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Numerical Investigation for Solving Two-Point Fuzzy Boundary Value Problems by Reproducing Kernel Approach

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Abstract: In this paper, we implement a relatively recent analytical technique, called iterative reproducing kernel method (IRKM), to obtain a computational solution for fuzzy two-point boundary value problem based on a generalized differentiability concept. The technique methodology is based on construct a solution in the form of a rabidly convergent series with minimum size of calculations using symbolic computation software. The proposed technique is fully compatible with the complexity of such problem, while the obtained results are highly encouraging. Efficacious computational experiments are provided to guarantee the procedure and to illustrate the theoretical statements of the present method in order to show its potentiality, generality and superiority for solving such fuzzy equation.

Keywords: Numerical solutions, Fuzzy boundary value problem, Generalized differentiability, Reproducing kernel approach

1 Introduction

Fuzzy differential equations (FDEs) are extensively used in modeling of complex phenomena arising in applied mathematics, physics, and engineering including fuzzy control theory, quantum optics, atmosphere, artificial intelligence, image processing and dynamical systems [1, 2, 3, 4, 5, 6]. In general, the data collection and analysis for physical phenomena is provided under uncertainty, which may arise in the experiment part and measurement process. Historically, the first approach was the use of Hukuhara differentiability for fuzzy number valued functions. After while, Bede [7] defined the generalized differentiability of fuzzy number valued functions, presented a counterexample that shows a fuzzy two-point boundary value problem is not equivalent to a fuzzy integral equation by using Green's function under the Hukuhara differentiability with fuzzy Aumann-type integral in the integral equation as well as he proved that a fuzzy two-point boundary value problem is usually very complex and hard to be solved analytically under the Hukuhara differentiability concept, in contrast with the main results in [8, 9]. However, there exists no method that yields an explicit solution for FDEs due to the complexities of uncertain parameters involving these equations. Anyhow, in most cases, analytical solutions cannot be found, where the solutions of such equations are always in demand due to practical interests. Therefore, an efficient reliable computer stimulation is required. To deal with this in more realistic situations, FDEs are commonly solved approximately using numerical techniques [10, 11, 12, 13, 14].

The aim of this paper is to extend the application of the iterative reproducing kernel method under the assumption of strongly generalized differentiability to provide a numerical approximate solution for fuzzy two-point boundary value problem [15, 16] in the following form

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t), 0 \le t \le 1, \quad (1)$$

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with fuzzy boundary conditions

$$y(0) = \gamma_0, y(1) = \gamma_1, \tag{2}$$

where $\gamma_0, \gamma_1 \in R_F$ and $p(t), q(t), g(t) : R_F \to R_F$ are continuous fuzzy-valued functions and $y : [0,1] \to R_F \in W_2^3[0,1]$ is unknown function to be determined, in which R_F denote the set of fuzzy numbers on $\mathbb{R}$.

Reproducing kernel Hilbert space method is an analytical as well as numerical method based on the reproducing kernel theory, which has important application in numerical analysis for handling different kinds of differential equations [17,18], integral and integro-differential equations [19,20,21,22,23], probability and statistics and others [24,25]. The proposed method is found to be an effective and smart technique for finding a series solution for strongly linear and nonlinear equations without linearization, perturbation, or discretization. Here, it is possible to pick any point in the interval of integration and as well the approximate solutions and their derivatives will be applicable, where the IRKM is not affected by computation round off errors. On the other hand, many applications for different problems by using other numerical algorithms can be found in [26,27,28,29,30,31].

This article is organized as follows. In the next section, we revisit briefly some necessary definitions and preliminary results of fuzzy calculus theory including the strongly generalized differentiability. Formulation of the solution for handling such problem is presented in Section 3 under the concept of generalized differentiability. In Section 4, the IRK algorithm is built and introduced to illustrate the capability of proposed approach with a numerical experiment and simulation results. The last section is devoted to a short conclusion.

2 Preliminaries

The material in this section is basic in certain sense. For the reader's convenience, we present some necessary definitions and notations from fuzzy calculus theory which be used throughout the paper. A fuzzy number u is a fuzzy subset of the real line with a normal, convex, and upper semicontinuous membership function of bounded support.

Let R_F denote the space of fuzzy real number, i.e., the set normal, fuzzy convex, upper semicontinuous, compactly supported fuzzy sets $u : \mathbb{R} \to [0,1]$. For $0 < \alpha \leq 1$ set $[u]_\alpha = \{s \in \mathbb{R} \mid u(s) \geq \alpha\}$ and $[u]_0 = \overline{\{s \in \mathbb{R} \mid u(s) > 0\}}$ (the closure of $\{s \in \mathbb{R} \mid u(s) > 0\}$). Then the α-level set $[u]_\alpha$ is a non-empty compact interval for all $0 \leq \alpha \leq 1$ and any $u \in R_F$. The notation $[u]_\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$ denotes explicitly the α-level set of u. We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches on u, respectively.

Remark 2.1 [32] The sufficient and necessary conditions for $[\underline{u}, \overline{u}]$ to define the parametric form of a fuzzy number are as follows:
(i) $\underline{u}$ is a bounded monotonic increasing (nondecreasing) left-continuous function $\forall \alpha \in (0,1]$ and right-continuous for $\alpha = 0$.
(ii) $\overline{u}$ is a bounded monotonic decreasing (nonincreasing) left-continuous function $\forall \alpha \in (0,1]$ and right-continuous for $\alpha = 0$.
(iii) $\underline{u} \leq \overline{u}$ for $0 \leq \alpha \leq 1$.

For $u, v \in R_F$ and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product λu are defined by $[u+v]_\alpha = [u]_\alpha + [v]_\alpha$, $[\lambda u]_\alpha = \lambda [u]_\alpha$, $\forall \alpha \in [0,1]$, where $[u]_\alpha + [v]_\alpha$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda [u]_\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$. The metric structure is given by the Hausdorff distance $D : R_F \times R_F \to \mathbb{R}^+ \cup \{0\}$ such that

$$D(u,v) = \sup_{\alpha \in [0,1]} \max\{| \underline{u} - \underline{v} |, | \overline{u} - \overline{v} |\}.$$

Definition 2.1 [32] Let $u, v \in R_F$. If there exists $w \in R_F$ such that $u = v + w$, then w is called the H-difference of u, v and it is denoted $u \ominus v$.

Definition 2.2 [32] Let $F : (a,b) \to R_F$ be a fuzzy function. We say F is differentiable at $t_0 \in (a,b)$, if there exists an element $F'(t_0) \in R_F$ such that the limits

$$\lim_{h \longrightarrow 0^+} \frac{F(t_0+h) \ominus F(t_0)}{h} \text{ and } \lim_{h \longrightarrow 0^+} \frac{F(t_0) \ominus F(t_0-h)}{h},$$

exist and are equal to $F'(t_0)$. Here the limits are taken in the metric space (R_F, D).

By using the H-differentiability *"Hukuhara differentiability concepts"* many existence and uniqueness results are obtained for the fuzzy Cauchy problem

$$y'(t) = f(t, y(t)),\ y(t_0) = y_0,$$

where $f : [t_0, \infty) \times R_F \to R_F$. These results are based on the fact that the Hukuhara-type Cauchy problem is equivalent to an Aumann-type integral equation similar to the classical case. The authors in **[9]** had tried to extend this correspondence to the case of fuzzy two-point boundary value problem. For the purpose of this analysis, we consider the following fuzzy two-point boundary value problem

$$y''(t) = f(t, y(t), y'(t)), \tag{3}$$

$$y(0) = \gamma_0,\ y(1) = \gamma_1,$$

where $\gamma_0, \gamma_1 \in R_F$, $f(t, y(t), y'(t)) = -p(t)y'(t) - q(t)y(t) + g(t)$ and $f : [t_0, \infty) \times R_F \times R_F \longrightarrow R_F$ is a continuous fuzzy function.

