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Wave Equation with Logarithmic Nonlinearities in Kirchhoff Type

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Abstract: In this paper, we study a viscoelastic wave equations of the Kirchhoff type

$$u'' - \phi(x)\left(M(\|\nabla_x u\|_2^2)\Delta_x u - \int_0^t g(t-s)\Delta_x u(s)ds\right) = au\ln|u|^k \qquad (1)$$

defined in any spaces dimension. It is well known that from a class of nonlinearities the logarithmic nonlinearity is distinguished by several interesting physical properties. We use weighted spaces to establish the long-time behavior of solution of (1). Furthermore, under convenient hypotheses on g and the initial data, the local-in-time existence of solution is established.

Keywords: Lyapunov function, viscoelasticity, Kirchhoff type, density, decay rate, weighted spaces, Logarithmic nonlinearities.

1 Introduction

In this paper, we consider the wave equation with logarithmic nonlinearity (1), where $x \in \mathbb{R}^n, t > 0, n \geq 2, k, a > 0$ and M is a positive C^1 function satisfying for $s \geq 0, m_0 > 0, m_1 \geq 0, \gamma \geq 1$, $M(s) = m_0 + m_1 s^\gamma$ and the scalar function $g(s)$ (so-called relaxation kernel) is assumed to satisfy (A1).
It is well known that from a class of nonlinearities, the logarithmic nonlinearity is distinguished by several interesting physical properties. In recent years, there has been a growing interest in the viscoelastic wave equation, its properties and variants of the problem can be found in [3], [14], [21], [22], [23], [25], [27] and [28].
The model here considered are well known ones and refer to materials with memory as they are termed in the wide literature which is concerned about their physical, mechanical behavior and the many interesting analytical problems. The physical characteristic property of such materials is that their behavior depends on time not only through the present time but also through their past history.

Eq. (1) is equipped by the following initial data.

$$u(0,x) = u_0(x) \in \mathscr{H}(\mathbb{R}^n), u'(0,x) = u_1(x) \in L^2_\rho(\mathbb{R}^n), \quad (2)$$

where the weighted spaces $\mathscr{H}$ is given in Definition 1 and the density function $\phi(x) > 0, \forall x \in \mathbb{R}^n, (\phi(x))^{-1} = \rho(x)$ satisfies

$$\rho : \mathbb{R}^n \to \mathbb{R}^*_+, \quad \rho(x) \in C^{0,\tilde{\gamma}}(\mathbb{R}^n) \quad (3)$$

with $\tilde{\gamma} \in (0,1)$ and $\rho \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s = \frac{2n}{2n-qn+2q}$.
This kind of systems appears in the models of nonlinear Kirchhoff-type. It is a generalization of a model introduced by Kirchhoff [13] in the case $n = 1$ this type of problem describes a small amplitude vibration of an elastic string. The original equation is:

$$\rho h u_{tt} + \tau u_t = \left(P_0 + \frac{Eh}{2L}\int_0^L |u_x(x,t)|^2 ds\right) u_{xx} + f, \quad (4)$$

where $0 \leq x \leq L$ and $t > 0, u(x,t)$ is the lateral displacement at the space coordinate x and the time t, ρ the mass density, h the cross-section area, L the length, P_0 the initial axial tension, τ the resistance modulus, E the

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Young modulus and f the external force (for example the action of gravity).
For the decay rate in $\mathbb{R}^n$, we quote essentially the results of [1], [10], [11], [12], [20]. In [11], authors showed that, for compactly supported initial data and for an exponentially decaying relaxation function, the decay of the energy of solution of a linear Cauchy problem (1), (2) with $\rho(x) = 1, M \equiv 1, a = 0$ is polynomial. The finite-speed propagation is used to compensate for the lack of Poincare's inequality. In the case $M \equiv 1, a = 0$, in [10], author looked into a linear Cauchy viscoelastic problem with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate for the lack of Poincare's inequality. The same problem treated in [10], was considered in [12], where they consider a Cauchy problem for a viscoelastic wave equation. Under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. Conditions used, on the relaxation function g and its derivative g' are different from the usual ones.

The problem (1),(2) without source, for the case $\rho(x) = 1, M \equiv 1$, in a bounded domain $\Omega \subset \mathbb{R}^n, (n \geq 1)$ with a smooth boundary $\partial\Omega$ and g is a positive nonincreasing function was considered in [20], where they established an explicit and general decay rate result for relaxation functions satisfying:

$$g'(t) \leq -H(g(t)), t \geq 0, \quad H(0) = 0 \tag{5}$$

for a positive function $H \in C^1(\mathbb{R}^+)$ and H is linear or strictly increasing and strictly convex C^2 function on $(0, r], 1 > r$. This improves the conditions considered in [1] on the relaxation functions

$$g'(t) \leq -\chi(g(t)), \quad \chi(0) = \chi'(0) = 0 \tag{6}$$

where χ is a non-negative function, strictly increasing and strictly convex on $(0, k_0], k_0 > 0$.
The goal of the present paper is to establish the existence of solution to the problem (1)-(2). We obtain also, a fast decay results.

2 Material, Assumptions and technical lemmas

The constants c used throughout this paper are positive generic constants which may be different in various occurrences also the functions considered are all real valued, here $u' = du(t)/dt$ and $u'' = d^2u(t)/dt^2$. For simplicity reason, we take $a = 1$
We recall and make use the following hypothesis on the function g as:

(A1) We assume that the function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is of class C^1 satisfying:

$$m_0 - \overline{g} = l > 0, \quad g(0) = g_0 > 0 \tag{7}$$

where $\overline{g} = \int_0^\infty g(t)dt$.

(A2) There exists a positive function $H \in C^1(\mathbb{R}^+)$ such that

$$g'(t) + H(g(t)) \leq 0, t \geq 0, \quad H(0) = 0 \tag{8}$$

and H is linear or strictly increasing and strictly convex C^2 function on $(0, r], 1 > r$.

