Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100621

Error Bounds for General Variational Inclusion Involving Difference of Operators

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Received: 2 Jul. 2016, Revised: 15 Aug. 2016, Accepted: 1 Sep. 2016
Published online: 1 Nov. 2016

Abstract: In this paper, we introduce some new classes of merit functions for general variational inclusion involving difference of two monotone operators. Using these merit functions, we obtain the error bounds for the solution of the general variational inclusion. Several special cases are also investigated. Results proved in this paper continue to hold for these cases. Results obtained in this paper can be viewed as significant contribution in this field and may motivate further research.

Keywords: Merit functions, Error bounds, Fixed-point, variational inclusion
AMS Subject Classification: 49J40, 90C23

1 Introduction

Variational inclusions are the natural generalization of variational inequalities having applications to many fields, for example, mechanics, physics, optimization and control theory, nonlinear programming, economics and engineering sciences. For details, see [1]-[31] and references therein. Variational inclusions involving the sum of monotone operators have been studied widely in recent years. It is known that the sum of two or more monotone operators is again a monotone operator but difference is not. Due to this fact, the problem of finding a zero of the difference of two monotone operators is very difficult as compared to finding the zeros of monotone operators, see Noor [22] and Stampacchia [29].

A novel and innovative technique for solving variational inclusion is via merit functions. Using this powerful technique, we reformulate the variational inclusion problems into equivalent optimization problems. Thus all the problems which can be solved in the frame work of variational inclusions can be discussed using optimization theory. It can be considered to discuss the convergence of iterative methods. Auslender [2] suggested the first merit function for variational inequalities. This merit function is not differentiable. Auslender [2] has shown that if the set is strongly convex, then the function is differentiable. But due to this strict condition many important applications can not be considered via Auslender merit function.

Merit functions are very useful in suggesting globally convergent algorithms for solving variational inclusion. These play a significant part to investigate the rate of convergence of iterative methods. Error bounds are responsible for providing the distance between the solution set and arbitrary point. Hence error bounds play a significant role in evaluating the global and local convergence analysis of algorithms of variational inclusions. There are so many merit functions for solving variational inclusions and its variant forms which comprises residual merit function, regularized merit functions and D-merit functions and many more . It is well-known that the residual merit function is not differentiable, which is a serious drawback. To over come this drawback, Fukushima [6] suggested and studied a regularized differentiable merit function under some suitable conditions for variational inequalities. It is called regularized merit function. It enables us to calculate the local error bound. Peng [24] and Yamashita et al. [31] introduced D-merit function, independently. D-merit function helps us to derive the global error bound. It also gives unconstrained optimization reformulation for variational inequalities, see [30]. Noor [17] and Noor [18] introduced and studied various merit functions such as: regularized merit function and D-merit functions for general variational inequalities and quasi variational

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inequalities, respectively. He has calculated the error bounds for both general and quasi variational inequalities. For recent applications, see [5,9] and references therein.

In this paper, we suggest some new merit functions for general variational inclusion. Using these merit functions, we derive the error bounds for general variational inclusion.

2 Formulation and Basic Results

Let $\mathscr{H}$ be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively. For given monotone operators $T,A,g:\mathscr{H}\to\mathscr{H}$, consider a problem of finding $u\in\mathscr{H}: g(u)\in\mathscr{H}$, such that

$$0\in A(g(u))-Tu. \tag{1}$$

The problem of type (1) is called general variational inclusion involving difference of monotone operators. This problem is considered by Noor et al. [22]. For recent developments and other aspects of general variational inclusion, see [15,22].
We now discuss some applications of the general variational inclusions (1).

2.1 Applications

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(I) If $g\equiv I$, the identity operator, then problem (1) is equivalent to finding $u\in\mathscr{H}$ such that

$$0\in A(u)-Tu, \tag{2}$$

a problem considered by Noor *et al.* [20,21] and Moudafi [11] recently using two different techniques.
(II) If $A(.)\equiv\partial\phi(.)$, the subdifferential of a proper, convex and lower-semicontinuous function $\phi:\mathscr{H}\to R\cup\infty$, then problem (1) is equivalent to finding $u\in\mathscr{H}$ such that

$$0\in\partial\phi(g(u))-Tu, \tag{3}$$

a problem considered and studied by Adly and Oettli [1]. They have discussed the existence result and considered an iterative method for solving the general variational inclusion problem of type (3).
We note that problem (3) can be written as: find $u\in\mathscr{H}: g(u)\in\mathscr{H}$ such that