Definition 2.3 [32] Let $F : (a,b) \to R_F$ and $t_0 \in (a,b)$. We say F is $(1)-$differentiable at t_0, if there exists an element $F'(t_0) \in R_F$ such that for all $h > 0$ sufficiently near to 0, exist $F(t_0+h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0-h)$ and the limits (in the metric D)

$$\lim_{h\to 0^+} \frac{F(t_0+h) \ominus F(t_0)}{h} =$$
$$\lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0-h)}{h} = F'(t_0).$$

In this case, we denote $F'(t_0)$ by $D_1^1 F(t_0)$. Also, F is $(2)-$differentiable if for all $h > 0$ sufficiently near to 0, exist $F(t_0+h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0-h)$ and the limits (in the metric D)

$$\lim_{h\to 0^-} \frac{F(t_0+h) \ominus F(t_0)}{h} =$$
$$\lim_{h\to 0^-} \frac{F(t_0) \ominus F(t_0-h)}{h} = F'(t_0).$$

In this case, we denote $F'(t_0)$ by $D_2^1 F(t_0)$.

3 Formulation of fuzzy two-point BVPs

In this section, we study the fuzzy BVPs under the concept of strongly generalized differentiability in which the fuzzy differential equation is converted into equivalent system of crisp system of BVPs for each type of differentiability. These can be done if the boundary value is fuzzy number, the solution is fuzzy function, and consequently the derivative must be considered as fuzzy derivative. Furthermore, a computational algorithm is provided to guarantee the procedure and to confirm the performance of the proposed technique. For more details, we refer to **[32,33,34]** and references therein.

Theorem 3.1 Let $F:(a,b) \to R_F$ be a fuzzy function such that $[F(t)]_\alpha = [\underline{f}(t), \overline{f}(t)]$ for each $\alpha \in [0,1]$. Thus, we have that

(i) If F is $(1)-$differentiable, then $\underline{f}$ and $\overline{f}$ are differentiable functions with $[D_1^1 F(t)]_\alpha = [\underline{f}'(t), \overline{f}'(t)]$.

(ii) If F is $(2)-$differentiable, then $\underline{f}$ and $\overline{f}$ are differentiable functions with $[D_2^1 F(t)]_\alpha = [\overline{f}'(t), \underline{f}'(t)]$.

Definition 3.1 Let $F:(a,b) \to R_F$ and $n,m = 1,2$. Then, F is $(n,m)-$differentiable at $t_0 \in (a,b)$, if $D_n^1 F$ exist on a neighborhood of t_0 as a fuzzy function and it is $(m)-$differentiable at t_0. The second derivatives of F is denoted by $D_{n,m}^2 F(t_0)$ for $n,m = 1,2$.

Theorem 3.2 Let $D_1^1 F : (a,b) \to R_F$ and $D_2^1 F : (a,b) \to R_F$ be fuzzy functions such that $[F(t)]_\alpha = [\underline{f}(t), \overline{f}(t)]$, then

(i) If $D_1^1 F$ is $(1)-$differentiable, then $\underline{f}'(t)$ and $\overline{f}'(t)$ are differentiable functions such that $[D_{1,1}^2 F(t)]_\alpha = [\underline{f}''(t), \overline{f}''(t)]$.

(ii) If $D_1^1 F$ is $(2)-$differentiable, then $\underline{f}'(t)$ and $\overline{f}'(t)$ are differentiable functions such that $[D_{1,2}^2 F(t)]_\alpha = [\overline{f}''(t), \underline{f}''(t)]$.

(iii) If $D_2^1 F$ is $(1)-$differentiable, then $\underline{f}'(t)$ and $\overline{f}'(t)$ are differentiable functions such that $[D_{2,1}^2 F(t)]_\alpha = [\overline{f}''(t), \underline{f}''(t)]$.

(iv) If $D_2^1 F$ is $(2)-$differentiable, then $\underline{f}'(t)$ and $\overline{f}'(t)$ are differentiable functions such that $[D_{2,2}^2 F(t)]_\alpha = [\underline{f}''(t), \overline{f}''(t)]$.

Definition 3.2 Let $y:[0,1] \to R_F$ be a fuzzy function and $n,m \in \{1,2\}$. Then, we say that y is a $(n,m)-$solution for problem (3) on $[0,1]$, if $D_n^1 y$ and $D_{n,m}^2 y$ exist on $[0,1]$, where $D_{n,m}^2 y(t) = f(t, y(t), D_n^1 y(t))$, $y(0) = \gamma_0$, $y(1) = \gamma_1$ and $\gamma_0, \gamma_1 \in R_F$.

Definition 3.4 Let $y:[0,1] \to R_F$ be a fuzzy function and $n,m \in \{1,2\}$. Then, we say that y is a $(n,m)-$solution for problem (3) on an interval $I \subset [0,1]$, if $D_n^1 y$, $D_{n,m}^2 y$ exist such that $D_{n,m}^2 y(t) = f(t, y(t), D_n^1 y(t))$ on I.

Definition 3.5 Let $n, m, n^{`}, m^{`} \in \{1,2\}$. Suppose $y:[0,1] \to R_F$ and $t_0 \in (0,1)$ such that $y(0) = \gamma_0$, if y is a $(n,m)-$solution of (3) on $(0,t_0)$ as well as y is a $(n^{`}, m^{`})-$solution of (3) on $(t_0,1)$ with $y(1) = \gamma_1$. Then, we say that y is a generalized solution of the FBVP (3).

Let $y = [\underline{y}, \overline{y}]$ be a $(n,m)-$solution for the FBVP (3) with $\gamma_0 = [\underline{\gamma}_0, \overline{\gamma}_0]$ and $\gamma_1 = [\underline{\gamma}_1, \overline{\gamma}_1]$. Hereafter, y is called the corresponding $(n,m)-$system for the FBVP (3). However, y has to be converted into crisp systems of BVPs. That is, there are four possible crisp systems for (3) that can be represented as follow:

$(1,1)-$system

$$\begin{aligned}
\underline{y}''(t) &= \underline{f}(t, y(t), D_1^1 y(t)),\\
\overline{y}''(t) &= \overline{f}(t, y(t), D_1^1 y(t)),\\
\underline{y}(0) &= \underline{\gamma}_0, \overline{y}(0) = \overline{\gamma}_0,\\
\underline{y}(1) &= \underline{\gamma}_1, \overline{y}(1) = \overline{\gamma}_1.
\end{aligned}$$

$(1,2)-$system

$$\begin{aligned}
\overline{y}''(t) &= \underline{f}(t, y(t), D_1^1 y(t)),\\
\underline{y}''(t) &= \overline{f}(t, y(t), D_1^1 y(t)),\\
\underline{y}(0) &= \underline{\gamma}_0, \overline{y}(0) = \overline{\gamma}_0,\\
\underline{y}(1) &= \underline{\gamma}_1, \overline{y}(1) = \overline{\gamma}_1.
\end{aligned}$$

$(2,1)-$system

$$\begin{aligned}
\overline{y}''(t) &= \underline{f}(t, y(t), D_2^1 y(t)),\\
\underline{y}''(t) &= \overline{f}(t, y(t), D_2^1 y(t)),\\
\underline{y}(0) &= \underline{\gamma}_0, \overline{y}(0) = \overline{\gamma}_0,\\
\underline{y}(1) &= \underline{\gamma}_1, \overline{y}(1) = \overline{\gamma}_1.
\end{aligned}$$

$(2,2)-$system

$$\begin{aligned}\underline{y}''(t) &= \underline{f}(t, y(t), D_2^1 y(t)),\\ \overline{y}''(t) &= \overline{f}(t, y(t), D_2^1 y(t)),\\ \underline{y}(0) &= \underline{\gamma}_0, \overline{y}(0) = \overline{\gamma}_0,\\ \underline{y}(1) &= \underline{\gamma}_1, \overline{y}(1) = \overline{\gamma}_1.\end{aligned}$$

4 Application and numerical simulation

Numerical technique is widely used by scientists and engineers to solve their problems. A major advantage for numerical technique is that a numerical answer can be obtained even when a problem has no analytical solution. However, result from numerical analysis is an approximation, in general, which can be made as accurate as desired. The reliability of the numerical result will depend on an error estimate and bound, therefore the analysis of error and the sources of error in numerical methods is also a critically important part of the study of numerical technique. In this section, we derive an error bounds for the present method in order to capture the behavior of the solutions.