(A3) According to results in [20], we have
1- We can deduce that there exists $t_1 > 0$ large enough such that:

1) $\forall t \geq t_1$: We have $\lim_{s\to+\infty} g(s) = 0$, which implies that $\lim_{s\to+\infty} -g'(s)$ cannot be positive, so $\lim_{s\to+\infty} -g'(s) = 0$. Then $g(t_1) > 0$ and

$$\max\{g(s), -g'(s)\} < \min\{r, H(r), H_0(r)\}, \tag{9}$$

where $H_0(t) = H(D(t))$ provided that D is a positive C^1 function, with $D(0) = 0$, for which H_0 is strictly increasing and strictly convex C^2 function on $(0, r]$ and

$$\int_0^{+\infty} g(s)H_0(-g'(s))ds < +\infty.$$

2) $\forall t \in [0, t_1]$: As g is nonincreasing, $g(0) > 0$ and $g(t_1) > 0$ then $g(t) > 0$ and

$$g(0) \geq g(t) \geq g(t_1) > 0.$$

Therefore, since H is a positive continuous function, then

$$a \leq H(g(t)) \leq b$$

for some positive constants a and b. Consequently,

$$g'(t) \leq -H(g(t)) \leq -kg(t), \quad k > 0$$

which gives

$$g'(t) \leq -kg(t), k > 0 \tag{10}$$

2- Let H_0^* be the convex conjugate of H_0 in the sense of Young (see [2], pages 61-64), then

$$H_0^*(s) = s(H_0')^{-1}(s) - H_0[(H_0')^{-1}(s)], \quad s \in (0, H_0'(r))$$

and satisfies the following Young's inequality

$$AB \leq H_0^*(A) + H_0(B), \quad A \in (0, H_0'(r)), B \in (0, r]. \tag{11}$$

The space $\mathscr{H}(\mathbb{R}^n)$ is defined as the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the norm $\|u\|^2_{\mathscr{H}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\nabla_x u|^2 dx$. It is defined in the next definition

Definition 1([23]). *We define the function spaces of our problem and its norm as follows:*

$$\mathscr{H}(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla_x f \in (L^2(\mathbb{R}^n))^n \right\} \tag{12}$$

and that $\mathscr{H}$ is embedded continuously in $L^{2n/(n-2)}$.
The space $L^2_\rho(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(f, h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx.$$

For $1 < q < \infty$, if f is a measurable function on $\mathbb{R}^n$, we define

$$\|f\|_{L^q_\rho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \rho |f|^q dx \right)^{1/q}. \tag{13}$$

*Remark.*The space $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space.

The following technical Lemmas will play an important role in the sequel.

Lemma 1.*[4] (Lemma 1.1) For any two functions $g,\ v \in C^1(\mathbb{R})$ and $\theta \in [0,1]$ we have*

$$v'(t)\int_0^t g(t-s)v(s)ds = -\frac{1}{2}\frac{d}{dt}\int_0^t g(t-s)|v(t)-v(s)|^2 ds + \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s)ds\right)|v(t)|^2 + \frac{1}{2}\int_0^t g'(t-s)|v(t)-v(s)|^2 ds - \frac{1}{2}g(t)|v(t)|^2.$$

and

$$\left|\int_0^t g(t-s)(v(t)-v(s))ds\right|^2 \leq \left(\int_0^t |g(s)|^{2(1-\theta)}ds\right)\left(\int_0^t |g(t-s)|^{2\theta}|v(t)-v(s)|^2 ds\right)$$

The next Lemma can be easily shown (see [14], [15]).

Lemma 2.*Let ρ satisfies (3), then for any $u \in \mathscr{H}(\mathbb{R}^n)$*

$$\|u\|_{L^q_\rho(\mathbb{R}^n)} \leq \|\rho\|_{L^s(\mathbb{R}^n)}\|\nabla_x u\|_{L^2(\mathbb{R}^n)}$$

with $s = \frac{2n}{2n-qn+2q}, 2 \leq q \leq \frac{2n}{n-2}$

Now, using lemma 2, we give the following Lemma concerning Logarithmic Sobolev inequality.

Lemma 3.*(see [7], [18], [24]) Let $u \in \mathscr{H}(\mathbb{R}^n)$ be any function and $c_1, c_2 > 0$ be any numbers. Then*

$$2\int_{\mathbb{R}^n}\rho(x)|u|^2\ln\left(\frac{|u|}{\|u\|^2_{L^2_\rho}}\right)dx + n(1+c_1)\|u\|^2_{L^2_\rho} \leq c_2\frac{\|\rho\|^2_{L^2}}{\pi}\|\nabla_x u\|_2^2$$

Definition 2.*By the weak solution of (1) over $[0,T]$ we mean a function*

$$u \in C([0,T],\mathscr{H}(\mathbb{R}^n)) \cap C^1([0,T],L^2_\rho(\mathbb{R}^n)) \cap C^2([0,T],\mathscr{H}^{-1}(\mathbb{R}^n))$$

with $u' \in L^2([0,T],\mathscr{H}(\mathbb{R}^n))$, such that $u(0) = u_0, u'(0) = u_1$ and for all $v \in \mathscr{H}, t \in [0,T]$,

$$\int_{\mathbb{R}^n}\rho(x)u\ln|u|^k v dx = \int_{\mathbb{R}^n}\rho(x)u'' v dx + M(\|\nabla_x u\|_2^2)\int_{\mathbb{R}^n}\nabla_x u \nabla_x v dx - \int_{\mathbb{R}^n}\int_0^t g(t-s)\nabla_x u(s) ds \nabla_x v dx$$

Multiplying the equation (1) by $\rho(x)u'$, and integrating by parts over $\mathbb{R}^n$, we have the energy of u at time t is given by

$$E(t) = \frac{1}{2}\Big(\|u'\|^2_{L^2_\rho} + \Big(m_0 - \int_0^t g(s)ds\Big)\|\nabla_x u\|_2^2 + (g\circ\nabla_x u) - \int_{\mathbb{R}^n}\rho(x)u^2\ln|u|^k dx\Big) + \frac{k}{4}\|u\|^2_{L^2_\rho} + \frac{m_1}{2(\gamma+1)}\|\nabla_x u\|_2^{2(\gamma+1)} \quad (14)$$

and the following energy functional law holds:

$$E'(t) = \frac{1}{2}(g'\circ\nabla_x u)(t) - \frac{1}{2}g(t)\|\nabla_x u(t)\|_2^2, \forall t \geq 0. \quad (15)$$

which means that, our energy is uniformly bounded and decreasing along the trajectories. The following notation will be used throughout this paper

$$(g\circ\nabla_x u)(t) = \int_0^t g(t-\tau)\,\|\nabla_x u(t) - \nabla_x u(\tau)\|_2^2 d\tau, \quad (16)$$

for $u(t) \in \mathscr{H}(\mathbb{R}^n), t \geq 0$.

