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G^+-algebra, Filters and Upper sets

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Abstract: In this paper, a new notion named G^+-algebra is introduced as a dual G-algebra. Then the relation between G^+-algebra with number of abstract algebras including dual *BCK*-algebra, implication algebra, *BE*-algebra and *J*-algebra is investigated and some properties of G^+-algebra are studied. Moreover, a binary relation on G^+-algebra is defined and proved that it forms a partially ordered relation. Finally, considering G^+-algebra, filters and upper sets are studied and related properties are discussed providing an equivalent condition of a filter and finding the relation between filters and upper sets.

Keywords: G^+-algebra, filters, upper sets

1 Introduction

The notion of dual *BCK*-algebra was considered in 2006 by R. Borzooei and S. Shoar. They showed that the implication algebra is equivalent to the dual implicative *BCK*-algebra [1]. Based on this notion, K. Kim and Y. Yon in 2007 studied the properties of dual *BCK*-algebra and proved that *MV*-algebra is equivalent to the bounded commutative dual *BCK*-algebra [2]. Later in 2008, A. Walendziak investigated the relation between *BE*-algebra, implicative algebra and *J*-algebra and stated that they are equivalent to the commutative dual *BCK*-algebra [3]. In 2012, Y. Yon and K. Kim showed that a commutative Heyting algebra is equivalent to a bounded implicative dual *BCK*-algebra.

The concept of filters were studied deeply in some algebras as the filters in general provides a precise language to locate elements that are large enough to satisfy some criterion, which is useful in analysis, general topology and logic. One of the algebras studied, was *BE*-algebra where B. L. Meng gave a procedure used to generate a filter by a subset in a transistive *BE*-algebra and constructed the quotient algebra of a transitive *BE*-algebra via a filter of it [4] and Rough filters are established [5].

Then the idea of a generalized upper sets in *BE*-algebra were developed, extended upper sets of *BE*-algebra were introduced and some relations with filters were obtained (see [6] and [7] repectively). In 2011, filters of *CI*-algebra are considered and an equivalent condition of filters using the notion of upper sets is provided [8]. In this paper we continue to study filter theory. In particular, for a new notion, named G^+-algebra obtained from G-algebra (we refer the reader to [9] for more information on G-algebra). Such algebra is a generalization of *CI*/*BE*-algebra in the sense that every *CI*/*BE*-algebra is a G^+-algebra but not vice versa. In [10], it is proved that any *CI*-algebra is equivalent to dual Q-algebra. Thus, G^+-algebra is considered as a generalization of dual Q-algebra as well. The paper is organized as follows. We start in Section 2 by giving preliminary definitions. In Section 3, we show that any dual *BCK*-algebra is a G^+-algebra and any implication algebra is a G^+-algebra. For the converse relation between dual *BCK*-algebra and G^+-algebra we need extra conditions that will be shown further in Theorem 3.2. Throughout the section, we study properties of G^+-algebra. Then we define a binary operation $\leq$ by $x \leq y$ if and only if $x \circ y = 1$ and show that every G^+-algebra determines a partially ordered set (a poset). Finally, in Section 4, we consider the definition of filters and upper sets. We give in Theorem 4.1 an equivalent condition for filters in G^+-algebra and we generalize the condition in Corollary 4.1. Through an example we note that upper sets are not filters in general and we obtain relations between them in Theorems 4.5, 4.6, 4.7, 4.8.

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2 Preliminaries

We recall some definition from [1], [9], [11], [12], [13]. Note that the bottom and top elements is denoted by 0 and 1, respectively.

Definition 2.1. [9] A G-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the axioms, for all $x,y \in X$:

(1) $x * x = 0$,
(2) $x * (x * y) = y$.

Definition 2.2. [1] A dual BCK-algebra is an algebra $(X,\circ,1)$ of type $(2,0)$ satisfying for all $x,y,z \in X$:

(1) $(x \circ y) \circ ((y \circ z) \circ (x \circ z)) = 1$,
(2) $x \circ ((x \circ y) \circ y) = 1$,
(3) $x \circ x = 1$,
(4) $x \circ y = 1$ and $y \circ x = 1$ imply $x = y$,
(5) $x \circ 1 = 1$.

