is $\left(1,\frac{16}{17}\right)$-convex.

Hence, by Theorem 6, we get that $f:[0,+\infty)\to\mathbb{R}$, defined by

$$f(x)=g(x)+cx^2=\frac{1}{12}\left(x^4-5x^3-5x\right)+\left(\frac{9}{12}+c\right)x^2$$

is strongly $\left(1,\frac{16}{17}\right)$-convex function with modulus $c>0$.

2 Inequalities for strongly (s,m)-convex function, modulus $c>0$

The following results generalize results in [28].

Theorem 7. *Let $f:[0,+\infty)\to\mathbb{R}$ be a strongly (s,m)-convex function, modulus c, where $m\in(0,1]$ and let $a,b\in[0,+\infty)$ with $a<b$. Suppose that $f\in L_1[a,b]$ and that $g:[a,b]\to\mathbb{R}$ is a nonnegative, integrable function which is symmetric with respect to $\frac{a+b}{2}$. Then*

$$
\begin{aligned}
&\int_a^b f(x)g(x)dx\\
&\le \frac{f(b)+f(a)}{2}\int_a^b\left(\frac{x-a}{b-a}\right)^s g(x)dx\\
&\quad+\frac{m\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)}{2}\int_a^b\left(\frac{b-x}{b-a}\right)^s g(x)dx\\
&\quad-\frac{c}{2}\left[\left|b-\frac{a}{m}\right|^2+\left|a-\frac{b}{m}\right|^2\right]\int_a^b\frac{x-a}{b-a}\cdot\frac{b-x}{b-a}g(x)dx.
\end{aligned}
$$

Proof. Let f and g as in the statement of the theorem. Then

$$
\begin{aligned}
&\int_a^b f(x)g(x)dx\\
&=\frac12\left[\int_a^b f(x)g(x)dx+\int_a^b f(a+b-x)g(a+b-x)dx\right]\\
&=\frac12\left[\int_a^b f(x)g(x)dx+\int_a^b f(a+b-x)g(x)dx\right]\\
&=\frac12\left[\int_a^b (f(x)+f(a+b-x))g(x)dx\right]\\
&\le\frac12\left[\int_a^b\left(\frac{x-a}{b-a}\right)^s f(b)+m\left(\frac{b-x}{b-a}\right)^s f\left(\frac{a}{m}\right)\right.\\
&\quad -c\cdot\frac{x-a}{b-a}\cdot\frac{b-x}{b-a}\left|b-\frac{a}{m}\right|^2\left(\frac{x-a}{b-a}\right)^s f(a)\\
&\quad \left.+m\left(\frac{b-x}{b-a}\right)^s f\left(\frac{b}{m}\right)+-c\cdot\frac{x-a}{b-a}\cdot\frac{b-x}{b-a}\left|a-\frac{b}{m}\right|^2\right]g(x)dx\\
&=\frac12\left[\int_a^b\left(\frac{x-a}{b-a}\right)^s(f(b)+f(a))+m\left(\frac{b-x}{b-a}\right)^s f\left(\frac{a}{m}\right)\right.\\
&\quad +m\left(\frac{b-x}{b-a}\right)^s f\left(\frac{b}{m}\right)\\
&\quad \left.-c\cdot\frac{x-a}{b-a}\cdot\frac{b-x}{b-a}\left[\left|b-\frac{a}{m}\right|^2+\left|a-\frac{b}{m}\right|^2\right]\right]g(x)dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{f(b)+f(a)}{2}\int_a^b \left(\frac{x-a}{b-a}\right)^s g(x)dx \\
&+\frac{m\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)}{2}\int_a^b \left(\frac{b-x}{b-a}\right)^s g(x)dx \\
&-\frac{c}{2}\left[\left|b-\frac{a}{m}\right|^2+\left|a-\frac{b}{m}\right|^2\right]\int_a^b \frac{x-a}{b-a}\cdot\frac{b-x}{b-a}g(x)dx.
\end{aligned}
$$

Theorem 8. *Let $f:[0,+\infty)\to\mathbb{R}$ be a strongly (s,m)-convex function, modulus c , where $s,m\in(0,1]$, and let $a,b\in[0,+\infty)$ with $a<b$. Suppose that $f\in L_1[a,b]$, and that $g:[a,b]\to\mathbb{R}$ is a nonnegative, integrable function which is symmetric with respect to $\frac{a+b}{2}$. Then*

$$
\begin{aligned}
&2^s f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx - m\int_a^b f\left(\frac{x}{m}\right)g(x)dx \\
&+2^{s-2}c\int_a^b \left|a+b-x-\frac{x}{m}\right|^2 g(x)dx \\
&\le \int_a^b f(x)g(x)dx.
\end{aligned}
$$