3 Global existence in time

According to logarithmic Sobolev inequality and similar to the proof in ([5], [6], [7], [24], [26]), we have the following result.

Theorem 1.*(Local existence) Let $u_0(x) \in \mathscr{H}(\mathbb{R}^n), u_1(x) \in L^2_\rho(\mathbb{R}^n)$ be given. Then, under hypothesis $(A1), (A2)$ and (3), the problem (1) has a unique local solution*

$$u \in C([0,T],\mathscr{H}(\mathbb{R}^n)) \cap C^1([0,T],L^2_\rho(\mathbb{R}^n))$$

Now, we introduce two functionals

$$J(t) = \frac{1}{2}\Big(\Big(m_0 - \int_0^t g(s)ds\Big)\|\nabla_x u\|_2^2 + (g\circ\nabla_x u) - \int_{\mathbb{R}^n}\rho(x)u^2\ln|u|^k dx\Big) + \frac{k}{4}\|u\|^2_{L^2_\rho} + \frac{m_1}{2(\gamma+1)}\|\nabla_x u\|_2^{2(\gamma+1)} \quad (17)$$

and

$$I(t) = \Big(m_0 - \int_0^t g(s)ds\Big)\|\nabla_x u\|_2^2 + (g\circ\nabla_x u) - \int_{\mathbb{R}^n}\rho(x)u^2\ln|u|^k dx + \frac{m_1}{2(\gamma+1)}\|\nabla_x u\|_2^{2(\gamma+1)} \quad (18)$$

Then,

$$J(t) = \frac{1}{2}I(t) + \frac{k}{4}\|u\|^2_{L^2_\rho} \quad (19)$$

As in ([9]) to establish the corresponding method of potential wells which is related to the logarithmic nonlinear term, we introduce the stable set as follows:

$$W = \{u \in \mathscr{H}(\mathbb{R}^n) : I(t) > 0, J(t) < d\} \cup \{0\} \quad (20)$$

*Remark.*We notice that the mountain pass level d given in (20) defined by

$$d = \inf\{\sup_{u\in\mathscr{H}(\mathbb{R}^n)\setminus\{0\}\mu\geq 0} J(\mu u)\}, \tag{21}$$

Also, by introducing the so called "Nehari manifold"
$\mathscr{N} = \{u\in\mathscr{H}(\mathbb{R}^n)\setminus\{0\} : I(t) = 0\}$
Similar to results in [29], it is readily seen that the potential depth d is also characterized by

$$d = \inf_{u\in\mathscr{N}} J(t). \tag{22}$$

This characterization of d shows that

$$dist\,(0,\mathscr{N}) = \min_{u\in\mathscr{N}} \|u\|_{\mathscr{H}(\mathbb{R}^n)} \tag{23}$$

By the fact that (15), we will prove the invariance of the set W. That is if for some $t_0 > 0$ if $u(t_0)\in W$, then $u(t)\in W$, $\forall t\geq t_0$, let us beginning by giving the existence Lemma of the potential depth. (See [7] Lemma 2.4)

Lemma 4.*d is positive constant.*

Lemma 5.*Let $u\in\mathscr{H}(\mathbb{R}^n)$ and $\beta = e^{\frac{1}{2}n(1+c_1)}$. if $0 < \|u\|^2_{L^2_\rho} < \beta$, then $I(t) > 0$; if $I(t) = 0, \|u\|_2^2 \neq 0$, then $\|u\|^2_{L^2_\rho} > \beta$.*

*Proof.*By (A1), (18) and Lemma3, we have

$$\begin{aligned}
I(t) &= \left(m_0 - \int_0^t g(s)ds\right)\|\nabla_x u\|_2^2 + (g\circ\nabla_x u)\\
&- \int_{\mathbb{R}^n}\rho(x)u^2\ln|u|^k dx + \frac{m_1}{2(\gamma+1)}\|\nabla_x u\|_2^{2(\gamma+1)}\\
&\geq l\|\nabla_x u\|_2^2 - k\int_{\mathbb{R}^n}\rho(x)u^2\left(\ln\frac{|u|}{\|u\|^2_{L^2_\rho}} + \ln\|u\|^2_{L^2_\rho}\right)dx\\
&\geq \left(l - \frac{kc_2}{2\pi}\|\rho\|^2_{L^2_\rho}\right)\|\nabla_x u\|_2^2 + \frac{1}{2}kn(1+c_1)\|u\|^2_{L^2_\rho}\\
&- k\|u\|^2_{L^2_\rho}\ln\|u\|^2_{L^2_\rho}
\end{aligned}$$

Choosing c_2 such that $l > \frac{kc_2}{2\pi}\|\rho\|^2_{L^2_\rho}$, then

$$I(t) \geq k\left(\frac{1}{2}n(1+c_1) - \ln\|u\|^2_{L^2_\rho}\right)\|u\|^2_{L^2_\rho}$$

Therefore, if $0 < \|u\|^2_{L^2_\rho} < \beta$, then $I(t) > 0$; if $I(t) = 0, \|u\|_2^2 \neq 0$, we have $\beta < \|u\|^2_{L^2_\rho}$ then, $\|u\|^2_{L^2_\rho} > \beta$.