In order to solve the FBVP (3) in the reproducing kernel space, we firstly need to convert the nonhomogeneous boundary conditions into homogeneous ones throughout the cases of previous $(n,m)-$systems.

Letting $\underline{u}(t) = \underline{y}(t) - t(\underline{\gamma}_1 - \underline{\gamma}_0) - \underline{\gamma}_0$ and $\overline{u}(t) = \overline{y}(t) - t(\overline{\gamma}_1 - \overline{\gamma}_0) - \overline{\gamma}_0$, then $(1,1)-$system can be converted into the following form

$$\begin{aligned}\underline{u}''(t) &= \underline{f}(t, u(t), D_1^1 u(t)),\\ \overline{u}''(t) &= \overline{f}(t, u(t), D_1^1 u(t)),\\ \underline{u}(0) &= 0,\ \overline{u}(0) = 0,\\ \underline{u}(1) &= 0,\ \overline{u}(1) = 0.\end{aligned}$$

Similarly the $(1,2)-$system, $(2,1)-$system and $(2,2)-$system have to be converted.

Definition 4.1 [35] Let E be a nonempty abstract set and $\mathbb{C}$ be the set of complex numbers. A function $K : E \times E \to \mathbb{C}$ is a reproducing kernel of the Hilbert space H if

(i) for each $x \in E$, $K(\cdot, x) \in H$,

(ii) for each $x \in E$ and $\varphi \in H$, $\langle \varphi(\cdot), K(\cdot, x)\rangle = \varphi(x)$.

Let $W_2^3[0,1]$ be a Hilbert space **[35]**, which is defined as follows

$$W_2^3[0,1] = \{u(t) \mid u''(t) \text{ is absolutely continuous},\ u'''(t) \in L_2[0,1],\ u(0) = u(1) = 0,\ t \in [0,1]\},$$

while the inner product and the norm of $W_2^3[0,1]$ are defined, respectively, as follows: For any functions $u(t)$, $v(t) \in W_2^3[0,1]$, we have

$$\langle u, v\rangle = \sum_{i=0}^{2} u^{(i)}(0) v^{(i)}(0) + \int_0^1 u'''(t) v'''(t) dt,$$
$$\|u\| = \sqrt{\langle u, u\rangle}.$$

Remark 4.1 [35] The space $W_2^3[0,1]$ is a reproducing kernel space if and only if for any $t \in [0,1]$, $I : f \to f(t)$ is a bounded functional in $W_2^3[0,1]$.

Theorem 4.1 The space $W_2^3[0,1]$ is a complete reproducing kernel space and the reproducing kernel function can be written as

$$R_t(s) = \begin{cases} \sum_{i=0}^{5} p_i(t) s^i,\ s \le t,\\ \sum_{i=0}^{5} q_i(t) s^i,\ s > t.\end{cases}$$

Proof. The proof of the completeness and reproducing property of $W_2^3[0,1]$ is similar to the proof in **[35]**. Now, suppose $R_t(s)$ is the reproducing kernel function of the space $W_2^3[0,1]$, then for each fixed $t \in [0,1]$ and any $u(s) \in W_2^3[0,1]$, $s \in [0,1]$, we have that $\langle u(s), R_t(s)\rangle = u(t)$. Thus,

$$\langle u(s), R_t(s)\rangle_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(0) R_t^{(i)}(0) + \int_0^1 u'''(\zeta) R_t'''(\zeta) d\zeta. \quad (4)$$

By applying the integration by parts for the second scheme of the right-hand of Equation (4), we obtain that

$$\int_0^1 u'''(\zeta) R_t'''(\zeta) d\zeta = \sum_{i=0}^{2} (-1)^i u^{(2-i)}(\zeta) R_t^{(3+i)}(\zeta) |_{\zeta=0}^{\zeta=1} + \int_0^1 (-1)^3 u(\zeta) R_t^{(6)}(\zeta) d\zeta.$$

Let $j = 2 - i$, the first term of the right-hand side of the above formula can be rewritten as

$$\sum_{j=0}^{2} (-1)^{2-j} u^{(j)}(\zeta) R_t^{(5-j)}(\zeta) |_{\zeta=0}^{\zeta=1}.$$

After some simplification, Equation (4) became's

$$\begin{aligned}\langle u(s), R_t(s)\rangle &= \sum_{i=0}^{2} u^{(i)}(0) \left(R_t^{(i)}(0) - (-1)^{2-i} R_t^{(5-i)}(0)\right)\\ &+ \sum_{i=0}^{2} (-1)^{2-i} u^{(i)}(1) R_t^{(5-i)}(1)\\ &+ \int_0^1 (-1)^3 u(\zeta) R_t^{(6)}(\zeta) d\zeta.\end{aligned}$$

Since $R_t(s), u(s) \in W_2^3[0,1]$, it follows that

$$\begin{aligned}&R_t^{(i)}(0) - (-1)^{2-i} R_t^{(5-i)}(0) = 0,\\ &R_t^{(5-i)}(1) = 0,\ i = 0,1,2.\end{aligned}$$

So,

$$\langle u(s), R_t(s)\rangle_{W_2^3[0,1]} = \int_0^1 u(\zeta)\left((-1)^3 R_t^{(6)}(\zeta)\right)d\zeta$$

Now, for each $t \in [0,1]$, if $R_t(s)$ satisfies $(-1)^3 R_t^{(6)}(s) = \delta(t-s)$, where δ is dirac-delta function, then

$$\begin{aligned}\langle u(s), R_t(s)\rangle_{W_2^3[0,1]} &= \int_0^1 u(s)\,\delta(t-s)\,dy\\ &= u(t).\end{aligned}$$

Obviously, $R_t(s)$ is the reproducing kernel of the space $W_2^3[0,1]$. Therefore, $R_t(s)$ is the solution of the following generalized differential equations:

$$\begin{cases}(-1)^3 R_t^{(6)}(s) = \delta(t-s),\\ R_t^{(i)}(0) - (-1)^{2-i} R_t^{(5-i)}(0) = 0, i = 0,1,2,\\ R_t^{(5-i)}(1) = 0, i = 0,1,2.\end{cases} \quad (5)$$

While $t \neq s$

$$(-1)^3 R_t^{(6)}(s) = 0. \quad (6)$$

with the boundary conditions (BC's)

$$\begin{aligned}&R_t^{(i)}(0) - (-1)^{2-i} R_t^{(5-i)}(0) = 0,\\ &R_t^{(5-i)}(1) = 0, i = 0,1,2.\end{aligned} \quad (7)$$

The characteristic equation of Equation (6) is $\lambda^6 = 0$, and their characteristic values are $\lambda = 0$ with 6 multiple roots. So, the general solution of Equation (6) is as follows

$$R_t(s) = \begin{cases}\sum_{i=0}^{5} p_i(t)s^i,\ s \le t;\\ \sum_{i=0}^{5} q_i(t)s^i,\ s > t.\end{cases} \quad (8)$$