$$\langle -Tu,g(v)-g(u)\rangle+\phi(g(v))-\phi(g(u))\geq 0,\quad \forall v\in\mathscr{H}, \tag{4}$$

which is known as the general mixed variational inequality or the variational inequality of the second kind.
(III) If ϕ is the indicator function of a closed and convex set K in a real Hilbert space, then problem (4) is equivalent to finding $u\in\mathscr{H}: g(u)\in K$ such that

$$\langle Tu,g(v)-g(u)\rangle\leq 0,\qquad \forall v\in\mathscr{H}: g(v)\in K, \tag{5}$$

which is known as the general variational inequality, introduced and studied by Noor [14] in 1988.

(IV) If $g=I$, the identity operator, then problem (5) reduces to: find $u\in K$ such that

$$\langle Tu,v-u\rangle\leq 0,\qquad \forall v\in K, \tag{6}$$

which is known as the classical variational inequalities, introduced and studied by Stampacchia [29] in 1964. For the applications, numerical methods and other aspects of these mixed variational inequalities, see [1,19] and the references therein.

We also need the following well-known fundamental results and concepts.

Definition 1.*An operator $T:\mathscr{H}\to\mathscr{H}$ is said to be strongly g-antimonotone if there exists a constant $\alpha>0$ such that*

$$\langle Tu-Tv,g(u)-g(v)\rangle\leq-\alpha\|g(u)-g(v)\|^2,\quad \forall u,v\in\mathscr{H}.$$

Definition 2.*An operator $T:\mathscr{H}\to\mathscr{H}$ is said to be strongly non-expanding if there exists a constant $\tau>0$ such that*

$$\|Tu-Tv\|\geq\tau\|u-v\|,\quad \forall u,v\in\mathscr{H}.$$

Definition 3.*An operator $T:\mathscr{H}\to\mathscr{H}$ is said to be Lipschitz continuous if there exists a constant $\beta>0$ such that*

$$\|Tu-Tv\|\leq\beta\|u-v\|,\quad \forall u,v\in\mathscr{H}.$$

From the definitions 2 and 3, it is clear that $\tau\leq\beta$.

Definition 4.*[4] If A is a maximal monotone operator on $\mathscr{H}$, then, for a constant $\rho>0$, the resolvent operator associated with A is defined by*

$$J_A(u)=(I+\rho A)^{-1}(u),\qquad \forall u\in\mathscr{H},$$

where I is the identity operator.

It is well known that a monotone operator is maximal if and only if its resolvent operator is defined everywhere. In addition, the resolvent operator is a single-valued and nonexpensive, that is,

$$\|J_A(u)-J_A(v)\|\leq\|u-v\|,\qquad \forall u,v\in\mathscr{H}.$$

It is known that $\partial f(\cdot)$, the subdifferential of a proper, convex and lower semicontinuous function, is a maximal monotone operator. The resolvent operator associated with $\partial f(\cdot)$ is defined as

$$J_{\partial f}(u)=(I+\rho\partial f(\cdot))^{-1}(u),\qquad \forall u\in\mathscr{H}, \tag{7}$$

Lemma 1.*[4] For a given $z\in\mathscr{H}, u\in\mathscr{H}$ satisfies the inequality*

$$\langle u-z,v-u\rangle+\rho\phi(v)-\rho\phi(u)\geq 0,\quad \forall v\in\mathscr{H},$$

if and only if

$$u=J_\phi z,$$

where $J_\phi=(I+\rho\partial\phi)^{-1}$ is the resolvent operator.

If the function $\phi(\cdot)$ is the indicator function of a closed convex set K in $\mathscr{H}$, then it is well known that $J_\phi = P_K$, the projection operator of $\mathscr{H}$ onto the closed convex set K.

Definition 5.*[17]A function $\mathscr{M} : \mathscr{H} \to \mathscr{R} \cup \{+\infty\}$ is called a merit (gap) function for the general variation inclusion (1) , if and only if*

(i)$\mathscr{M}(u) \geq 0, \quad \forall u \in \mathscr{H} : g(u) \in \mathscr{H}(u)$.
(ii)$\mathscr{M}(\bar{u}) = 0$, if and only if, $\bar{u} \in \mathscr{H} : g(\bar{u}) \in \mathscr{H}(u)$ solves (1).