Theorem 2.*(Global Existence) Let $u_0(x)\in\mathscr{H}(\mathbb{R}^n), u_1(x)\in L^2_\rho(\mathbb{R}^n)$ and $0 < E(0) < d, I(0) > 0$. Then, under hypothesis $(A1),(A2)$ and conditions (3), the problem (1) has a global solution in time.*

*Proof.*From the definition of energy for solution and by (15), we have

$$\frac{1}{2}\|u'\|^2_{L^2_\rho} + J(t) \leq \frac{1}{2}\|u_1\|^2_{L^2_\rho} + J(0), \quad \forall t\in[0,T_{max}) \tag{24}$$

where T_{max} is the maximal existence time of solution of u. Then, by the definition of the stable set and using Lemma 5, we have $u\in W$, $\quad\forall t\in[0,T_{max})$

4 Decay estimates

We apply the multiplier techniques to obtain useful estimates and prepare some functionals associated with the nature of our problem to introduce an appropriate Lyapunov functions. For this purpose, we introduce the functionals

$$\psi_1(t) = \int_{\mathbb{R}^n}\rho(x)uu'dx, \tag{25}$$

Lemma 6.*Under the hypothesis (A1) and (A2), the functional ψ_1 satisfies, along the solution of (1),(2)*

$$\begin{aligned}
&\psi_1'(t) \leq \|u'\|^2_{L^2_\rho} + m_1\|\nabla_x u\|_2^{2(\gamma+1)} + \frac{(1-l)}{4\sigma}(g\circ\nabla_x u)\\
&+\left[\left(\sigma + \frac{kc_2}{2\pi}\|\rho\|^2_{L^2} - l\right) + k\|\rho\|^2_{L^2}\left(\ln\|u\|^2_{L^2_\rho} - \frac{1}{2}n(1+c_1)\right)\right]\|\nabla u\|_2^2.
\end{aligned}$$

*Proof.*From (25), integrate over $\mathbb{R}^n$, we have

$$\begin{aligned}
&\psi_1'(t) = \int_{\mathbb{R}^n}\rho(x)|u'|^2dx + \int_{\mathbb{R}^n}\rho(x)uu''dx\\
&= \int_{\mathbb{R}^n}\left(\rho(x)|u'|^2 + M(\|\nabla_x u\|_2^2)u\Delta_x u - u\int_0^t g(t-s)\Delta_x u(s,x)ds\right)dx\\
&+\int_{\mathbb{R}^n}\rho(x)u^2\ln|u|^k dx\\
&\leq \|u'\|^2_{L^2_\rho(\mathbb{R}^n)} + m_1\|\nabla_x u\|_2^{2(\gamma+1)} - l\|\nabla_x u\|_2^2\\
&+k\int_{\mathbb{R}^n}\rho(x)u^2\left(\ln\left(\frac{|u|}{\|u\|^2_{L^2_\rho}}\right) + \ln\|u\|^2_{L^2_\rho}\right)dx\\
&+\int_{\mathbb{R}^n}\nabla_x u\int_0^t g(t-s)(\nabla_x u(s) - \nabla_x u(t))dsdx.
\end{aligned}$$

We have by using the Logarithmic Sobolev inequality in Lemma 3 and generalized version of Poincare's inequality in Lemma2 Using Young's inequality and Lemma 1 for $\theta = 1/2$, we obtain

$$\begin{aligned}
&\psi_1'(t) \leq \|u'\|^2_{L^2_\rho} + m_1\|\nabla_x u\|_2^{2(\gamma+1)} + \left(\frac{kc_2}{2\pi}\|\rho\|^2_{L^2} - l\right)\|\nabla_x u\|_2^2\\
&+k\|u\|^2_{L^2_\rho}\ln\|u\|^2_{L^2_\rho}\\
&+\sigma\|\nabla_x u\|_2^2 + \frac{1}{4\sigma}\int_{\mathbb{R}^n}\left(\int_0^t g(t-s)|\nabla_x u(s) - \nabla_x u(t)|ds\right)^2 dx\\
&-\frac{1}{2}kn(1+c_1)\|u\|^2_{L^2_\rho}\\
&\leq \|u'\|^2_{L^2_\rho} + m_1\|\nabla_x u\|_2^{2(\gamma+1)} + \left(\sigma + \frac{kc_2}{2\pi}\|\rho\|^2_{L^2} - l\right)\|\nabla_x u\|_2^2\\
&+\frac{(1-l)}{4\sigma}(g\circ\nabla_x u) + k\left(\ln\|u\|^2_{L^2_\rho} - \frac{1}{2}n(1+c_1)\right)\|u\|^2_{L^2_\rho}.
\end{aligned}$$

Then

$$\begin{aligned}
&\psi_1'(t) \leq \|u'\|^2_{L^2_\rho} + m_1\|\nabla_x u\|_2^{2(\gamma+1)} + \frac{(1-l)}{4\sigma}(g\circ\nabla_x u)\\
&+\left[\left(\sigma + \frac{kc_2}{2\pi}\|\rho\|^2_{L^2} - l\right) + k\|\rho\|^2_{L^2}\left(\ln\|u\|^2_{L^2_\rho} - \frac{1}{2}n(1+c_1)\right)\right]\|\nabla u\|_2^2.
\end{aligned}$$

The existence of the memory term forces us to make second modification of the associate energy functional. Set

$$\psi_2(t) = -\int_{\mathbb{R}^n}\rho(x)u'\int_0^t g(t-s)(u(t)-u(s))dsdx. \tag{26}$$

Lemma 7. *Under the hypothesis (A1) and (A2), the functional ψ_2 satisfies, along the solution of (1),(2), for any $\sigma \in (0, m_0)$*

$$\psi_2'(t) \le \left[\sigma + k\Big(\sigma\frac{c_2}{2\pi} + \ln\|u\|^2_{L^2_\rho} - \frac{n(1+c_1)}{2}\Big)\right]\|\nabla_x u\|_2^2$$
$$+cm_1\|\nabla_x u\|_2^{2(\gamma+1)} + c_\sigma(1+(k\frac{c_2}{2\pi}+1)\|\rho\|^2_{L^2})(g\circ\nabla_x u)$$
$$-c_\sigma\|\rho\|^2_{L^2}(g'\circ\nabla_x u) + \left(\sigma - \int_0^t g(s)ds\right)\|u'\|^2_{L^2_\rho}.$$