Definition 2.3. [11] An implication algebra is a set X with a binary operation $*$ which satisfies the following axioms, for all $x,y,z \in X$:

(1) $(x * y) * x = x$,
(2) $(x * y) * y = (y * x) * x$,
(3) $x * (y * z) = y * (x * z)$.

Proposition 2.1. [1] If $(X,*)$ is an implication algebra, then $(X,*,1)$ is a dual BCK-algebra.

Definition 2.4. [12] An algebra $(X,*)$ is said to be a J-algebra if, for any $x,y \in X$,

$$x * (x * (y * (y * x))) = y * (y * (x * (x * y))).$$

Definition 2.5. [13] An algebra $(X;*,1)$ is called a BE-algebra if, for all $x,y,z \in X$:

(1) $x * x = 1$,
(2) $x * 1 = 1$,
(3) $1 * x = x$,
(4) $x * (y * z) = y * (x * z)$.

Proposition 2.2. [11] Every BE-algebra is a J-algebra.

3 On G^+-algebra

In this section we introduce the notion of G^+-algebra and study its relation with dual BCK-algebra, implication algebra, BE-algebra and J-algebra. We also study some properties of G^+-algebra. Moreover, we show that every G^+-algebra determines a partially ordered set.

From Definition 2.1, we introduce the definition of G^+-algebra as follows.

Definition 3.1. Let $(X,*,0)$ be a G-algebra and a binary operation $*$ defined on X as follows: $x * y = y \circ x$. Then $(X,\circ,1)$ of type $(2,0)$ is called a G^+-algebra, if for any $x,y \in X$, it satisfies the following conditions:

(1) $x \circ x = 1$,
(2) $(y \circ x) \circ x = y$.

Proposition 3.1. Let $(X,\circ,1)$ be a G^+-algebra, then

$$y \circ ((y \circ x) \circ x) = 1.$$

Proof. $y \circ ((y \circ x) \circ x) = y \circ y = 1$.

Proposition 3.2. Any dual BCK-algebra is a G^+-algebra.

Proof. Obvious.

Corollary 3.1. G^+-algebra is a generalization of dual BCK-algebra.

Corollary 3.2. Any implication algebra is a G^+-algebra.

Proof. Follows directly from Propositions 2.1 and 3.1.

Proposition 3.3. Let $(X,\circ,1)$ be a G^+-algebra, then the following statements hold, for all $x \in X$:

(1) $1 \circ x = x$,
(2) $(x \circ 1) \circ 1 = x$.

Proof. Let $x \in X$, using Definition 3.1, we have $1 \circ x = (x \circ x) \circ x = x$. We can get the second equation by putting $y = x$ and $x = 1$ in (2).

Theorem 3.1. Let $(X,\circ,1)$ be a BE-algebra and $x,y \in X$ then $(X,\circ,1)$ is a G^+-algebra.

Proof. By using condition (1) and (4) of BE-algebra in Definition 2.5 we get $y \circ ((y \circ x) \circ x) = (y \circ x) \circ (y \circ x) = 1$ which is the second axiom of G^+-algebra. Therefore, $(X,\circ,1)$ is a G^+-algebra.

Proposition 3.4. Let $(X,\circ,1)$ be a G^+-algebra and $x,y,z \in X$, such that $x \circ (y \circ z) = y \circ (x \circ z)$ and $x \circ 1 = 1$, then $(X,\circ,1)$ is a BE-algebra.

Proof. Direct from Definition 2.5 and Proposition 3.3(1).

Corollary 3.3. Let $(X,\circ,1)$ be a G^+-algebra and $x,y,z \in X$, such that $x \circ (y \circ z) = y \circ (x \circ z)$ and $x \circ 1 = 1$, then $(X,\circ,1)$ is a J-algebra.

Proof. Follows directly from Proposition 3.4 and Proposition 2.2.

Proposition 3.5. Let $(X,\circ,1)$ be a G^+-algebra. Then for any $x,y \in X$:

(1) If $y \circ x = 1$ then $y = x$,
(2) If $x \circ 1 = y \circ 1$ then $y = x$.