Proof. In this case we have

$$
\begin{aligned}
& f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \\
&= \int_a^b f\left(\frac{a+b-x}{2}+\frac{m}{2}\frac{x}{m}\right)g(x)dx \\
&\le \int_a^b \left(\frac{1}{2^s}f(a+b-x)+\frac{m}{2^s}f\left(\frac{x}{m}\right)-\frac{c}{4}\left|a+b-x-\frac{x}{m}\right|^2\right)g(x)dx \\
&= \frac{1}{2^s}\int_a^b f(a+b-x)g(x)dx+\frac{m}{2^s}\int_a^b f\left(\frac{x}{m}\right)g(x)dx \\
&-\frac{c}{4}\int_a^b \left|a+b-x-\frac{x}{m}\right|^2 g(x)dx \\
&= \frac{1}{2^s}\int_a^b f(a+b-x)g(a+b-x)dx+\frac{m}{2^s}\int_a^b f\left(\frac{x}{m}\right)g(x)dx \\
&-\frac{c}{4}\int_a^b \left|a+b-x-\frac{x}{m}\right|^2 g(x)dx \\
&= \frac{1}{2^s}\int_a^b f(x)g(x)dx+\frac{m}{2^s}\int_a^b f\left(\frac{x}{m}\right)g(x)dx \\
&\quad -\frac{c}{4}\int_a^b \left|a+b-x-\frac{x}{m}\right|^2 g(x)dx,
\end{aligned}
$$

thus obtaining the required inequality.

Theorem 9. *Let $f,g:[0,+\infty)\to[0,+\infty)$ be such that $fg\in L_1([a,b])$, where $0\le a<b<\infty$. If f is (s_1,m_1)-strongly convex function, modulus c_1 and g is (s_2,m_2)-strongly convex function, modulus c_2 on $[a,b]$, for some fixed $m_1,m_2,s_1,s_2\in(0,1]$, then*

$$
\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \min\{M_1,M_2\},
$$

where

$$
\begin{aligned}
M_1 &:= \frac{1}{s_1+s_2+1}\left(f(a)g(a)+m_1m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right) \\
&+\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)}m_2 f(a)g\left(\frac{b}{m_2}\right) \\
&+\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)}m_1 f\left(\frac{b}{m_1}\right)g(a) \\
&-\frac{1}{(s_1+3)(s_1+2)}c_2\left|a-\frac{b}{m_2}\right|^2\left(f(a)+m_1 f\left(\frac{b}{m_1}\right)\right) \\
&-\frac{1}{(s_2+3)(s_2+2)}c_1\left|a-\frac{b}{m_1}\right|^2\left(g(a)+m_2 g\left(\frac{b}{m_2}\right)\right) \\
&+c_1c_2\frac{1}{30}\left|a-\frac{b}{m_1}\right|^2\left|a-\frac{b}{m_2}\right|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
M_2 &:= \frac{1}{s_1+s_2+1}\left(f(b)g(b)+m_1m_2 f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right) \\
&+\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)}m_2 f(b)g\left(\frac{a}{m_2}\right) \\
&+\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)}m_1 f\left(\frac{a}{m_1}\right)g(b) \\
&-c_2\left|\frac{a}{m_2}-b\right|^2\frac{1}{(s_1+3)(s_1+2)}\left(f(b)+m_1 f\left(\frac{a}{m_1}\right)\right) \\
&-c_1\left|\frac{a}{m_1}-b\right|^2\frac{1}{(s_2+3)(s_2+2)}\left(g(b)+m_2 g\left(\frac{a}{m_2}\right)\right) \\
&+c_1c_2\left|\frac{a}{m_1}-b\right|^2\left|\frac{a}{m_2}-b\right|^2\frac{1}{30}.
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
f(tx+(1-t)y) &\le t^{s_1}f(x)+m_1(1-t)^{s_1}f\left(\frac{y}{m_1}\right) \\
&\quad -c_1t(1-t)\left|x-\frac{y}{m_1}\right|^2
\end{aligned}
$$

and

$$
\begin{aligned}
g(tx+(1-t)y) &\le t^{s_2}g(x)+m_2(1-t)^{s_2}g\left(\frac{y}{m_2}\right) \\
&\quad -c_2t(1-t)\left|x-\frac{y}{m_2}\right|^2,
\end{aligned}
$$