Proof. Exploiting Eq. (1), (26) to get

$$\psi_2'(t) = -\int_{\mathbb{R}^n}\rho(x)u''\int_0^t g(t-s)(u(t)-u(s))dsdx$$
$$-\int_{\mathbb{R}^n}\rho(x)u'\int_0^t g'(t-s)(u(t)-u(s))dsdx$$
$$-\int_0^t g(s)ds\|u'\|^2_{L^2_\rho}$$
$$= \int_{\mathbb{R}^n} M(\|\nabla u\|_2^2)\nabla_x u\int_0^t g(t-s)(\nabla_x u(t)-\nabla_x u(s))dsdx$$
$$-\int_{\mathbb{R}^n}\rho(x)u\ln|u|^k\int_0^t g(t-s)(u(t)-u(s))dsdx$$
$$-\int_{\mathbb{R}^n}\left(\int_0^t g(t-s)\nabla_x u(s,x)ds\right)\times$$
$$\left(\int_0^t g(t-s)(\nabla_x u(t)-\nabla_x u(s))ds\right)dx$$
$$-\int_{\mathbb{R}^n}\rho(x)u'\int_0^t g'(t-s)(u(t)-u(s))dsdx$$
$$-\int_0^t g(s)ds\|u'\|^2_{L^2_\rho}$$

By (A1), we have

$$\psi_2'(t) = \left(m_0 - \int_0^t g(s)ds\right)\times$$
$$\int_{\mathbb{R}^n}\nabla_x u\int_0^t g(t-s)(\nabla_x u(t)-\nabla_x u(s))dsdx$$
$$+\int_{\mathbb{R}^n}\left(\int_0^t g(t-s)(\nabla_x u(t)-\nabla_x u(s))ds\right)^2dx$$
$$+cm_1\|\nabla_x u\|_2^{2(\gamma+1)}$$
$$-\int_{\mathbb{R}^n}\rho(x)u\ln|u|^k\int_0^t g(t-s)(u(t)-u(s))dsdx$$
$$-\int_{\mathbb{R}^n}\rho(x)u'\int_0^t g'(t-s)(u(t)-u(s))dsdx$$
$$-\int_0^t g(s)ds\|u'\|^2_{L^2_\rho} + c(g\circ\nabla_x u)(t).$$

By Holder's and Young's inequalities and Lemma 2, we estimate

$$-\int_{\mathbb{R}^n}\rho(x)u'\int_0^t g'(t-s)(u(t)-u(s))dsdx$$
$$\le \left(\int_{\mathbb{R}^n}\rho(x)|u'|^2dx\right)^{1/2}\times$$
$$\left(\int_{\mathbb{R}^n}\rho(x)\left|\int_0^t g'(t-s)(u(t)-u(s))ds\right|^2\right)^{1/2}$$
$$\le \sigma\|u'\|^2_{L^2_\rho} + c_\sigma\left\|\int_0^t -g'(t-s)(u(t)-u(s))ds\right\|^2_{L^2_\rho}$$
$$\le \sigma\|u'\|^2_{L^2_\rho} - c_\sigma\|\rho\|^2_{L^2}(g'\circ\nabla_x u)(t).$$

and

$$\int_{\mathbb{R}^n}\rho(x)u'\int_0^t g(t-s)(u(t)-u(s))dsdx$$
$$\le \sigma\|u'\|^2_{L^2_\rho} + c_\sigma\|\rho\|^2_{L^2}(g\circ\nabla_x u)(t).$$

and by Lemma 2 and Lemma 3 and conditions in Lemma 5, we have

$$-\int_{\mathbb{R}^n}\rho(x)\ln|u|^k u\int_0^t g(t-s)(u(t)-u(s))dsdx$$
$$\le k\int_{\mathbb{R}^n}\rho(x)\left(\ln\Big(\frac{|u|}{\|u\|^2_{L^2_\rho}}\Big) + \ln\|u\|^2_{L^2_\rho}\right)u\times$$
$$\int_0^t g(t-s)(u(t)-u(s))dsdx$$
$$\le k\left(\ln\|u\|^2_{L^2_\rho} - \frac{n(1+c_1)}{2}\right)\|u\|^2_{L^2_\rho}$$
$$+k\frac{c_2}{2\pi}\left\|u\int_0^t g(t-s)(u(t)-u(s))ds\right\|^2_{L^2_\rho}$$
$$\le k\left(\ln\|u\|^2_{L^2_\rho} - \frac{n(1+c_1)}{2}\right)\|\rho\|^2_{L^2}\|\nabla_x u\|_2^2$$
$$+k\frac{c_2}{2\pi}\|\rho\|^2_{L^2}\left\|\nabla u\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right\|^2_{L^2_\rho}$$
$$\le k\left(\sigma\frac{c_2}{2\pi} + \ln\|u\|^2_{L^2_\rho} - \frac{n(1+c_1)}{2}\right)\|\rho\|^2_{L^2}\|\nabla_x u\|_2^2$$
$$+c_\sigma k\frac{c_2}{2\pi}\|\rho\|^2_{L^2}(g\circ\nabla_x u).$$

Using Young's and Poincare's inequalities and Lemma 1 for $\theta = 1/2$, we obtain

$$\psi_2'(t) \le \left[\sigma + k\Big(\sigma\frac{c_2}{2\pi} + \ln\|u\|^2_{L^2_\rho} - \frac{n(1+c_1)}{2}\Big)\right]\|\nabla_x u\|_2^2$$
$$+cm_1\|\nabla_x u\|_2^{2(\gamma+1)}$$
$$+c_\sigma(1+(k\frac{c_2}{2\pi}+1)\|\rho\|^2_{L^2})(g\circ\nabla_x u) - c_\sigma\|\rho\|^2_{L^2}(g'\circ\nabla_x u)$$
$$+\left(\sigma - \int_0^t g(s)ds\right)\|u'\|^2_{L^2_\rho}.$$

Now, let us define

$$L(t) = \xi_1 E(t) + \psi_1(t) + \xi_2\psi_2(t) \tag{27}$$

for $\xi_1, \xi_2 > 1$. We need the next Lemma, which means that there is equivalent between the Lyapunov and energy functions, that is for $\xi_1, \xi_2 > 1$, we have

$$\beta_1 L(t) \le E(t) \le \beta_2 L(t)$$

holds for two positive constants β_1 and β_2.