Proof. Let $x,y \in X$.
(1) If $y \circ x = 1$, we have $y = (y \circ x) \circ x = 1 \circ x = x$.
(2) If $x \circ 1 = y \circ 1$, then $(x \circ 1) \circ 1 = (y \circ 1) \circ 1$. Therefore, $x = y$, from axiom (2) of Proposition 3.3.

Proposition 3.6. Let $(X,\circ,1)$ be a G^+-algebra. Let $x,y,z \in X$, such that $x \circ (y \circ z) = y \circ (x \circ z)$ and $x \circ 1 = 1$. Then $x \circ (y \circ x) = 1$.

Proof. We have, $x \circ (y \circ x) = y \circ (x \circ x) = y \circ 1 = 1$.

Define a binary operation ”$\vee$ ” on X as the following: for any $x,y\in X$, $x\vee y=(x\circ y)\circ y$.

Definition 3.2. For any algebra X, we say that X is commutative if $x\vee y=y\vee x$, $\forall x,y\in X$.

Corollary 3.4. In G^+-algebra X, $x\vee y=x,\forall x,y\in X$.

Corollary 3.5. A G^+-algebra X is not commutative in general.

Proof. Let $x,y\in X$. Then $x\vee y=(x\circ y)\circ y=x$ and $y\vee x=(y\circ x)\circ x=y$.

Definition 3.3. A G^+-algebra $(X,\circ,1)$ is a self-distributive if the operation $\circ$ is:

(1) left self-distributed if, for all $x,y,z\in X$, $x\circ(y\circ z)=(x\circ y)\circ(x\circ z)$,
(2) right self-distributed if, for all $x,y,z\in X$, $(x\circ y)\circ z=(x\circ z)\circ(y\circ z)$.

Proposition 3.7. In G^+-algebra, $x\circ(x\vee y)=1$.

Proof. We have, $x\circ(x\vee y)=x\circ x=1$ from Definition 3.1.

Proposition 3.8. Let $(X,\circ,1)$ be a G^+-algebra, then:

(1) $(x\circ y)\vee z=(x\vee z)\circ(y\vee z)$,
(2) $(x\vee y)\circ z=(x\circ z)\vee(y\circ z)$.

Proof. Direct.

Proposition 3.9. In a right self-distributive G^+-algebra,

(1) $x\circ y=1$ imply $(x\vee z)\circ(y\vee z)=1$,
(2) $x\circ y=y\circ z=1$ imply $(x\vee a)\circ(z\vee a)=1$.

Proof. (1) $(x\vee z)\circ(y\vee z)=(x\circ y)\vee z=(1\vee z)=1$.
(2) Let $x\circ y=1$ and $y\circ z=1$. As $x\circ z=((x\circ y)\circ y)\circ z=(1\circ y)\circ z=y\circ z=1$. Therefore from (1), $(x\vee a)\circ(z\vee a)=1$.

Theorem 3.2. Let $(X,\circ,1)$ be a G^+-algebra and $x\circ(y\circ z)=y\circ(x\circ z)$ and $x\circ 1=1$, then $(X,\circ,1)$ is a dual BCK-algebra.

Proof. Let $x\circ y=1$ and $y\circ x=1$. Then from Proposition 3.5 (1), $x=y$. To prove that $(x\circ y)\circ((y\circ z)\circ(x\circ z))=1$. Consider $(y\circ z)\circ(x\circ z)$, using $x\circ(y\circ z)=y\circ(x\circ z)$ we get, $(y\circ z)\circ(x\circ z)=x\circ((y\circ z)\circ z)=x\circ y$. Hence, $(x\circ y)\circ((y\circ z)\circ(x\circ z))=(x\circ y)\circ(x\circ y)=1$. Therefore, G^+-algebra is a dual BCK-algebra.

The right cancellation holds in a G^+-algebra whereas the left cancellation holds with an extra axiom as shown in the next Propositions.

Proposition 3.10. Let $(X,\circ,1)$ be a G^+-algebra and $x\circ z=y\circ z$, for any $x,y,z\in X$. Then $x=y$.

Proof. Let $x\circ z=y\circ z$. We have, from Definition 3.1 (2), $x=(x\circ z)\circ z=(y\circ z)\circ z=y$.