3 Main Results

In this section, we introduce some merit functions associated with the problem (1). Using these merit functions, we obtain some error bounds for problem (1). For this purpose, we need the following result.

Lemma 2.*[22] Let ϕ be a maximal monotone operator. Then function $u \in \mathscr{H} : g(u) \in \mathscr{H}$, is a solution of the general variational inclusion (1), if and only if, $u \in \mathscr{H} : g(u) \in \mathscr{H}$, satisfies the relation*

$$g(u) = J_\phi[g(u) + \rho Tu], \tag{8}$$

where $J_\phi = (I + \rho\phi)^{-1}$ is the resolvent operator and $\rho > 0$ is a constant.

It is well known that the resolvent operator J_ϕ is a nonexpansive operator, that is,

$$\left\|J_\phi(u) - J_\phi(v)\right\| \leq \|u - v\|, \quad \forall u, v \in \mathscr{H}.$$

If ϕ is the indicator function of a closed and convex set K in a real Hilbert space, then $J_\phi = P_K$ the projection operator, the equation (8) will become

$$g(u) = P_K[g(u) + \rho Tu]. \tag{9}$$

From Lemma 2, it follows that the problem (1) is equivalent to a fixed point problem (8). This equivalent formulation plays a crucial part in developing several iterative methods.

We now define the residue vector

$$\mathscr{R}_\rho(u) \equiv \mathscr{R}(u) = g(u) - J_\phi[g(u) + \rho Tu]. \tag{10}$$

It is clear from Lemma 2 that problem (1) has a solution $u \in \mathscr{H} : g(u) \in \mathscr{H}$, if and only if $u \in \mathscr{H} : g(u) \in \mathscr{H}$ is zero of the equation

$$\mathscr{R}_\rho(u) \equiv \mathscr{R}(u) = 0. \tag{11}$$

We now show that $\mathscr{R}_\rho(u)$ plays the role of natural residue vector for general variational inclusion (1).

Theorem 1.*Let $\rho > 0$ be arbitrary. An element $u \in \mathscr{H}$ solves general variational inclusion problem (1) if, and only if, $\mathscr{R}_\rho(u) = 0$.*

*Proof.*Let $\mathscr{R}_\rho(u) = 0$. Then $g(u) = J_\phi[g(u) + \rho Tu]$, which is equivalent to

$$g(u) = arg \quad min_{v\in\mathscr{H}}\{\phi(v) + \frac{1}{2\alpha}\|v - (g(u) + \rho Tu)\|^2\}.$$

By optimality conditions, this is equivalent to

$$0 \in \partial\phi(g(u)) + \frac{1}{\rho}(g(u) - (g(u) + \rho Tu)) = \partial\phi(g(u)) - Tu,$$

which in turn is equivalent, by definition of the subgradient, to

$$\langle Tu, g(u) - g(v)\rangle + \phi(g(v)) - \phi(g(u)) \geq 0, \qquad \forall v \in \mathscr{H} \tag{12}$$

which means that $u \in \mathscr{H}$ solves general variational inclusion (1). This completes the proof. □

*Remark.*It ie easy to see the normal residue vector $\| \mathscr{R}_\rho(u) \|$ is a gap function for general variational inclusion (1).

Now by using normal residual vector $\| \mathscr{R}_\rho(u) \|$, we derive the error bounds for the solution of general variational inclusion (1).

Theorem 2.*Assume that $\bar{u} \in \mathscr{H}$ be a solution of general variational inclusion problem (1). Let the operator T be strongly g-antimonotone and Lipschitz continuous with constants $\alpha, \beta > 0$, respectively. Let g be Lipschitz continuous with constant $\beta_1 > 0$ and nonexpanding with constant $\tau > 0$. Then for any $u \in \mathscr{H}$ and $\rho > 0$, we have*

$$\frac{1}{c_1} \| \mathscr{R}_\rho(u) \| \leq \| u - \bar{u} \| \leq c_2 \| \mathscr{R}_\rho(u) \|, \tag{13}$$

where $c_1 = (2\beta_1 + \rho\beta)$ and $c_2 = \frac{1}{\alpha\tau}(\beta + \frac{\beta_1}{\rho})$.