for all $t\in[0,1]$. f and g are nonnegative, hence

$$
\begin{aligned}
& f(ta+(1-t)b)\cdot g(ta+(1-t)b) \\
\leq\ & t^{s_1+s_2} f(a)g(a)+m_2 t^{s_1}(1-t)^{s_2} f(a)g\left(\frac{b}{m_2}\right) \\
& -c_2 t^{s_1+1}(1-t)f(a)\left|a-\frac{b}{m_2}\right|^2+m_1(1-t)^{s_1} f\left(\frac{b}{m_1}\right)t^{s_2}g(a) \\
& +m_1(1-t)^{s_1} f\left(\frac{b}{m_1}\right)m_2(1-t)^{s_2}g\left(\frac{b}{m_2}\right) \\
& -m_1c_2t(1-t)^{s_1+1} f\left(\frac{b}{m_1}\right)\left|a-\frac{b}{m_2}\right|^2 \\
& -c_1t^{s_2+1}(1-t)\left|a-\frac{b}{m_1}\right|^2 g(a) \\
& -c_1t(1-t)^{s_2+1}\left|a-\frac{b}{m_1}\right|^2 m_2g\left(\frac{b}{m_2}\right) \\
& +c_1c_2t^2(1-t)^2\left|a-\frac{b}{m_1}\right|^2\left|a-\frac{b}{m_2}\right|^2.
\end{aligned}
$$

Integrating both sides of the above inequality over $[0,1]$, we obtain

$$
\begin{aligned}
& \frac{1}{b-a}\int_a^b f(x)g(x)dx=\int_0^1 f(ta-(1-t)b)g(ta+(1-t)b)dt \\
\leq\ & f(a)g(a)\int_0^1 t^{s_1+s_2}dt+m_2 f(a)g\left(\frac{b}{m_2}\right)\int_0^1 t^{s_1}(1-t)^{s_2} \\
& -c_2 f(a)\left|a-\frac{b}{m_2}\right|^2\int_0^1 t^{s_1+1}(1-t)dt \\
& +m_1 f\left(\frac{b}{m_1}\right)g(a)\int_0^1(1-t)^{s_1}t^{s_2}dt \\
& +m_1m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\int_0^1(1-t)^{s_1+s_2}dt \\
& -m_1c_2 f\left(\frac{b}{m_1}\right)\left|a-\frac{b}{m_2}\right|^2\int_0^1 t(1-t)^{s_1+1}dt \\
& -c_1\left|a-\frac{b}{m_1}\right|^2 g(a)\int_0^1 t^{s_2+1}(1-t)dt \\
& -c_1\left|a-\frac{b}{m_1}\right|^2 m_2g\left(\frac{b}{m_2}\right)\int_0^1 t(1-t)^{s_2+1}dt \\
& +c_1c_2\left|a-\frac{b}{m_1}\right|^2\left|a-\frac{b}{m_2}\right|^2\int_0^1 t^2(1-t)^2dt \\
=\ & f(a)g(a)B(s_1+s_2+1,1)+m_2 f(a)g\left(\frac{b}{m_2}\right)B(s_1+1,s_2+1) \\
& -c_2 f(a)\left|a-\frac{b}{m_2}\right|^2 B(s_1+2,2) \\
& +m_1 f\left(\frac{b}{m_1}\right)g(a)B(s_2+1,s_1+1) \\
& +m_1m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)B(1,s_1+s_2+1) \\
& -m_1c_2 f\left(\frac{b}{m_1}\right)\left|a-\frac{b}{m_2}\right|^2 B(2,s_1+2) \\
& -c_1\left|a-\frac{b}{m_1}\right|^2 g(a)B(s_2+2,2) \\
& -c_1\left|a-\frac{b}{m_1}\right|^2 m_2g\left(\frac{b}{m_2}\right)B(2,s_2+2) \\
& +c_1c_2\left|a-\frac{b}{m_1}\right|^2\left|a-\frac{b}{m_2}\right|^2 B(3,3),
\end{aligned}
$$

where B is Eulers Beta-function.