On the other hand, since $(-1)^3 R_t^{(3)}(s) = \delta(t-s)$, we have

$$R_t^{(i)}(t+0) = R_t^{(i)}(t-0),\ i = 0,1,2,3,4. \quad (9)$$

Integrating $(-1)^3 R_t^{(6)}(s) = \delta(t-s)$ from $t-\varepsilon$ to $t+\varepsilon$ with respect to s and let $\varepsilon \to 0$, we have the jump degree of $R_t^{(5)}(s)$ at $t = s$ given by

$$\left(R_t^{(5)}(t+0) - R_t^{(5)}(t-0)\right) = -1. \quad (10)$$

So, the representation of the reproducing kernel function $R_t(s)$ in $W_2^3[0,1]$, using Mathematica software package, is provided by

$$R_t(s) = \begin{cases}1 + \frac{s^5}{120} + \frac{1}{12}t^2 s^2(3+s) + ts(1 - \frac{s^4}{24}), s \le t,\\ 1 + \frac{t^5}{120} + \frac{1}{12}t^2 s^2(3+t) + ts(1 - \frac{t^4}{24}), s > t.\end{cases}$$

Remark 4.2 If a Hilbert space H of functions on a set E admits a reproducing kernel, then the reproducing kernel $R_t(s)$ is uniquely determined by the Hilbert space H.

Let us assume that we have the system of second-order differential equations in the form

$$\begin{aligned}\underline{u}''(t) &= f(t, \underline{u}(t), \underline{u}'(t), \overline{u}(t), \overline{u}'(t)),\ 0 < t < 1,\\ \overline{u}''(t) &= g(t, \underline{u}(t), \underline{u}'(t), \overline{u}(t), \overline{u}'(t)),\ 0 < t < 1,\\ \underline{u}(0) &= 0,\ \underline{u}(1) = 0, \overline{u}(0) = 0,\ \overline{u}(1) = 0.\end{aligned} \quad (1)$$

and let $Lu = u''$, $L : W_2^3[0,1] \to W_2^1[0,1]$, then, system (1) can be converted into the following form

$$\begin{aligned}L\underline{u}(t) &= f(t, \underline{u}(t), \underline{u}'(t), \overline{u}(t), \overline{u}'(t)),\ 0 < t < 1,\\ L\overline{u}(t) &= g(t, \underline{u}(t), \underline{u}'(t), \overline{u}(t), \overline{u}'(t)),\ 0 < t < 1,\end{aligned} \quad (12)$$

where $\underline{u}(t)$, $\overline{u}(t) \in W_2^3[0,1]$ and f, $g \in W_2^1[0,1]$. Clear that L is a bounded linear operator.

Now, we construct an orthogonal system of the space $W_2^3[0,1]$. Let $\Phi_i(t) = R_{t_i}(t)$ and $\psi_i(t) = L^*\Phi_i(t)$, where L^* is the conjugate operator of L. In terms of the properties of reproducing kernel $R_t(s)$, one obtains that

$$\begin{aligned}\langle u(t), \psi_i(t)\rangle_{W_2^3} &= \langle u(t), L^*\Phi_i(t)\rangle_{W_2^3}\\ &= \langle Lu(t), \Phi_i(t)\rangle_{W_2^1} = Lu(t_i),\\ &i = 1,2,....\end{aligned}$$

Lemma 4.1 If $\{t_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\psi_i(t)\}_{i=1}^{\infty}$ is a complete system of $W_2^3[0,1]$ if L^{-1} in (12) existent and $\psi_i(t) = L_s R_t(s)_{|s=t_i}$, then

$$\psi_i(t) = \frac{d^2 R_t(s)}{dt^2}\ |_{s=t_i}\ . \quad (13)$$

Proof. For each fixed $u(t) \in W_2^3[0,1]$. If $\langle u(t), \psi_i(t)\rangle_{W_2^3} = 0$, $i = 1,2,...$, then

$$\begin{aligned}\langle u(t), \psi_i(t)\rangle_{W_2^3} &= \langle u(t), L^*\Phi_i(t)\rangle_{W_2^3}\\ &= \langle Lu(t), \Phi_i(t)\rangle_{W_2^1} = Lu(t_i) = 0.\end{aligned}$$

Note that $\{t_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, therefore, $Lu(t) = 0$. It follows that $u(t) = 0$ from the existence of L^{-1} and the continuity of $u(t)$.

Moreover, the orthonormal system of $\{\overline{\psi}_i(t)\}_{i=1}^{\infty}$ in $W_2^3[0,1]$ can be derived by using Gram-Schmidt orthogonalization process of $\{\psi_i(t)\}_{i=1}^{\infty}$ as follows

$$\overline{\psi}_i(t) = \sum_{k=1}^{i} \beta_{ik}\psi_k(t), \quad (14)$$

where β_{ik} are orthogonalization coefficients, $\beta_{ii} > 0$, $i = 1, 2, ..., n$, and

$$\beta_{11} = \frac{1}{\|\psi_1\|},$$

$$\beta_{ij} = \frac{-\sum_{k=j}^{i-1} c_{ik}\beta_{kj}}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1}(c_{ik})^2}} \ (j < i),$$

$$\beta_{ii} = \frac{1}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1}(c_{ik})^2}} \ (i > 1).$$

in which $c_{ik} = \langle \psi_i, \overline{\psi}_k \rangle_{W_2^3[0,1]}$ and $\{\overline{\psi}_i(t)\}_{i=1}^{\infty}$ is the orthonormal system in the space $W_2^3[0,1]$.

Lemma 4.2 [36] If $u(t) \in W_2^3[0,1]$, then there exists $M > 0$, such that $\|u\|_{C^2[0,1]} \leq \|u\|_{W_2^3[0,1]}$, where

$$\|u\|_{C^2[0,1]} = \max_{t\in[0,1]} |u(t)| + \max_{t\in[0,1]} |u'(t)| + \max_{t\in[0,1]} |u''(t)| .$$

Lemma 4.3 [36] If $\|\underline{u}_n - \underline{u}\|_{W_2^3} \to 0$, $\|\overline{u}_n - \overline{u}\|_{W_2^3} \to 0$, $t_n \to t$, $(n \to \infty)$ and $f(t, \underline{u}(t), \underline{u}'(t), \overline{u}(t), \overline{u}'(t))$, $g(t, \underline{u}(t), \underline{u}'(t), \overline{u}(t), \overline{u}'(t))$ for $t \in [0,1]$ are continuous with respect to t, then

$$f(t_n, \underline{u}_{n-1}(t_n), \underline{u}'_{n-1}(t_n), \bar{u}_{n-1}(t_n), \bar{u}'_{n-1}(t_n)) \to f(t, \underline{u}(t), \underline{u}'(t), \overline{u}(t), \overline{u}'(t)) \text{ as } n \to \infty,$$

$$g(t_n, \underline{u}_{n-1}(t_n), \underline{u}'_{n-1}(t_n), \bar{u}_{n-1}(t_n), \bar{u}'_{n-1}(t_n)) \to g(t, \underline{u}(t), \underline{u}'(t), \overline{u}(t), \overline{u}'(t)) \text{ as } n \to \infty.$$

In the next theorem, we will give the presentation of the exact solutions of system (12) in the IRKM.