Proposition 3.11. Let $(X,\circ,1)$ be a G^+-algebra and $(z\circ 1)\circ(z\circ x)=x$, for any $x,y,z\in X$. Then $z\circ x=z\circ y$ imply $x=y$.

Proof. Let $z\circ x=z\circ y$. Then $(z\circ 1)\circ(z\circ x)=(z\circ 1)\circ(z\circ y)$. Hence $x=y$.

Proposition 3.12. In G^+-algebra X, the following are equivalent for any $x,y,z\in X$:

(1) $z\circ(y\circ x)=y\circ(z\circ x)$,
(2) $(z\circ x)\circ(y\circ x)=y\circ z$.

Proof. To prove that (1) implies (2). Assume $z\circ(y\circ x)=y\circ(z\circ x)$. Then $(z\circ x)\circ(y\circ x)=y\circ((z\circ x)\circ x)=y\circ z$. Conversely, suppose that $(z\circ x)\circ(y\circ x)=y\circ z$. Then we have, $z\circ(y\circ x)=((y\circ x)\circ x)\circ(z\circ x)=y\circ(z\circ x)$.

Proposition 3.13. In G^+-algebra X, for $x,y,z\in X$ if $y=z$, then the following are equivalent:

(1) $(z\circ y)\circ(z\circ x)=y\circ x$,
(2) $(z\circ x)\circ(y\circ x)=y\circ z$.

Proof. Suppose that $(z\circ y)\circ(z\circ x)=y\circ x$. We have $(z\circ y)\circ(z\circ x)=[(y\circ x)\circ(z\circ x)]\circ(z\circ x)$. By the right cancellation law in Proposition 3.10 we get, $z\circ y=(y\circ x)\circ(z\circ x)$. Hence, $y\circ z=(z\circ x)\circ(y\circ x)$. On the other hand, assume that $(z\circ x)\circ(y\circ x)=y\circ z$. We have, $[(z\circ x)\circ(y\circ x)]\circ(y\circ x)=(y\circ z)\circ(y\circ x)$. Thus, $z\circ x=(y\circ z)\circ(y\circ x)$. By putting $z=y$, we get (1).

Definition 3.4. Let $(X,\circ,1)$ be a G^+-algebra. Define a binary relation $\leq$ on X as follows: $x\leq y$ if and only if $x\circ y=1$.
Every G^+-algebra determines a partially ordered set (a poset), with the binary operation $\leq$ defined above.

Theorem 3.3. If X is a G^+-algebra, then the relation $\leq$ is a partial order on X where $x\leq y$ if and only if $x\circ y=1$.

Proof. (i) Since $x\circ x=1$, we get $x\leq x$. Thus $\leq$ is reflexive.
(ii) Let $x\leq y$ and $y\leq x$. Then, $y\circ x=1$ and $x\circ y=1$. If $y\circ x=1$ we get $y=x$ from Proposition 3.5. Hence $\leq$ is anti-symmetric.
(iii) Let $x\leq y$ and $y\leq z$. Then, $x\circ y=1$ and $y\circ z=1$. Since $x\circ z=((x\circ y)\circ y)\circ z=(1\circ y)\circ z=y\circ z=1$, we have $x\leq z$. Hence $\leq$ is transitive.

If a G^+-algebra is a poset on X, then we have the following Theorem.

Theorem 3.4. Let $(X,\circ,1)$ be a G^+-algebra. Then $G^+_{\leq}$-algebra is the algebra generated by a partial order relation $\leq$ which satisfies the following axioms for all $x,y\in X$:

(1) $x\leq x$,
(2) $y\leq((y\circ x)\circ x)$.

Proposition 3.14. $1\leq x$ imply $x=1$.

Proof. As $1\circ x=1$, Proposition 3.5 proves the axiom.

Proposition 3.15. Let $(X,\circ,1)$ be a left self-distributive G^+-algebra and $x\circ 1=1$. If $x\leq y$ then $z\circ x\leq z\circ y$.

Table 1: Cayley table

$\circ$	1	a	b
1	1	a	b
a	a	1	a
b	b	b	1

Proof. Given that $x \circ y = 1$ and $x \circ 1 = 1$. We have that, $(z \circ x) \circ (z \circ y) = z \circ (x \circ y) = z \circ 1 = 1$.