Proof. Let $\bar{u} \in \mathscr{H} : g(\bar{u}) \in \mathscr{H}$ be a solution of general variational inequality (4), then

$$\langle -T\bar{u}, g(v) - g(\bar{u})\rangle + \phi(g(v)) - \phi(g(\bar{u})) \geq 0.$$

Taking $g(v) = J_\phi[g(u) + \rho Tu]$ in above inequality, we get

$$\begin{aligned}&\left\langle -T\bar{u}, J_\phi[g(u) + \rho Tu] - g(\bar{u})\right\rangle \\ &+\phi(J_\phi[g(u) + \rho Tu]) - \phi(g(\bar{u})) \geq 0.\end{aligned} \tag{14}$$

Fix any $u \in \mathscr{H} : g(u) \in \mathscr{H}$ and $\rho > 0$. By the definition of J_ϕ, we have that $J_\phi[g(u) + \rho Tu]$ satisfies

$$Tu + \frac{1}{\rho}(g(u) - J_\phi[g(u) + \rho Tu]) \in \partial\phi(J_\phi[g(u) + \rho Tu]),$$

Thus $u \in \mathscr{H} : g(u) \in \mathscr{H}$, satisfies the inequality

$$\left\langle -Tu - \frac{1}{\rho}(g(u) - J_\phi[g(u) + \rho Tu]), g(v) - g(J_\phi[g(u) + \rho Tu])\right\rangle + \phi(g(v)) - \phi(J_\phi[g(u) + \rho Tu]) \geq 0, \quad v \in \mathscr{H} : g(v) \in \mathscr{H}.$$

Letting $v = \bar{u}$ in above inequality, we have

$$\begin{aligned}&\left\langle -Tu - \frac{1}{\rho}(g(u) - J_\phi[g(u) + \rho Tu]), g(\bar{u}) - g(J_\phi[g(u) + \rho Tu])\right\rangle \\ &+ \phi(g(\bar{u}) - \phi(J_\phi[g(u) + \rho Tu]) \geq 0.\end{aligned} \tag{15}$$

Adding equation (14) and (15), we obtain

$$\langle T\bar{u}-Tu, g(\bar{u})-g(J_\phi[g(u)+\rho Tu])\rangle \geq \frac{1}{\rho}\langle g(u)-J_\phi[g(u)+\rho Tu]), g(\bar{u})-g(J_\phi[g(u)+\rho Tu])\rangle. \quad (16)$$

Since T is strongly g-antimonotone with constant $\alpha > 0$ and g be strongly nonexpanding with constant $\tau > 0$, we have

$$\begin{aligned} & -\alpha\tau \parallel \bar{u}-u \parallel \\ \geq\ & -\alpha \parallel g(\bar{u})-g(u) \parallel \geq \langle T\bar{u}-Tu, g(\bar{u})-g(u)\rangle \\ =\ & \langle T\bar{u}-Tu, g(\bar{u})-g(J_\phi[g(u)+\rho Tu])\rangle \\ & +\langle T\bar{u}-Tu, g(J_\phi[g(u)+\rho Tu])-g(u)\rangle \\ \geq\ & \frac{1}{\rho}\langle g(u)-J_\phi[g(u)+\rho Tu]), g(\bar{u})-g(J_\phi[g(u)+\rho Tu])\rangle \\ & +\langle T\bar{u}-Tu, g(J_\phi[g(u)+\rho Tu])-g(u)\rangle \\ =\ & \frac{1}{\rho}\langle \mathscr{R}_\rho(u), g(\bar{u})-g(u)\rangle \\ +\ & \frac{1}{\rho}\langle \mathscr{R}_\rho(u), g(u)-g(J_\phi[g(u)+\rho Tu])\rangle \\ & -\langle T\bar{u}-Tu, g(u)-g(J_\phi[g(u)+\rho Tu])\rangle \\ \geq\ & \frac{1}{\rho}\parallel \mathscr{R}_\rho(u)\parallel^2 - \parallel T\bar{u}-Tu \parallel\parallel \mathscr{R}_\rho(u)\parallel \\ & -\frac{1}{\rho}\parallel g(u)-g(\bar{u})\parallel\parallel \mathscr{R}_\rho(u)\parallel \\ \geq\ & -(\beta+\frac{\beta_1}{\rho})\parallel \bar{u}-u \parallel\parallel \mathscr{R}_\rho(u)\parallel, \end{aligned} \quad (17)$$

where third inequality comes from equation (16), in fifth step we used Cauchy Schwarz inequality and last expression is obtained by using Lipschitz continuity of T and g with constants $\beta > 0$ and $\beta_1 > 0$ respectively.
Hence inequality (17) takes the form