Theorem 4.2. If $\{t_i\}_{i=1}^{\infty}$ is dense on $[0,1]$ and $\underline{u}(t), \overline{u}(t) \in W_2^3[0,1]$ are the solutions of (12), then $\underline{u}(t), \overline{u}(t)$ satisfy the following form, respectively

$$\underline{u}(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(t, \underline{u}(t), \underline{u}'(t), \overline{u}(t), \overline{u}'(t))\,|_{t=t_k}\ \overline{\psi}_i(t), \tag{15}$$

$$\overline{u}(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} g(t, \underline{u}(t), \underline{u}'(t), \overline{u}(t), \overline{u}'(t))\,|_{t=t_k}\ \overline{\psi}_i(t), \tag{16}$$

while the approximate solutions can be obtained by

$$\underline{u}_n(t) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(t_k, \underline{u}_{k-1}(t_k), ..., \overline{u}'_{k-1}(t_k))\overline{\psi}_i(t), \tag{17}$$

$$\overline{u}_n(t) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} g(t_k, \underline{u}_{k-1}(t_k), ..., \overline{u}'_{k-1}(t_k))\overline{\psi}_i(t), \tag{18}$$

where $\underline{u}_0(t), \overline{u}_0(t)$ (fixed) $\in W_2^3[0,1]$.

Proof. Clear that $\{\overline{\psi}_i(t)\}_{i=1}^{\infty}$ is the complete orthonormal basis in $W_2^3[0,1]$. Since $\underline{u}(t) \in W_2^3[0,1]$, then it can be expanded in the form of Fourier series about $\{\overline{\psi}_i(t)\}_{i=1}^{\infty}$ such that

$$\begin{aligned}
\underline{u}(t) &= \sum_{i=1}^{\infty} \langle \underline{u}(t), \overline{\psi}_i(t)\rangle_{W_2^3} \overline{\psi}_i(t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle \underline{u}(t), \psi_k(t)\rangle_{W_2^3} \overline{\psi}_i(t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle \underline{u}(t), L^*\Phi_k(t)\rangle_{W_2^3} \overline{\psi}_i(t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle L\underline{u}(t), \Phi_k(t)\rangle_{W_2^1} \overline{\psi}_i(t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle f(t, \underline{u}(t), ..., \overline{u}'(t)), \Phi_k(t)\rangle_{W_2^1} \overline{\psi}_i(t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(t_k, \underline{u}(t_k), \underline{u}'(t_k), \overline{u}(t_k), \overline{u}'(t_k))\overline{\psi}_i(t).
\end{aligned}$$

In the same way, we can get that $\overline{u}(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} g(t_k, \underline{u}(t_k), \underline{u}'(t_k), \overline{u}(t_k), \overline{u}'(t_k))\overline{\psi}_i(t)$. The approximate solutions can be also obtained by the n-term intercept of the exact solutions $\underline{u}(t)$ and $\overline{u}(t)$ such that

$$\underline{u}_n(t) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(t_k, \underline{u}_{k-1}(t_k), ..., \overline{u}'_{k-1}(t_k))\overline{\psi}_i(t),$$

$$\overline{u}_n(t) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} g(t_k, \underline{u}_{k-1}(t_k), ..., \overline{u}'_{k-1}(t_k))\overline{\psi}_i(t),$$

where $\underline{u}_0(t) = 0$, $\overline{u}_0(t) = 0$ such that $\underline{u}_0(t)$, $\overline{u}_0(t)$ (Fixed) $\in W_2^3[0,1]$.

Next, we show that $\underline{u}_n(t)$ and $\overline{u}_n(t)$ in iterative formulas (17) and (18) are convergent uniformly to the exact solutions $\underline{u}(t)$ and $\overline{u}(t)$ of system (12), respectively.

Theorem 4.3 Suppose the following conditions are satisfied:

(i) $\|\underline{u}_n\|_{W_2^3}$, $\|\overline{u}_n\|_{W_2^3}$ are bounded.

(ii) $\{t_i\}_{i=1}^{\infty}$ is dense on $[0,1]$.

(iii) $f(t, a_1, a_2, a_3, a_4), g(t, a_1, a_2, a_3, a_4) \in W_2^1[0,1]$ for any $a_1 = \underline{u}(t)$, $a_2 = \underline{u}'(t)$, $a_3 = \overline{u}(t)$, $a_4 = \overline{u}'(t) \in W_2^3[0,1]$.

Then $\underline{u}_n(t)$, $\overline{u}_n(t)$ in iterative formulas (17) and (18) are convergent to the exact solutions $\underline{u}(t)$, $\overline{u}(t)$ of system (12) in $W_2^3[0,1]$ and

$$\underline{u}(t)=\sum_{i=1}^{\infty}A_i\overline{\psi}_i,\ \overline{u}(t)=\sum_{i=1}^{\infty}B_i\overline{\psi}_i,$$

where

$$A_i=\sum_{k=1}^{i}\beta_{ik}f(t,\underline{u}_{k-1}(t),...,\overline{u}'_{k-1}(t))\,|_{t=t_k}\ ,$$

$$B_i=\sum_{k=1}^{i}\beta_{ik}g(t,\underline{u}_{k-1}(t),...,\overline{u}'_{k-1}(t))\,|_{t=t_k}\ .$$

Proof. First of all, we will prove the convergence of $\underline{u}_n(t)$, $\overline{u}_n(t)$. From Equations (17) and (18), we have that

$$\underline{u}_{n+1}(t)=\underline{u}_n(t)+A_{n+1}\overline{\psi}_{n+1}(t)\,,$$
$$\overline{u}_{n+1}(t)=\overline{u}_n(t)+B_{n+1}\overline{\psi}_{n+1}(t)\,.$$

By the orthogonality of $\{\overline{\psi}_i(t)\}_{i=1}^{\infty}$, it follows that

$$\begin{aligned}||\underline{u}_{n+1}||^2_{W_2^3}&=||\underline{u}_n||^2_{W_2^3}+(A_{n+1})^2\\&=||\underline{u}_{n-1}||^2_{W_2^3}+(A_n)^2+(A_{n+1})^2\\&\cdots\\&=||\underline{u}_0||^2_{W_2^3}+\sum_{i=1}^{n+1}(A_i)^2,\\||\overline{u}_{n+1}||^2_{W_2^3}&=||\overline{u}_n||^2_{W_2^3}+(B_{n+1})^2\\&=||\overline{u}_{n-1}||^2_{W_2^3}+(B_n)^2+(B_{n+1})^2\\&\cdots\\&=||\overline{u}_0||^2_{W_2^3}+\sum_{i=1}^{n+1}(B_i)^2.\end{aligned}$$

From the boundedness of $||\underline{u}_n||_{W_2^3}$ and $||\overline{u}_n||_{W_2^3}$, we have $\sum_{i=1}^{\infty}(A_i)^2<\infty$, $\sum_{i=1}^{\infty}(B_i)^2<\infty$, that is, $\{A_i\}_{i=1}^{\infty}$, $\{B_i\}_{i=1}^{\infty}\in l^2\ (i=1,2,...)$.