Lemma 8. *For $\xi_1, \xi_2 > 1$, we have*

$$L(t) \sim E(t).$$

Proof. By (27) we have

$$\begin{aligned} &|L(t)-\xi_1 E(t)| \le |\psi_1(t)| + \xi_2|\psi_2(t)| \\ &\le \int_{\mathbb{R}^n} |\rho(x)uu'|\,dx \\ &+\xi_2 \int_{\mathbb{R}^n} \left|\rho(x)u' \int_0^t g(t-s)(u(t)-u(s))ds\right| dx. \end{aligned}$$

Thanks to Holder and Young's inequalities, we have by using Lemma 2

$$\begin{aligned} &\int_{\mathbb{R}^n} |\rho(x)uu'|\,dx \\ &\le \left(\int_{\mathbb{R}^n} \rho(x)|u|^2dx\right)^{1/2} \left(\int_{\mathbb{R}^n} \rho(x)|u'|^2dx\right)^{1/2} \\ &\le \frac{1}{2}\left(\int_{\mathbb{R}^n} \rho(x)|u|^2dx\right) + \frac{1}{2}\left(\int_{\mathbb{R}^n} \rho(x)|u'|^2dx\right) \\ &\le c\|u'\|^2_{L^2_\rho} + c\|\rho\|^2_{L^2}\|\nabla_x u\|^2_2 \end{aligned}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^n} \left| \left(\rho(x)^{\frac{1}{2}}u'\right)\left(\rho(x)^{\frac{1}{2}}\int_0^t g(t-s)(u(t)-u(s))ds\right)\right| dx \\ &\le \left(\int_{\mathbb{R}^n} \rho(x)|u'|^2dx\right)^{1/2} \times \\ &\left(\int_{\mathbb{R}^n} \rho(x)\left|\int_0^t g(t-s)(u(t)-u(s))ds\right|^2 dx\right)^{1/2} \\ &\le \frac{1}{2}\|u'\|^2_{L^2_\rho} + \frac{1}{2}\left\|\int_0^t g(t-s)(u(t)-u(s))ds\right\|^2_{L^2_\rho} \\ &\le \frac{1}{2}\|u'\|^2_{L^2_\rho} + \frac{1}{2}\|\rho\|^2_{L^2}(g\circ\nabla_x u). \end{aligned}$$

Then,

$$|L(t)-\xi_1 E(t)| \le cE(t).$$

Therefore, we can choose ξ_1 so that

$$L(t) \sim E(t). \tag{28}$$

Lemma 9. *For all $t \ge t_1 > 0$, we have*

$$\int_{t_1}^t (g\circ\nabla_x u)(s)ds \le H_0^{-1}\Big(-\int_{t_1}^t H_0(-g'(s))g'(s) \times \int_{\mathbb{R}^n} g(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2dxds\Big).$$

where H_0 introduced in (9).

Proof. By (15) and (A3), we have for all $t \ge t_1$

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_0^{t_1} g(t-s)|\nabla_x u(t)-\nabla_x u(s)|^2dsdx \\ &\le -\frac{1}{k}\int_{\mathbb{R}^n}\int_0^{t_1} g(t-s)|\nabla_x u(t)-\nabla_x u(s)|^2dsdx \\ &\le -cE'(t). \end{aligned}$$

Now, we define

$$I(t) = \int_{t_1}^t H_0(-g'(s))(g\circ\nabla_x u)(t)ds. \tag{29}$$

Since $\int_0^{+\infty} H_0(-g'(s))g(s)ds < +\infty$, from (15) we have

$$\begin{aligned} I(t) &= \int_{t_1}^t H_0(-g'(s))\int_{\mathbb{R}^n} g(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2dxds \\ &\le 2\int_{t_1}^t H_0(-g'(s))g(s)\int_{\mathbb{R}^n} |\nabla_x u(t)|^2+|\nabla_x u(t-s)|^2dxds \\ &\le cE(0)\int_{t_1}^t H_0(-g'(s))g(s)ds < 1. \end{aligned} \tag{30}$$

We define again a new functional $\lambda(t)$ related with $I(t)$ as

$$\lambda(t) = -\int_{t_1}^t H_0(-g'(s))g'(s)\int_{\mathbb{R}^n} g(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2dxds.$$

From (A1)-(A3) and , we get

$$H_0(-g'(s))g(s) \le H_0(H(g(s)))g(s) = D(g(s))g(s) \le k_0.$$

for some positive constant k_0. Then, for all $t \ge t_1$

$$\begin{aligned} \lambda(t) &\le -k_0\int_{t_1}^t g'(s)\int_{\mathbb{R}^n} |\nabla_x u(t)-\nabla_x u(t-s)|^2dxds \\ &\le -k_0\int_{t_1}^t g'(s)\int_{\mathbb{R}^n} |\nabla_x u(t)|^2+|\nabla_x u(t-s)|^2dxds \\ &\le -cE(0)\int_{t_1}^t g'(s)ds \\ &\le cE(0)g(t_1) \\ &< \min\{r, H(r), H_0(r)\}. \end{aligned} \tag{31}$$

Using the properties of H_0 (strictly convex in $(0,r], H_0(0)=0$), then for $x \in (0,r], \theta \in [0,1]$

$$H_0(\theta x) \le \theta H_0(x).$$

Using hypothesis in (A3), (30), (31) and Jensen's inequality leads to

$$\begin{aligned} &\lambda(t) = \frac{1}{I(t)}\int_{t_1}^t I(t)H_0[H_0^{-1}(-g'(s))]H_0(-g'(s))g'(s) \times \\ &\int_{\mathbb{R}^n} g(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2dxds \\ &\ge \frac{1}{I(t)}\int_{t_1}^t H_0[I(t)H_0^{-1}(-g'(s))]H_0(-g'(s))g'(s) \times \\ &\int_{\mathbb{R}^n} g(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2dxds \\ &\ge H_0\Big(\frac{1}{I(t)}\int_{t_1}^t I(t)H_0^{-1}(-g'(s))H_0(-g'(s))g'(s) \times \\ &\int_{\mathbb{R}^n} g(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2dxds\Big) \\ &\ge H_0\left(\int_{t_1}^t\int_{\mathbb{R}^n} g(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2dxds\right) \end{aligned}$$

which implies

$$\int_{t_1}^{t}\int_{\mathbb{R}^n} g(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds \le H_0^{-1}(\lambda(t)).$$

Our next main result reads as follows.