Then,

$$
\begin{aligned}
& \frac{1}{b-a}\int_a^b f(x)g(x)dx \\
\leq\ & f(a)g(a)\frac{\Gamma(s_1+s_2+1)\Gamma(1)}{\Gamma(s_1+s_2+2)} \\
& +m_2 f(a)g\left(\frac{b}{m_2}\right)\frac{\Gamma(s_1+1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)} \\
& -c_2 f(a)\left|a-\frac{b}{m_2}\right|^2\frac{\Gamma(s_1+2)\Gamma(2)}{\Gamma(s_1+4)} \\
& +m_1 f\left(\frac{b}{m_1}\right)g(a)\frac{\Gamma(s_2+1)\Gamma(s_1+1)}{\Gamma(s_1+s_2+2)} \\
& +m_1m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\frac{\Gamma(1)\Gamma(s_1+s_2+1)}{\Gamma(s_1+s_2+2)} \\
& -m_1c_2 f\left(\frac{b}{m_1}\right)\left|a-\frac{b}{m_2}\right|^2\frac{\Gamma(2)\Gamma(s_1+2)}{\Gamma(s_1+4)} \\
& -c_1\left|a-\frac{b}{m_1}\right|^2 g(a)\frac{\Gamma(s_2+2)\Gamma(2)}{\Gamma(s_2+4)} \\
& -c_1\left|a-\frac{b}{m_1}\right|^2 m_2g\left(\frac{b}{m_2}\right)\frac{\Gamma(2)\Gamma(s_2+2)}{\Gamma(s_2+4)} \\
& +c_1c_2\left|a-\frac{b}{m_1}\right|^2\left|a-\frac{b}{m_2}\right|^2\frac{\Gamma(3)\Gamma(3)}{\Gamma(6)} \\
=\ & f(a)g(a)\frac{(s_1+s_2)\Gamma(s_1+s_2)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)} \\
& +m_2 f(a)g\left(\frac{b}{m_2}\right)\frac{s_1\Gamma(s_1)s_2\Gamma(s_2)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)} \\
& -c_2 f(a)\left|a-\frac{b}{m_2}\right|^2\frac{(s_1+1)s_1\Gamma(s_1)}{(s_1+3)(s_1+2)(s_1+1)s_1\Gamma(s_1)} \\
& +m_1 f\left(\frac{b}{m_1}\right)g(a)\frac{s_2\Gamma(s_2)s_1\Gamma(s_1)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)} \\
& +m_1m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\frac{(s_1+s_2)\Gamma(s_1+s_2)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)} \\
& -m_1c_2 f\left(\frac{b}{m_1}\right)\left|a-\frac{b}{m_2}\right|^2\frac{(s_1+1)s_1\Gamma(s_1)}{(s_1+3)(s_1+2)(s_1+1)s_1\Gamma(s_1)} \\
& -c_1\left|a-\frac{b}{m_1}\right|^2 g(a)\frac{(s_2+1)s_2\Gamma(s_2)}{(s_2+3)(s_2+2)(s_2+1)s_2\Gamma(s_2)} \\
& -c_1\left|a-\frac{b}{m_1}\right|^2 m_2g\left(\frac{b}{m_2}\right)\frac{(s_2+1)s_2\Gamma(s_2)}{(s_2+3)(s_2+2)(s_2+1)s_2\Gamma(s_2)} \\
& +c_1c_2\left|a-\frac{b}{m_1}\right|^2\left|a-\frac{b}{m_2}\right|^2\frac{2!\cdot 2!}{5!}
\end{aligned}
$$

$$
\begin{aligned}
&= f(a)g(a)\frac{1}{(s_1+s_2+1)} \\
&\quad +m_2 f(a)g\left(\frac{b}{m_2}\right)\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)} \\
&\quad -c_2 f(a)\left|a-\frac{b}{m_2}\right|^2\frac{1}{(s_1+3)(s_1+2)} \\
&\quad +m_1 f\left(\frac{b}{m_1}\right)g(a)\frac{s_1s_2\Gamma(s_2)\Gamma(s_1)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)} \\
&\quad +m_1m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\frac{1}{(s_1+s_2+1)} \\
&\quad -m_1c_2 f\left(\frac{b}{m_1}\right)\left|a-\frac{b}{m_2}\right|^2\frac{1}{(s_1+3)(s_1+2)} \\
&\quad -c_1\left|a-\frac{b}{m_1}\right|^2 g(a)\frac{1}{(s_2+3)(s_2+2)} \\
&\quad -c_1\left|a-\frac{b}{m_1}\right|^2 m_2 g\left(\frac{b}{m_2}\right)\frac{1}{(s_2+3)(s_2+2)} \\
&\quad +c_1c_2\left|a-\frac{b}{m_1}\right|^2\left|a-\frac{b}{m_2}\right|^2\frac{1}{30} \\
&= \frac{1}{s_1+s_2+1}\left(f(a)g(a)+m_1m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right) \\
&\quad +\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{(s_1+s_2+1)(s_1+s_2)\Gamma(s_1+s_2)}\left(m_2 f(a)g\left(\frac{b}{m_2}\right)\right. \\
&\quad \left.+m_1 f\left(\frac{b}{m_1}\right)g(a)\right) \\
&\quad -\frac{1}{(s_1+3)(s_1+2)}c_2\left|a-\frac{b}{m_2}\right|^2\left(f(a)+m_1 f\left(\frac{b}{m_1}\right)\right) \\
&\quad -\frac{1}{(s_2+3)(s_2+2)}c_1\left|a-\frac{b}{m_1}\right|^2\left(g(a)+m_2 g\left(\frac{b}{m_2}\right)\right) \\
&\quad +c_1c_2\frac{1}{30}\left|a-\frac{b}{m_1}\right|^2\left|a-\frac{b}{m_2}\right|^2.
\end{aligned}
$$