Let $m>n$, for $(\underline{u}_m-\underline{u}_{m-1})\perp(\underline{u}_{m-1}-\underline{u}_{m-2})$ $\perp...\perp(\underline{u}_{n+1}-\underline{u}_n)$, $(\overline{u}_m-\overline{u}_{m-1})\perp(\overline{u}_{m-1}-\overline{u}_{m-2})$ $\perp...\perp(\overline{u}_{n+1}-\overline{u}_n)$, it follows that

$$\begin{aligned}||\underline{u}_m(t)-\underline{u}_n(t)||^2_{W_2^3}&=\|\,\underline{u}_m(t)-\underline{u}_{m-1}(t)+\underline{u}_{m-1}(t)\\&\quad-...+\underline{u}_{n+1}(t)-\underline{u}_n(t)\,\|^2_{W_2^3}\\&=||\underline{u}_m(t)-\underline{u}_{m-1}(t)||^2_{W_2^3}+...\\&\quad+||\underline{u}_{n+1}(t)-\underline{u}_n(t)||^2_{W_2^3}\\&=\sum_{i=n+1}^{m}(A_i)^2\longrightarrow0,\ (n\to\infty),\end{aligned}$$

$$\begin{aligned}||\overline{u}_m(t)-\overline{u}_n(t)||^2_{W_2^3}&=\|\,\overline{u}_m(t)-\overline{u}_{m-1}(t)+\overline{u}_{m-1}(t)\\&\quad-...+\overline{u}_{n+1}(t)-\overline{u}_n(t)\,\|^2_{W_2^3}\\&=||\overline{u}_m(t)-\overline{u}_{m-1}(t)||^2_{W_2^3}+...\\&\quad+||\overline{u}_{n+1}(t)-\overline{u}_n(t)||^2_{W_2^3}\\&=\sum_{i=n+1}^{m}(B_i)^2\longrightarrow0,\ (n\to\infty).\end{aligned}$$

Considering the completeness of $W_2^3[0,1]$, there exists $\underline{u}(t),\overline{u}(t)\in W_2^3[0,1]$ such that $\underline{u}_n(t)\to\underline{u}(t)$ as $n\to\infty$ in sense of the norm of $W_2^3[0,1]$ and $\overline{u}_n(t)\to\overline{u}(t)$ as $n\to\infty$ in sense of the norm of $W_2^3[0,1]$.

Next, we will prove that $\underline{u}(t)$ and $\overline{u}(t)$ are the solutions of system (12). Since $\{t_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, we know that $\underline{u}_n(t)$ and $\overline{u}_n(t)$ converge uniformly to $\underline{u}(t)$ and $\overline{u}(t)$, respectively. By taking limits of Equations (17) and (18), it follows that

$$\underline{u}(t)=\sum_{i=1}^{\infty}A_i\overline{\psi}_i,\ \overline{u}(t)=\sum_{i=1}^{\infty}B_i\overline{\psi}_i.$$

Since

$$\begin{aligned}(L\underline{u})(t_j)&=\sum_{i=1}^{\infty}A_i\left\langle L\overline{\psi}_i(t),\Phi_j(t)\right\rangle_{W_2^1}\\&=\sum_{i=1}^{\infty}A_i\left\langle\overline{\psi}_i(t),L^*\Phi_j(t)\right\rangle_{W_2^3}\\&=\sum_{i=1}^{\infty}A_i\left\langle\overline{\psi}_i(t),\psi_j(t)\right\rangle_{W_2^3},\end{aligned}$$

and

$$\begin{aligned}(L\overline{u})(t_j)&=\sum_{i=1}^{\infty}B_i\left\langle L\overline{\psi}_i(t),\Phi_j(t)\right\rangle_{W_2^1}\\&=\sum_{i=1}^{\infty}B_i\left\langle\overline{\psi}_i(t),L^*\Phi_j(t)\right\rangle_{W_2^3}\\&=\sum_{i=1}^{\infty}B_i\left\langle\overline{\psi}_i(t),\psi_j(t)\right\rangle_{W_2^3}.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{j=1}^{n}\beta_{nj}(L\underline{u})(t_j)&=\sum_{i=1}^{\infty}A_i\left\langle\overline{\psi}_i(t),\sum_{j=1}^{n}\beta_{nj}\psi_j(t)\right\rangle_{W_2^3}\\&=\sum_{i=1}^{\infty}A_i\left\langle\overline{\psi}_i(t),\overline{\psi}_n(t)\right\rangle_{W_2^3}=A_n,\end{aligned}$$

and

$$\begin{aligned}\sum_{j=1}^{n}\beta_{nj}(L\overline{u})(t_j)&=\sum_{i=1}^{\infty}B_i\left\langle\overline{\psi}_i(t),\sum_{j=1}^{n}\beta_{nj}\psi_j(t)\right\rangle_{W_2^3}\\&=\sum_{i=1}^{\infty}B_i\left\langle\overline{\psi}_i(t),\overline{\psi}_n(t)\right\rangle_{W_2^3}=B_n.\end{aligned}$$

If $n = 1$, then

$$(L\underline{u})(t_1) = f(t_1, \underline{u}_0(t_1), \underline{u}_0'(t_1), \overline{u}_0(t_1), \overline{u}_0'(t_1)),$$
$$(L\overline{u})(t_1) = g(t_1, \underline{u}_0(t_1), \underline{u}_0'(t_1), \overline{u}_0(t_1), \overline{u}_0'(t_1)).$$

If $n = 2$, then

$$(L\underline{u})(t_2) = f(t_2, \underline{u}_1(t_2), \underline{u}_1'(t_2), \overline{u}_1(t_2), \overline{u}_1'(t_2)),$$
$$(L\overline{u})(t_2) = g(t_2, \underline{u}_1(t_2), \underline{u}_1'(t_2), \overline{u}_1(t_2), \overline{u}_1'(t_2)).$$

Furthermore, by induction, it is easy to see that

$$(L\underline{u})(t_j) = f(t_j, \underline{u}_{j-1}(t_j), ..., \overline{u}_{j-1}'(t_j)),$$
$$(L\overline{u})(t_j) = g(t_j, \underline{u}_{j-1}(t_j), ..., \overline{u}_{j-1}'(t_j)).$$

Since $\{t_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, for any $y \in [0,1]$, there exists subsequence $\{t_{n_j}\}$ such that $t_{n_j} \to y$, as $j \to \infty$. Hence, let $j \to \infty$ in the last equations, thus by the convergence of $\underline{u}_n(t)$, $\overline{u}_n(t)$ and Lemma (4.3), we get that

$$(L\underline{u})(y) = f(y, \underline{u}(y), \underline{u}'(y), \overline{u}(y), \overline{u}'(y)),$$
$$(L\overline{u})(y) = g(y, \underline{u}(y), \underline{u}'(y), \overline{u}(y), \overline{u}'(y)).$$

That is, $\underline{u}(t)$ and $\overline{u}(t)$ are the solutions of system (12) with

$$\underline{u}(t) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i, \ \overline{u}(t) = \sum_{i=1}^{\infty} B_i \overline{\psi}_i.$$

Example 4.1 Let us consider the following fuzzy two-point boundary value problem

$$y''(t) = 2\gamma, \ y(0) = \frac{1}{8}\gamma, \ y(1) = \frac{3}{8}\gamma, \ t \in [0,1], \quad (19)$$

where γ is the triangular fuzzy number having $\alpha-$level sets $[\alpha - 1, 1 - \alpha]$, $\alpha \in [0,1]$.

In order to illustrate the performance of the IRKM for solving FBVP (19), we present the following four cases:

If y is a $(1,1)-$solution for Equation (19), then

$$\begin{aligned} &\underline{y}''(t) = 2(\alpha - 1), \\ &\underline{y}(0) = \frac{\alpha - 1}{8}, \ \underline{y}(1) = \frac{3(\alpha - 1)}{8}, \\ &\overline{y}''(t) = 2(1 - \alpha), \\ &\overline{y}(0) = \frac{1 - \alpha}{8}, \ \overline{y}(1) = \frac{3(1 - \alpha)}{8}. \end{aligned} \quad (20)$$

where the exact solutions are

$$\begin{aligned} \underline{y}(t) &= \frac{\alpha - 1}{8}(8t^2 - 6t + 1), \\ \overline{y}(t) &= \frac{1 - \alpha}{8}(8t^2 - 6t + 1). \end{aligned} \quad (21)$$

Using the RKHS method by taking $t_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$, and $n = 101$, the numerical results of system (20) are given in Table 1, Table 2 and Figure 1.