$$\parallel \bar{u}-u \parallel \leq c_2 \parallel \mathscr{R}_\rho(u)\parallel, \quad (18)$$

where

$$c_2 = \frac{1}{\alpha\tau}(\beta+\frac{\beta_1}{\rho}).$$

Now from equation (10) and using Lipschitz continuity of T and g, we have

$$\begin{aligned} & \parallel \mathscr{R}_\rho(u)\parallel \\ =\ & \parallel g(u)-J_\phi[g(u)+\rho Tu]\parallel \\ =\ & \parallel g(u)-g(\bar{u})+J_\phi[g(\bar{u})+\rho T\bar{u}]-J_\phi[g(u)+\rho Tu] \\ \leq\ & \parallel g(u)-g(\bar{u})\parallel + \parallel g(u)-g(\bar{u})+\rho(Tu-T\bar{u})\parallel \\ \leq\ & (2\beta_1+\rho\beta)\parallel u-\bar{u}\parallel, \end{aligned}$$

from which we have

$$\frac{1}{c_1}\parallel \mathscr{R}_\rho(u)\parallel \leq \parallel u-\bar{u}\parallel, \quad (19)$$

where $c_1 = (2\beta_1+\rho\beta)$. Combining (18) and (19) we have the required result (13).

Letting $u = 0$ in (13), we have

$$\frac{1}{c_1}\parallel \mathscr{R}_\rho(0)\parallel \leq \|\bar{u}\| \leq c_2 \left\|\mathscr{R}_\rho(0)\right\|. \quad (20)$$

Combining (13) and (20), we obtain a relative error bound for any point $u \in \mathscr{H}$.

Theorem 3. *Assume that all the assumptions of Theorem 2 hold. If $0 \neq \bar{u} \in \mathscr{H}$ is a solution of (1), then*

$$\begin{aligned} & k_1 \left\|\mathscr{R}_\rho(u)\right\| / \left\|\mathscr{R}_\rho(0)\right\| \\ \leq\ & \|u-\bar{u}\| / \|\bar{u}\| \leq k_2 \left\|\mathscr{R}_\rho(u)\right\| / \left\|\mathscr{R}_\rho(0)\right\|. \end{aligned}$$

Note that the normal residue vector (merit function) $\parallel \mathscr{R}_\rho(u)\parallel$ defined by (10) is nondifferentiable. To overcome this draw back, we consider an other merit function associated with problem (1). This merit function can be viewed as a regularized merit function, see [6].
We consider the function for all $u \in \mathscr{H} : g(u) \in \mathscr{H}$, such that

$$\begin{aligned} \mathscr{G}_\rho(u) =\ & max_{v\in\mathscr{H}, g(v)\in\mathscr{H}}\{\langle Tu, g(v)-g(u)\rangle - \phi(g(v))+\phi(g(u)) \\ & -\frac{1}{2\rho}\parallel g(u)-g(v)\parallel^2\}, \forall u \in \mathscr{H}, g(u) \in \mathscr{H}, \end{aligned} \quad (21)$$

which is finite valued everywhere and is differentiable whenever all operators involved in $\mathscr{G}_\rho(u)$, are differentiable.

Lemma 3. *For any $\rho > 0$, $\mathscr{G}_\rho(u)$ can be written as*

$$\begin{aligned} \mathscr{G}_\rho(u) =\ & \langle Tu, J_\phi[g(u)+\rho Tu]-g(u)\rangle - \phi(J_\phi[g(u)+\rho Tu]) \\ & +\phi(g(u)) - \frac{1}{2\rho}\parallel g(u)-J_\phi[g(u)+\rho Tu]\parallel^2, \\ & \forall u \in \mathscr{H}, g(u) \in \mathscr{H}. \end{aligned} \quad (22)$$

Proof. Using the technique of Solodov [27], one can easily prove this result. □

We now show that the function $\mathscr{G}_\rho(u)$ for $\rho > 0$ given by (21) is a gap function for general variational inclusion (1).

Theorem 4. *For all $u \in \mathscr{H} : g(u) \in \mathscr{H}$, we have*

$$\mathscr{G}_\rho(u) \geq \frac{1}{2\rho}\parallel \mathscr{R}_\rho(u)\parallel^2, \quad \forall u \in \mathscr{H} : g(u) \in \mathscr{H}.$$

In Particular, we have $\mathscr{G}_\rho(u) = 0$, if and only if $u \in \mathscr{H} : g(u) \in \mathscr{H}$ is a solution of the problem (1).