By interchanging a and b, in the same way we obtain

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq M_2,$$

hence

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \min\{M_1,M_2\}.$$

3 Applications

As immediate consequences of theorems 7 and 8, we get the following result.

Proposition 6. *Let $f:[0,+\infty)\to\mathbb{R}$ be a strongly $(s,1)$-convex function, modulus c , and let $a,b\in[0,+\infty)$ with $a<b$. Suppose that $f\in L_1[a,b]$ and that $g:[a,b]\to\mathbb{R}$ is a nonnegative, integrable function which is symmetric with respect to $\frac{a+b}{2}$. Then*

$$
\begin{aligned}
&\int_a^b f(x)g(x)dx \\
&\leq \frac{f(b)+f(a)}{2}\int_a^b\left[\left(\frac{x-a}{b-a}\right)^s+\left(\frac{b-x}{b-a}\right)^s\right]g(x)dx \\
&\quad -c\int_a^b (x-a)(b-x)g(x)dx.
\end{aligned}
$$

Proposition 7. *Let $f:[0,+\infty)\to\mathbb{R}$ be a strongly $(1,1)$-convex function, modulus c , and let $a,b\in[0,+\infty)$ with $a<b$. Then*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(b)+f(a)}{2}-\frac{c}{6}(b-a)^2.$$

Proposition 8. *Let $f:[0,+\infty)\to\mathbb{R}$ be a strongly (s,m)-convex function, modulus c , where $s,m\in(0,1]$, and let $a,b\in[0,+\infty)$ with $a<b$. Suppose that $f\in L_1([a,b])$, and that $g:[a,b]\to\mathbb{R}$ is a nonnegative, integrable function which is symmetric with respect to $\frac{a+b}{2}$. Then*

$$
\begin{aligned}
&2^s f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx - m\int_a^b f\left(\frac{x}{m}\right)g(x)dx \\
&\quad +2^{s-2}c\int_a^b\left|a+b-x-\frac{x}{m}\right|^2 g(x)dx \\
&\leq \int_a^b f(x)g(x)dx \\
&\leq \frac{f(b)+f(a)}{2}\int_a^b\left(\frac{x-a}{b-a}\right)^s g(x)dx \\
&\quad +\frac{m\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)}{2}\int_a^b\left(\frac{b-x}{b-a}\right)^s g(x)dx \\
&\quad -\frac{c}{2}\left[\left|b-\frac{a}{m}\right|^2+\left|a-\frac{b}{m}\right|^2\right]\int_a^b\frac{x-a}{b-a}\cdot\frac{b-x}{b-a}g(x)dx.
\end{aligned}
$$

4 Comments

The main contributions of this paper has been the introduction of a new class of function of generalized convexity, we have shown that these classes contain some previously known classes as special cases as well as Hermite-Hadamard type inequalities for these functions. We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly introduced functions in various fields of pure and applied sciences.

References

[1] A.G. Azpeitia, Convex functions and the Hadamard inequality, Rev. Colombiana Math., 28 1994, 7-12.

[2] M. Bessenyei and Zs. Páles. Characterization of convexity via Hadamard's inequality. *Math. Inequal. Appl. 9*, pages 53–62, 2006.

[3] M. Bracamonte, J. Giménez, N. Merentes and M. Vivas, Fejér type inequalities for $m-$convex functions. Submitted for publication.

[4] S.S. Dragomir, C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.

[5] S.S. Dragomir, G. Toader, Some inequalities for m-convex functions, Studia Univ. Babes- Bolyai, Mathematica, 38(1), 1993, 21-28.