If y is a $(1,2)-$solution for the Equation (19), then

$$\begin{aligned} &\underline{y}''(t) = 2(1 - \alpha), \\ &\underline{y}(0) = \frac{\alpha - 1}{8}, \ \underline{y}(1) = \frac{3(\alpha - 1)}{8}, \\ &\overline{y}''(t) = 2(\alpha - 1), \\ &\overline{y}(0) = \frac{1 - \alpha}{8}, \ \overline{y}(1) = \frac{3(1 - \alpha)}{8}. \end{aligned} \quad (22)$$

where the exact solutions are

$$\begin{aligned} \underline{y}(t) &= \frac{-(\alpha - 1)}{8}(8t^2 - 10t - 1), \\ \overline{y}(t) &= \frac{-(1 - \alpha)}{8}(8t^2 - 10t - 1). \end{aligned} \quad (23)$$

The numerical results of system (22) are given in Table 3, Table 4 and Figure 2.

If y is a $(2,2)-$solution for Equation (19), then

$$\begin{aligned} &\underline{y}''(t) = 2(\alpha - 1), \\ &\underline{y}(0) = \frac{\alpha - 1}{8}, \ \underline{y}(1) = \frac{3(\alpha - 1)}{8}, \\ &\overline{y}''(t) = 2(1 - \alpha), \\ &\overline{y}(0) = \frac{1 - \alpha}{8}, \ \overline{y}(1) = \frac{3(1 - \alpha)}{8}. \end{aligned} \quad (24)$$

where the exact solutions are

$$\begin{aligned} \underline{y}(t) &= \frac{\alpha - 1}{8}(8t^2 - 6t + 1), \\ \overline{y}(t) &= \frac{1 - \alpha}{8}(8t^2 - 6t + 1). \end{aligned} \quad (25)$$

The numerical results of system (24) are given in Table 5, Table 6 and Figure 3.

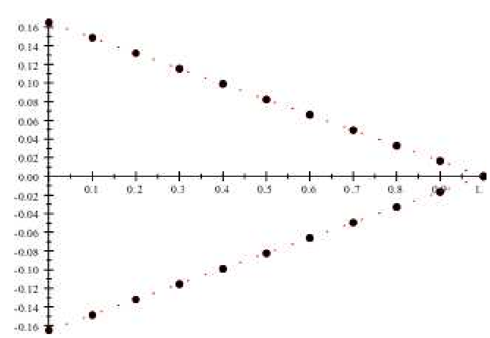

Fig. 1 The $(1,1)$-solution: exact (red) and numerical (black) solutions.

Table 1 Numerical results $\underline{y}$ for system (20) at $t = 0.8$.

α	Exact Solution $\underline{y}$	Numerical Solution	Absolute Error	Relative Error
0.0	-0.1650	-0.164999733	2.66665×10^{-7}	1.61615×10^{-6}
0.1	-0.1485	-0.148499760	2.39999×10^{-7}	1.61615×10^{-6}
0.2	-0.1320	-0.131999786	2.13332×10^{-7}	1.61615×10^{-6}
0.3	-0.1155	-0.115499813	1.86666×10^{-7}	1.61615×10^{-6}
0.4	-0.0990	-0.098999840	1.59999×10^{-7}	1.61615×10^{-6}
0.5	-0.0825	-0.082499999	1.33332×10^{-7}	1.61615×10^{-6}
0.6	-0.0660	-0.065999893	1.06666×10^{-7}	1.61615×10^{-6}
0.7	-0.0495	-0.049499920	7.99998×10^{-8}	1.61615×10^{-6}
0.8	-0.0330	-0.032999946	5.33331×10^{-8}	1.61615×10^{-6}
0.9	-0.0165	-0.016499973	2.66666×10^{-8}	1.61615×10^{-6}

Table 2 Numerical results $\overline{y}$ for system (20) at $t = 0.8$.

α	Exact Solution $\overline{y}$	Numerical Solution	Absolute Error	Relative Error
0.0	0.1650	0.164999733	2.66665×10^{-7}	1.61615×10^{-6}
0.1	0.1485	0.148499760	2.39999×10^{-7}	1.61615×10^{-6}
0.2	0.1320	0.131999786	2.13332×10^{-7}	1.61615×10^{-6}
0.3	0.1155	0.115499813	1.86666×10^{-7}	1.61615×10^{-6}
0.4	0.0990	0.098999840	1.59999×10^{-7}	1.61615×10^{-6}
0.5	0.0825	0.082499999	1.33332×10^{-7}	1.61615×10^{-6}
0.6	0.0660	0.065999893	1.06666×10^{-7}	1.61615×10^{-6}
0.7	0.0495	0.049499920	7.99998×10^{-8}	1.61615×10^{-6}
0.8	0.0330	0.032999946	5.33331×10^{-8}	1.61615×10^{-6}
0.9	0.0165	0.016499973	2.66666×10^{-8}	1.61615×10^{-6}

Table 3 Numerical results $\underline{y}$ for system (22) at $t = 0.1$.

α	Exact Solution $\underline{y}$	Numerical Solution	Absolute Error	Relative Error
0.0	-0.240	-0.239999762	2.37499×10^{-7}	9.89580×10^{-7}
0.1	-0.216	-0.215999786	2.13749×10^{-7}	9.89580×10^{-7}
0.2	-0.192	-0.191999810	1.89999×10^{-7}	9.89580×10^{-7}
0.3	-0.168	-0.167999833	1.66249×10^{-7}	9.89580×10^{-7}
0.4	-0.144	-0.143999857	1.42499×10^{-7}	9.89580×10^{-7}
0.5	-0.120	-0.119999881	1.18749×10^{-7}	9.89580×10^{-7}
0.6	-0.096	-0.095999905	9.49997×10^{-8}	9.89580×10^{-7}
0.7	-0.072	-0.071999928	7.12498×10^{-8}	9.89580×10^{-7}
0.8	-0.048	-0.047999952	4.74998×10^{-8}	9.89580×10^{-7}
0.9	-0.024	-0.023999976	2.37499×10^{-8}	9.89580×10^{-7}

If y is a $(2,1)-$solution for Equation (19), then

$$\begin{aligned}
&\underline{y}''(t) = 2(1-\alpha),\\
&\underline{y}(0) = \frac{\alpha-1}{8},\ \underline{y}(1) = \frac{3(\alpha-1)}{8},\\
&\overline{y}''(t) = 2(\alpha-1),\\
&\overline{y}(0) = \frac{1-\alpha}{8},\ \overline{y}(1) = \frac{3(1-\alpha)}{8}.
\end{aligned} \tag{26}$$

where the exact solutions are

$$\begin{aligned}
\underline{y}(t) &= \frac{-(\alpha-1)}{8}(8t^2-10t-1),\\
\overline{y}(t) &= \frac{-(1-\alpha)}{8}(8t^2-10t-1).
\end{aligned} \tag{27}$$

The numerical results of system (26) are given in Table 7, Table 8 and Figure 4.

Table 4 Numerical results $\overline{y}$ for system (22) at $t = 0.1$.

α	Exact Solution $\overline{y}$	Numerical Solution	Absolute Error	Relative Error
0.0	0.240	0.239999762	2.37499×10^{-7}	9.89580×10^{-7}
0.1	0.216	0.215999786	2.13749×10^{-7}	9.89580×10^{-7}
0.2	0.192	0.191999810	1.89999×10^{-7}	9.89580×10^{-7}
0.3	0.168	0.167999833	1.66249×10^{-7}	9.89580×10^{-7}
0.4	0.144	0.143999857	1.42499×10^{-7}	9.89580×10^{-7}
0.5	0.120	0.119999881	1.18749×10^{-7}	9.89580×10^{-7}
0.6	0.096	0.095999905	9.49997×10^{-8}	9.89580×10^{-7}
0.7	0.072	0.071999928	7.12498×10^{-8}	9.89580×10^{-7}
0.8	0.048	0.047999952	4.74998×10^{-8}	9.89580×10^{-7}
0.9	0.024	0.023999976	2.37499×10^{-8}	9.89580×10^{-7}

Table 5 Numerical results $\underline{y}$ for system (24) at $t = 0.2$.