Corollary 3.6. If $x \leq y$ then $z \leq (x \circ y)$.

Proposition 3.16. Let $(X, \circ, 1)$ be a G^+-algebra and suppose that $x \circ (y \circ z) = y \circ (x \circ z)$. If $x \leq y$ then $y \circ z \leq x \circ z$.

Proof. We have, $(y \circ z) \circ (x \circ z) = x \circ ((y \circ z) \circ z) = x \circ y = 1$.

Corollary 3.7. In a right self-distributive G^+-algebra, If $x \leq y$ then $(y \circ x) \leq z$.

4 Filters and Upper sets

In this section we consider the concepts of filters and upper sets. We study their properties and find out how filters and upper sets are related in G^+-algebra.

Definition 4.1. Let $(X, \circ, 1)$ be a G^+-algebra. A non-empty subset F of X is said to be a filter of X if:

(1) $1 \in F$,
(2) $x \circ y \in F$ and $x \in F$ imply $y \in F$.

Example 4.1. The algebra $(X, \circ, 1)$ where $X = \{1, a, b\}$ with Cayley table (Table 1) is a G^+-algebra.

Observe that $F_1 := \{1, a, b\}$ is a filter whereas $F_2 := \{1, a\}$ is not a filter as $a \circ b = a \in F_2$ and $a \in F_2$ but $b \notin F_2$.

Theorem 4.1. Let $(X, \circ, 1)$ be a G^+-algebra and let F be a set containing 1. Then F is a filter if and only if for any element $x \in F$, $x \circ y = 1$ implies $y \in F$.

Proof. Suppose that for any element $x \in F$, $x \circ y = 1$ implies $y \in F$. Given $1 \in F$, the first condition holds. Let $x \circ y \in F$ and $x \in F$, from Proposition 3.1 we have $x \circ ((x \circ y) \circ y) = 1$. Then $x \circ (1 \circ y) = 1$ and so $x \circ y = 1$ which implies $y \in F$, the second condition holds.
Now suppose that F is a filter and $x \circ y = 1$ for any $x, y \in F$. As F is a filter contains 1 we have $x \circ y \in F$. From Definition 4.1 (2), $y \in F$.

In the next corollary we will write x^n for x which occurs n times, for any $n \in \mathbb{N}$.

Corollary 4.1. Let F be a set containing 1. Then F is a filter if and only if for any element $x \in F$, $x^n \circ y = 1$ implies $y \in F$.

Proof. Straightforward by induction.

The proof of Corollary 4.2 follows directly from Proposition 3.5.

Corollary 4.2. Let X be G^+-algebra. If $x \circ y = 1$, then $a^n \circ x = 1$ implies $a^n \circ y = 1$, for all $x, y, a \in X$.

Definition 4.2. Let $(X, \circ, 1)$ be a G^+-algebra. Define the upper set of an element $x \in X$ by $U(x) := \{z \in X | x \circ z = 1\}$ and the upper set of two elements $x, y \in X$ by $U(x, y) := \{z \in X | x \circ (y \circ z) = 1\}$.

Example 4.2. Using the algebra in Example 4.1, we have $U(b, b) = \{1, a\}$, $U(a) = \{a\}$.

Remark. Upper sets need not to be filter in general as in Example 4.1 $U(b, b)$ is not a filter.

Proposition 4.1. Let X be G^+-algebra. If $x \circ y = 1$ then $U(x) = U(y)$.

Proof. Suppose $x \circ y = 1$. Let $z \in U(y)$, thus $y \circ z = 1$ and so from Proposition 3.5 (1), $y = z$. We also have $x = y$. Thus $x = y = z$. Therefore, $x \circ z = x \circ x = 1$ which implies $z \in U(x)$. Similarly, $z \in U(x)$ implies $z \in U(y)$. Hence $U(x) = U(y)$.

Theorem 4.3. In G^+-algebra, $x = y$ if and only if $U(x) = U(y)$.

Proof. Suppose $x = y$ and Let $z \in U(x)$. Then $x \circ z = 1 = y \circ z$ which implies that $z \in U(y)$. Similarly, we can show that $U(y) \subseteq U(x)$. It follows that $U(x) = U(y)$.
Suppose $U(x) = U(y)$, and let $z \in U(x)$. Thus $x \circ z = y \circ z = 1$. From Proposition 3.10, $x = y$.