[6] S. S. Dragomir. On some new inequalities of Hermite-Hadamard type for $m-$convex functions. *Tamkang Journal of Mathematics, Volume 33, Number 1, Spring*, pages 45–55, 2002.

[7] N. Eftekhari, Refinements of Hadamard type inequalities for (α,m)-convex functions, International Journal of Pure and Applied Mathematics, Volume 80 No. 5 2012, 673-681.

[8] N. Eftekhari, Some remarks on (s,m)-convexity in the second sense, Journal of mathematical inequalities, Volume 8, Number 3, 2014, 489-495.

[9] L. Fejér. Über die fourierreihen. *II, Math. Naturwiss. Anz Ungar. Akad.Wiss.*, pages 369–390, 1906.

[10] H. Angulo, J. Giménez, A. Moros and K. Nikodem, On strongly h-convex functions, Ann. Funct. Anal. 2, 2011, No 2, 85-91.

[11] A. Kilbas, H.M. Srivastava, and J. Trujillo. *Sur deux limites d'une intégrale définie*. Mathesis 3,82, 1883.

[12] J. Hadamard. étude sur les proprits des fonctions entières en particulier dune fonction considérée par Riemann. *J. Math. Pures Appl., 58*, pages 171–215, 1893.

[13] H. Hudzik and L. Maligranda, Some remarks on $s-$convex functions, Aequationes Mathematicae 48, University of Waterloo, 1994, 100-111.

[14] M. V. Jovanovič, A note on strongly convex and quasiconvex functions, Translation of : Mat. Zametki 60, 1996, 778-779.

[15] M. Klaricic Bacula, J. Pecaric, and M. Ribicic. Companion inequalities to jensen's inequality for $m-$convex and (α,m) convex functions. *J. Inequal. Pure and Appl. Math, 7(5), Art. 194*, pages 26–35, 2006.

[16] M. Kuczma. *An Introduction to the Theory of Functional Equations and Inequalities, Cauchy's Equation and Jensen's Inequality*. Second Edition. Birkhäuser, BaselBostonBerlin, 2009.

[17] T. Lara, N. Merentes, R. Quintero, E. Rosales, On Strongly m-Convex Functions, Mathematica Aeterna, Vol. 5, 2015, No. 3, 521-535.

[18] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, Dordrecht, 1993.

[19] N. Merentes, K. Nikodem, Remarks on Strongly Convex Functions, Ae- qationes. Math. 80, 2010, 193-199.

[20] L. Montrucchio, Lipschitz continuous policy function for strongly concave optimization problems, J. Math. Econom. 16 ,1987, 259-273.

[21] Muharrem T., Erhan S. and Selahattin M., Hermite-Hadamard type inequalities for Log-convex stochastic processes. Journal of New Theory. 2015, N 2, 23-32.

[22] Muharrem T., Erhan S. and Nurgül Okur B., Om Hermite-Hadamard-type inequalities for strongly-Log convex stochastic processes. Küresel Mühendislik Çalşmalar Dergisi Cilt 1 Say 2, 2014, 53-61.

[23] P. Niculescu, L-E. Persson, Convex Functions and their Applications. A contemporary approach CMS Books in Mathematics vol. 23, Springer-Verlag, New York, 2006.

[24] J. Pečarić, F. Proschan, and Y.L Tong. *Convex Functions, Partial Orderings and statistical applications*. Academic Press, Inc. Boston, 1992.

[25] E. S. Polovinkin, Strongly convex analysis, Sbornik: Math., 187 (2) 1996, 259-286.

[26] B. T. Polyak, Existence theorems and convergence of minimizing sequences in extremum problems with restrictions, Soviet Math. Dokl. 7, 1966, 72- 75.

[27] T. Rajba, Sz.Wasowicz, Probalistic characterization of strong convexity, Opuscula Math. 31, 2011, 97-103.

[28] M. Sarikaya, E. Set, M.E. Özdemir, On some new inequalities of Hadamard involving h-convex functions, Acta Math. Univ. Comenianae Vol. LXXIX, 2, 2010, pp. 265.272.

[29] G. Toader, Some generalization of the convexity, Proc. of the Collo. Approx. and Optim., Univ. Cluj-Napoca, 1984, 329-338.

[30] J. Vial, Strong convexity of set and functions. J. Math. Economy. 9, 1982, 187-205.

[31] C. Zalinescu. *Convex Analysis in General Vector Spaces*. World Scientific, New Jersey, 2002.

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