α	Exact Solution $\underline{y}$	Numerical Solution	Absolute Error	Relative Error
0.0	-0.0150	-0.014999600	3.99998×10^{-7}	2.66665×10^{-5}
0.1	-0.0135	-0.013499640	3.59999×10^{-7}	2.66665×10^{-5}
0.2	-0.0120	-0.011999680	3.19999×10^{-7}	2.66665×10^{-5}
0.3	-0.0105	-0.010499720	2.79999×10^{-7}	2.66665×10^{-5}
0.4	-0.0090	-0.008999760	2.39999×10^{-7}	2.66665×10^{-5}
0.5	-0.0075	-0.007499800	1.99999×10^{-7}	2.66665×10^{-5}
0.6	-0.0060	-0.005999840	1.59999×10^{-7}	2.66665×10^{-5}
0.7	-0.0045	-0.004499880	1.19999×10^{-7}	2.66665×10^{-5}
0.8	-0.0030	-0.002999920	7.99997×10^{-8}	2.66665×10^{-5}
0.9	-0.0015	-0.001499960	3.99998×10^{-8}	2.66665×10^{-5}

Table 6 Numerical results $\overline{y}$ for system (24) at $t = 0.2$.

α	Exact Solution $\overline{y}$	Numerical Solution	Absolute Error	Relative Error
0.0	0.0150	0.014999600	3.99998×10^{-7}	2.66665×10^{-5}
0.1	0.0135	0.013499640	3.59999×10^{-7}	2.66665×10^{-5}
0.2	0.0120	0.011999680	3.19999×10^{-7}	2.66665×10^{-5}
0.3	0.0105	0.010499720	2.79999×10^{-7}	2.66665×10^{-5}
0.4	0.0090	0.008999760	2.39999×10^{-7}	2.66665×10^{-5}
0.5	0.0075	0.007499800	1.99999×10^{-7}	2.66665×10^{-5}
0.6	0.0060	0.005999840	1.59999×10^{-7}	2.66665×10^{-5}
0.7	0.0045	0.004499880	1.19999×10^{-7}	2.66665×10^{-5}
0.8	0.0030	0.002999920	7.99997×10^{-8}	2.66665×10^{-5}
0.9	0.0015	0.001499960	3.99998×10^{-8}	2.66665×10^{-5}

5 Concluding remarks

In this paper, we introduce an algorithm for solving fuzzy two-point boundary value problem based on the use of the IRKM method in which a new constructed of the fuzzy two-point boundary conditions involved. The main characteristic feature of the IRKM method is that the global approximation can be established on the whole solution domain, in contrast with other numerical methods like onestep and multistep methods, and the convergence is uniform. Indeed, the present method is accurate, need less effort to achieve the results, and especially developed for nonlinear case. On the other hand, the derivatives of the approximate solutions are also uniformly convergent. Results obtained show that the numerical scheme is very effective and convenient for solving such problems. Additionally, we note that not only a computational method is presented but also the error of the approximate solutions are monotone decreasing in the sense of the norm of $W_2^3[0,1]$.

Table 7 Numerical results $\underline{y}$ for system (26) at $t = 0.9$.

α	Exact Solution $\underline{y}$	Numerical Solution	Absolute Error	Relative Error
0.0	-0.440	-0.439999862	1.37499×10^{-7}	3.12499×10^{-7}
0.1	-0.396	-0.395999876	1.23749×10^{-7}	3.12499×10^{-7}
0.2	-0.352	-0.351999890	1.09999×10^{-7}	3.12499×10^{-7}
0.3	-0.308	-0.307999903	9.62497×10^{-8}	3.12499×10^{-7}
0.4	-0.264	-0.263999917	8.24997×10^{-8}	3.12499×10^{-7}
0.5	-0.220	-0.219999931	6.87498×10^{-8}	3.12499×10^{-7}
0.6	-0.176	-0.175999945	5.49998×10^{-8}	3.12499×10^{-7}
0.7	-0.132	-0.131999958	4.12498×10^{-8}	3.12499×10^{-7}
0.8	-0.088	-0.087999972	2.74999×10^{-8}	3.12499×10^{-7}
0.9	-0.044	-0.043999986	1.37499×10^{-8}	3.12499×10^{-7}

Table 8 Numerical results $\overline{y}$ for system (26) at $t = 0.9$.

α	Exact Solution $\overline{y}$	Numerical Solution	Absolute Error	Relative Error
0.0	0.440	0.439999862	1.37499×10^{-7}	3.12499×10^{-7}
0.1	0.396	0.395999876	1.23749×10^{-7}	3.12499×10^{-7}
0.2	0.352	0.351999890	1.09999×10^{-7}	3.12499×10^{-7}
0.3	0.308	0.307999903	9.62497×10^{-8}	3.12499×10^{-7}
0.4	0.264	0.263999917	8.24997×10^{-8}	3.12499×10^{-7}
0.5	0.220	0.219999931	6.87498×10^{-8}	3.12499×10^{-7}
0.6	0.176	0.175999945	5.49998×10^{-8}	3.12499×10^{-7}
0.7	0.132	0.131999958	4.12498×10^{-8}	3.12499×10^{-7}
0.8	0.088	0.087999972	2.74999×10^{-8}	3.12499×10^{-7}
0.9	0.044	0.043999986	1.37499×10^{-8}	3.12499×10^{-7}

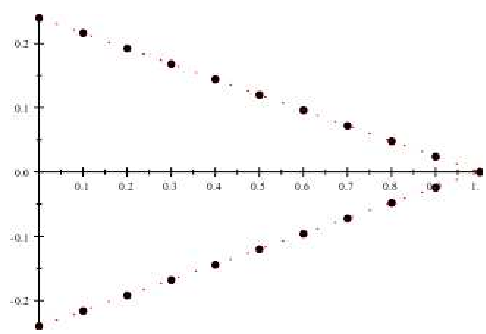

Fig. 2 The $(1,2)$-solution: exact (red) and numerical (black) solutions.

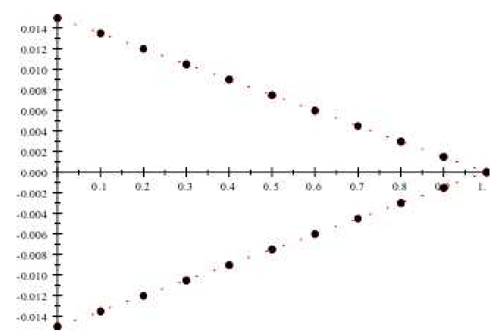

Fig. 3 The $(2,2)$-solution: exact (red) and numerical (black) solutions.

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Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

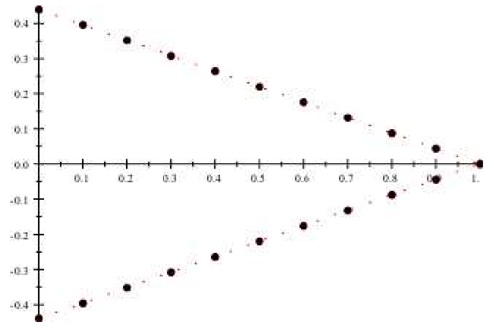

Fig. 4 The $(2,1)$-solution: exact (red) and numerical (black) solutions.

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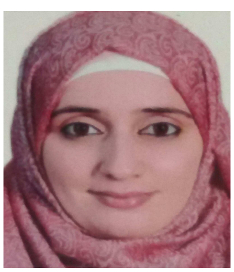

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