Theorem 4.4. In a left self-distributive G^+-algebra, if $a, b \in U(x, y)$ then $a \circ b \in U(x, y)$ and $b \circ a \in U(x, y)$.

Proof. Since $a, b \in U(x, y)$, we know that $x \circ (y \circ a) = 1$ and $x \circ (y \circ b) = 1$. Using the left self-distributive law twice we have, $x \circ (y \circ (a \circ b)) = x \circ ((y \circ a) \circ (y \circ b)) = (x \circ (y \circ a)) \circ (x \circ (y \circ b)) = 1 \circ 1 = 1$. In a similar way it can be shown that $b \circ a \in U(x, y)$.

Remark. $a \circ b \notin U(y, x)$ and $b \circ a \notin U(y, x)$.

Proposition 4.2. In G^+-algebra, if $x \circ 1 = 1$ then $1 \in U(x, y)$.

Proof. Let $x \circ 1 = 1$. So, $x \circ (y \circ 1) = 1$ this means that $1 \in U(x, y)$.

Theorem 4.5. In a left self-distributive G^+-algebra an upper set $U(x, y)$ is a filter if $x \circ 1 = 1$.

Proof. From Proposition 4.2, we know that $1 \in U(x, y)$. Let $a \circ b \in U(x, y)$ and $a \in U(x, y)$. Thus we know that $x \circ (y \circ (a \circ b)) = 1$ and $x \circ (y \circ a) = 1$. Starting with $x \circ (y \circ (a \circ b)) = 1$ and using left distribution law we have $x \circ (y \circ (a \circ b)) = x \circ ((y \circ a) \circ (y \circ b)) = (x \circ (y \circ a)) \circ (x \circ (y \circ b)) = 1 \circ (x \circ (y \circ b)) = x \circ (y \circ b) = 1$. Hence, $b \in U(x, y)$ i.e. the upper set $U(x, y)$ is a filter of X.

In the following part we give properties of upper sets. We omit obvious proofs.

Proposition 4.3. For any $x \in X$, $x \in U(x)$.

Proposition 4.4. For $1 \in X$, $1 \in U(1,1)$.

Proposition 4.5. For any $x \in X$, $U(x) = U(1,x)$.

Proof. Using the definitions of upper sets, $U(1,x) = \{z \in X | 1 \circ (x \circ z) = 1\} = \{z \in X | x \circ z = 1\} = U(x)$.

Proposition 4.6. If $y \circ 1 = 1$ then $U(x) \subseteq U(y,x)$.

Proof. Suppose that $y \circ 1 = 1$ and let $z \in U(x)$ i.e. $x \circ z = 1$. Then $y \circ (x \circ z) = y \circ 1 = 1$. Then obviously $z \in U(y,x)$. This proves that $U(x) \subseteq U(y,x)$.

Corollary 4.3. If $y \circ 1 \neq 1$ then $U(x) \setminus \{1\} \subseteq X \setminus U(y,x)$.

Proof. Suppose $y \circ 1 \neq 1$ and let $z \in U(x) \setminus \{1\}$. From Definition 4.2 $x \circ z = 1$. Then $y \circ (x \circ z) = y \circ 1 \neq 1$. Thus, $z \notin U(y,x)$. Hence, $z \in X \setminus U(y,x)$. Therefore, $U(x) \setminus \{1\} \subseteq X \setminus U(y,x)$.

Theorem 4.6. Let F be a subset of a G^+-algebra X. If F is a filter of X then $U(x,y) \subseteq F$ $\forall x,y \in F$. Similarly, $U(y,x) \subseteq F, \forall x,y \in F$.

Proof. Suppose F is a filter. Let $x,y \in F$ and $z \in U(x,y)$. Then $x \circ (y \circ z) = 1$. As $1 \in F$, we know that $x \circ (y \circ z) \in F$. Applying Definition 4.1 (2) twice we have $z \in F$. Thus $U(x,y) \subseteq F$.

Theorem 4.7. Let F be a subset of a G^+-algebra X and $x \circ 1 = 1$. If $U(x,y) \subseteq F, \forall x,y \in F$ then F is a filter.