Theorem 3. *Let $(u_0,u_1)\in\mathscr{H}(\mathbb{R}^n)\times L^2_\rho(\mathbb{R}^n)$ and suppose that (A1)- (A2) hold. Then there exist positive constants $\alpha_0,\alpha_1,\alpha_2,\alpha_3$ such that the energy of solution given by (1),(2) satisfies,*

$$E(t)\le \alpha_3 H_1^{-1}(\alpha_1 t+\alpha_2),\quad \text{for all}\quad t\ge 0,$$

where

$$H_1(t)=\int_t^1 (sH_0'(\alpha_0 s))^{-1}ds$$

Proof. From (15), results of Lemma 6 and Lemma 7, we have

$$\begin{aligned} L'(t) &= \xi_1 E'(t)+\psi_1'(t)+\xi_2\psi_2'(t)\\ &\le (\frac{1}{2}\xi_1 - c_\sigma\|\rho\|^2_{L^2}\xi_2)(g'\circ\nabla_x u)+M_0(g\circ\nabla_x u)\\ &- M_1\|u'\|^2_{L^2_\rho}-M_2\|\nabla_x u\|_2^2+(c\xi_2+1)m_1\|\nabla_x u\|_2^{2(\gamma+1)}\end{aligned}$$

where

$$M_0=\left(\xi_2 c_\sigma(1+(k\frac{c_2}{2\pi}+1)\|\rho\|^2_{L^2})+\frac{(1-l)}{4\sigma}\right)>0,$$

$$M_1=\left(\xi_2\left(\int_0^{t_1}g(s)ds-\sigma\right)-1\right),$$

$$\begin{aligned}M_2 &= \frac{1}{2}\xi_1 g(t_1)-\Big[\Big(\sigma+\frac{kc_2}{2\pi}\|\rho\|^2_{L^2}-l\Big)\\ &+k\|\rho\|^2_{L^2}\Big(\ln\|u\|^2_{L^2_\rho}-\frac{1}{2}n(1+c_1)\Big)\Big]\\ &-\xi_2\left[\sigma+k\Big(\sigma\frac{c_2}{2\pi}+\ln\|u\|^2_{L^2_\rho}-\frac{n(1+c_1)}{2}\Big)\right]\end{aligned}$$

and t_1 was introduced in (A3). We choose σ so small that $\xi_1>2c_\sigma\|\rho\|^2_{L^2}\xi_2$. Whence σ is fixed, we can choose

$$\xi_2>\left(\int_0^{t_1}g(s)ds-\sigma\right)^{-1}$$

and ξ_1 large enough so that $M_2>0$, which yields

$$L'(t)\le M_0(g\circ\nabla_x u)+(c\xi_2+1)m_1\|\nabla_x u\|_2^{2(\gamma+1)}-cE'(t),\quad \forall t\ge t_1.$$

Now we set $F(t)=L(t)+cE(t)$, which is equivalent to $E(t)$. Then, we get for some $c>2(c\xi_2+1)(\gamma+1)$

$$\begin{aligned}F'(t)&=L'(t)+cE'(t) \qquad (32)\\ &\le -cE(t)+c\int_{\mathbb{R}^n}\int_{t_1}^{t} g(t-s)|\nabla_x u(t)-\nabla_x u(s)|^2 dsdx,\\ &\quad \text{for all}\quad t\ge t_1.\end{aligned}$$

Using Lemma(9), we obtain

$$F'(t)\le -cE(t)+cH_0^{-1}(\lambda(t)),\quad \text{for all}\quad t\ge t_1.$$

Now, we will following the steps in ([20]) and using the fact that $E'\le 0, 0<H_0', 0<H_0''$ on $(0,r]$ to define the functional

$$F_1(t)=H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)F(t)+cE(t),\quad \alpha_0<r, 0<c,$$

where $F_1(t)\sim E(t)$ and

$$\begin{aligned}F_1'(t) &= \alpha_0\frac{E'(t)}{E(0)}H_0''\left(\alpha_0\frac{E(t)}{E(0)}\right)F(t)\\ &+ H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)F'(t)+cE'(t)\\ &\le -cE(t)H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)\\ &+c\,H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)H_0^{-1}(\lambda(t))+cE'(t).\end{aligned}$$

Let H_0^* given in (A3) and using Young's inequality (11) with $A=H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right), B=H_0^{-1}(\lambda(t))$, to get

$$\begin{aligned}F_1'(t)&\le -cE(t)H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)+cH_0^*\left(H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)\right)\\ &+c\lambda(t)+cE'(t)\\ &\le -cE(t)H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)+c\alpha_0\frac{E(t)}{E(0)}H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)\\ &-c'E'(t)+cE'(t).\end{aligned}$$

Choosing α_0,c,c', such that for all $t\ge t_1$ we have

$$\begin{aligned}F_1'(t)&\le -k\frac{E(t)}{E(0)}H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)\\ &=-kH_2\left(\frac{E(t)}{E(0)}\right),\end{aligned}$$

where $H_2(t)=tH_0'(\alpha_0 t)$. Using the strict convexity of H_0 on $(0,r]$, to find that H_2',H_2 are strict positives on $(0,1]$, then

$$R(t)=\tau\frac{k_1F_1(t)}{E(0)}\sim E(t),\quad \tau\in(0,1) \qquad (33)$$

and

$$R'(t)\le -\tau k_0 H_2(R(t)),\quad k_0\in(0,+\infty), t\ge t_1.$$

Then, a simple integration and a suitable choice of τ yield,

$$R(t)\le H_1^{-1}(\alpha_1 t+\alpha_2),\quad \alpha_1,\alpha_2\in(0,+\infty), t\ge t_1.$$

here $H_1(t)=\int_t^1 H_2^{-1}(s)ds$. From (33), for a positive constant α_3, we have

$$E(t)\le\alpha_3 H_1^{-1}(\alpha_1 t+\alpha_2),\quad \alpha_1,\alpha_2\in(0,+\infty), t\ge t_1.$$

The fact that H_1 is strictly decreasing function on $(0,1]$ and due to properties of H_2, we have

$$\lim_{t\to 0} H_1(t) = +\infty.$$

Then

$$E(t) \leq \alpha_3 H_1^{-1}(\alpha_1 t + \alpha_2), \quad \text{for all} \quad t \geq 0.$$

This completes the proof of Theorem 3.

5 Concluding comments

The coupled systems of wave equations abound in the world. One reason is that nature is full of those physical phenomenos. Another reason is that systems are often used to model a large class of engineering sciences, where propagation and transmission of informations or material are involved.

1- It will be also interesting to consider, derived from (1), and study the questions of asymptotic behavior of the related coupled system

$$\begin{cases} \left(|u_1'|^{l-2}u_1'\right)' + \phi(x)A\left(u_1 + \int_0^t g_1(s)u_1(t-s,x)ds\right) \\ = au_2 \ln|u_1|^k, \\ \left(|u_2'|^{l-2}u_2'\right)' + \phi(x)A\left(u_2 + \int_0^t g_2(s)u_2(t-s,x)ds\right) \\ = au_1 \ln|u_2|^k, \\ (u_1(0,x),u_2(0,x)) = (u_{10}(x),u_{20}(x)) \in (\mathscr{H}(\mathbb{R}^n))^2, \\ (u_1'(0,x),u_2'(0,x)) = (u_{11}(x),u_{21}(x)) \in (L^l_\rho(\mathbb{R}^n))^2, \end{cases}$$

where our weak coupling is given by the logarithmic nonlinearities terms for $a \neq 0, l, n \geq 2$ and A is a linear, selfadjoint operator in $L^2(\mathbb{R}^n)$.

2. Let us remark that, it is similar to study the question of existence and decay of solution of the same problem with the presence of weak-viscoelasticity in the form

$$\begin{cases} \left(|u_1'|^{l-2}u_1'\right)' + \phi(x)A\left(u_1 + \alpha_1(t)\int_0^t g_1(s)u_1(t-s,x)ds\right) \\ = au_2 \ln|u_1|^k, \\ \left(|u_2'|^{l-2}u_2'\right)' + \phi(x)A\left(u_2 + \alpha_2(t)\int_0^t g_2(s)u_2(t-s,x)ds\right) \\ = au_1 \ln|u_2|^k, \\ (u_1(0,x),u_2(0,x)) = (u_{10}(x),u_{20}(x)) \in (\mathscr{H}(\mathbb{R}^n))^2, \\ (u_1'(0,x),u_2'(0,x)) = (u_{11}(x),u_{21}(x)) \in (L^l_\rho(\mathbb{R}^n))^2, \end{cases}$$

where we should need additional, conditions on α as follows

$$1-\alpha_i(t)\int_0^t g_i(t)dt \geq k_i > 0, \int_0^\infty g_i(t)dt < +\infty, \alpha_i(t) > 0,$$

$$\lim_{t\to+\infty} \frac{-\alpha'(t)}{\alpha(t)\xi(t)} = 0 \tag{34}$$

where

$$\alpha(t) = \min\{\alpha_1(t), \alpha_2(t)\}, \quad \forall t \geq 0.$$

Which will be our next works. For the reader we shall develop here the next important technical Lemma.

Lemma 10. *For any $v \in C^1\left(0,T,H^1(\mathbb{R}^n)\right)$ we have*

$$\begin{aligned} &-\int_{\mathbb{R}^n}\alpha(t)\int_0^t g(t-s)Av(s)v'(t)dsdx \\ =& \frac{1}{2}\frac{d}{dt}\alpha(t)\left(g\circ A^{1/2}v\right)(t) \\ -& \frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(s)\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dxds\right] \\ &-\frac{1}{2}\alpha(t)\left(g'\circ A^{1/2}v\right)(t) + \frac{1}{2}\alpha(t)g(t)\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dxds \\ &-\frac{1}{2}\alpha'(t)\left(g\circ A^{1/2}v\right)(t) + \frac{1}{2}\alpha'(t)\int_0^t g(s)ds\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dxds. \end{aligned}$$

Proof. It's not hard to see

$$\begin{aligned} &\int_{\mathbb{R}^n}\alpha(t)\int_0^t g(t-s)Av(s)v'(t)dsdx \\ =& \alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n}A^{1/2}v'(t)A^{1/2}v(s)dxds \\ =& \alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n}A^{1/2}v'(t)\left[A^{1/2}v(s)-A^{1/2}v(t)\right]dxds \\ &+\alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n}A^{1/2}v'(t)A^{1/2}v(t)dxds. \end{aligned}$$

Consequently,

$$\begin{aligned} &\int_{\mathbb{R}^n}\alpha(t)\int_0^t g(t-s)Av(s)v'(t)dsdx \\ =& -\frac{1}{2}\alpha(t)\int_0^t g(t-s)\frac{d}{dt}\int_{\mathbb{R}^n}\left|A^{1/2}v(s)-A^{1/2}v(t)\right|^2 dxds \\ &+\alpha(t)\int_0^t g(s)\left(\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dx\right)ds \end{aligned}$$

which implies,

$$\begin{aligned} &\int_{\mathbb{R}^n}\alpha(t)\int_0^t g(t-s)Av(s)v'(t)dsdx \\ =& -\frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n}\left|A^{1/2}v(s)-A^{1/2}v(t)\right|^2 dxds\right] \\ &+\frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(s)\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dxds\right] \\ &+\frac{1}{2}\alpha(t)\int_0^t g'(t-s)\int_{\mathbb{R}^n}\left|A^{1/2}v(s)-A^{1/2}v(t)\right|^2 dxds \\ &-\frac{1}{2}\alpha(t)g(t)\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dxds. \\ &+\frac{1}{2}\alpha'(t)\int_0^t g(t-s)\int_{\mathbb{R}^n}\left|A^{1/2}v(s)-A^{1/2}v(t)\right|^2 dxds \\ &-\frac{1}{2}\alpha'(t)\int_0^s g(s)ds\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dxds. \end{aligned}$$

This completes the proof.

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