Proof. Suppose that for any $x,y \in F$, $U(x,y) \subseteq F$. Consider the element $x \in F, x \circ (x \circ 1) = 1$ and so $1 \in U(x,x) \subseteq F$. Let $x \circ y \in F$ and $x \in F$. From Definition 3.1 (1), $(x \circ y) \circ (x \circ y) = 1$, which implies $y \in U(x \circ y, x) \subseteq F$. This proves that F is a filter.

Corollary 4.4. If $U(x)$ is a filter of X that contains x,y then $U(x,y) \subseteq U(x)$ and $U(y,x) \subseteq U(x)$.

Proof. Suppose that $U(x)$ is a filter of X. Let $z \in U(x,y)$. Thus, $x \circ (y \circ z) = 1$. As $U(x)$ is a filter, we have $x \circ (y \circ z) \in U(x)$ and $y \in U(x)$ which implies that $z \in U(x)$. This proves that $U(x,y) \subseteq U(x)$. The second part is proved similarly.
We consider the generalization of upper sets in G^+-algebra $U_n(x,y) := \{z \in X | x^n \circ (y \circ z) = 1\}$. The next theorem is proved directly.

Theorem 4.8. If X is a left self-distributive G^+-algebra and $x \circ 1 = 1$ then the generalized upper set $U_n(x,y)$ is a filter.

Proposition 4.7. In G^+-algebra, if $z \circ (y \circ x) = y \circ (z \circ x)$ and $y \circ a = 1$ then $U_n(x,y) = U_n(y,x)$.

Proof. We have, $U_n(x,y) = x^n \circ (y \circ a) = x^n \circ 1 = 1$. On the other hand, $U_n(y,x) = y^n \circ (x \circ a) = y^{n-1} \circ (y \circ (x \circ a)) = y^{n-1} \circ (x \circ (y \circ a)) = y^{n-1} \circ (x \circ 1) = y^{n-1} \circ 1 = 1$.

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References

[1] R. A. Borzooei, and S. Khosravi Shoar. "Implication algebras are equivalent to the dual implicative *BCK*-algebras," Scientiae Mathematicae Japonicae **63**(3), 429-432 (2006).
[2] K. H. Kim, and Y. H. Yon, "Dual *BCK*-algebra and *MV*-algebra,"Scientiae Mathematicae Japonicae **66**(2) , 247-254 (2007).
[3] A. Walendziak, "On commutative *BE*-algebras," Scientiae Mathematicae Japonicae **69**(2), 585-588 (2008).
[4] B. L. Meng, "On filters in *BE*-algebras," Scientiae Mathematicae Japonicae **71**(2), 201-207 (2010).
[5] S. E. Kang, and S. S. Ahn, "Rough filters of *BE*-algebras," Journal of applied mathematics & informatics **30**(5-6), 1023-1030 (2012).
[6] S. S. Ahn, and K. S. So, "On generalized upper sets in *BE*-algebras," Bulletin of the Korean Mathematical Society **46**(2), 281-287 (2009).
[7] H. S. Kima and K. J. Leeb, "Extended upper sets in *BE*-algebras," Bulletin of the Malaysian Mathematical Sciences Society **34** (3), 511-520 (2011).
[8] B. Piekart, and A. Walendziak, "On filters and upper sets in *CI*-algebras," Algebra and Discrete Mathematics **11**(1), 109-115 (2011).
[9] R. K. Bandru, and N. Rafi, "On *G*-Algebras," Scientia Magna International Book Series **8**(3), 1-7 (2012).
[10] A. B. Saeid, "*CI*-algebra is equivalent to dual *Q*-algebra," Journal of the Egyptian Mathematical Society **21**(1), 1-2 (2013).
[11] J. C. Abbott, "Semi-boolean algebras," Matematički Vesnik **4**(19), 177-198 (1967).
[12] K. Iseki, H. S. Kim, and J. Neggers, "On *J*-algebras," Scientiae Mathematicae Japonicae **63**(3), 413-420 (2006).
[13] H. S. Kim, and Y. H. Kim, "On *BE*-algebras," Scientiae Mathematicae Japonicae **66**(1), 113-116 (2007).

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