Proof. Fix any $u \in \mathscr{H}$, $\rho > 0$. Observe that

$$\begin{aligned} & 0 \in \partial\phi(J_\phi[g(u)+\rho Tu]) + \frac{1}{\rho}(J_\phi[g(u)+\rho Tu] \\ & -(g(u)+\rho Tu)), \end{aligned}$$

which is equivalent to

$$Tu + \frac{1}{\rho}(g(u)-J_\phi[g(u)+\rho Tu]) \in \partial\phi(J_\phi[g(u)+\rho Tu]).$$

By definition of subdifferential, we have

$$\langle -Tu-\frac{1}{\rho}(g(v)-J_\phi[g(u)+\rho Tu]), g(v)-J_\phi[g(u)+\rho Tu]\rangle$$
$$+\phi(g(v))-\phi(J_\phi[g(u)+\rho Tu])\geq 0.$$

Taking $v=u$ in above inequality, we get

$$\langle -Tu-\frac{1}{\rho}(g(u)-J_\phi[g(u)+\rho Tu]), g(u)-J_\phi[g(u)+\rho Tu]\rangle$$
$$+\phi(g(u))-\phi(J_\phi[g(u)+\rho Tu])\geq 0,$$

$$\{\langle\ Tu,\ J_\phi[g(u)+\rho Tu]-g(u)\rangle+\phi(g(u))$$
$$-\ \ \phi(J_\phi[g(u)+\rho Tu])\}\geq\frac{1}{\rho}\langle\mathscr{R}_\rho(u),\mathscr{R}_\rho(u)\rangle. \quad (23)$$

Combining (22) with (23), we get

$$\mathscr{G}_\rho(u)\geq\frac{1}{\rho}\parallel\mathscr{R}_\rho(u)\parallel^2-\frac{1}{2\rho}\parallel\mathscr{R}_\rho(u)\parallel^2=\frac{1}{2\rho}\parallel\mathscr{R}_\rho(u)\parallel^2, \quad (24)$$

which is the required result. Clearly we have $\mathscr{G}_\rho(u)\geq 0$, $\forall u\in\mathscr{H}:g(u)\in\mathscr{H}$.
Now if $\mathscr{G}_\rho(u)=0$, then clearly $\mathscr{R}_\rho(u)=0$. Hence by Theorem 1 , we see that $u\in\mathscr{H}:g(u)\in\mathscr{H}$ is the solution of problem (1). Conversely if $u\in\mathscr{H}:g(u)\in\mathscr{H}$ is the solution of problem (1), then $g(u)=J_\phi[g(u)+\rho Tu]$ by Lemma 2. Consequently from (22), we see that $\mathscr{G}_\rho(u)=0$, the required result.

From Theorem 4, we see that the function $\mathscr{G}_\rho(u)$ defined by (21) is a merit function for the general variational inclusion (1). It is clear that the regularized merit function is differentiable whenever T,g and ϕ are differentiable. We now derive the error bounds without using the Lipschitz continuity of T,g and ϕ.

Theorem 5.*Let $\bar{u}\in\mathscr{H}:g(\bar{u})\in\mathscr{H}$ be a solution of the problem (1). Let T be a strongly anti g-monotone with constant $\alpha>0$. If g is strongly nonexpanding with a constant $\tau>0$, then*

$$\|u-\bar{u}\|\leq\frac{1}{\tau\sqrt{\alpha-\frac{1}{2\rho}}}\sqrt{\mathscr{G}_\rho(u)}, \forall u\in\mathscr{H},\quad \rho>\frac{1}{2\alpha}. \quad (25)$$

*Proof.*From (21), it can be written as

$$\mathscr{G}_\rho(u)\geq\langle Tu,g(\bar{u})-g(u)\rangle-\phi(g(\bar{u}))$$
$$+\phi(g(u))-\frac{1}{2\rho}\parallel g(u)-g(\bar{u})\parallel^2.$$

By using strongly anti g-monotonicity of T and nonexpandicity of g, we have

$$\mathscr{G}_\rho(u)\geq\langle T\bar{u},g(\bar{u})$$
$$-g(u)\rangle+\alpha\tau^2\parallel\bar{u}-u\parallel^2$$
$$-\phi(g(\bar{u}))+\phi(g(u))-\frac{\tau^2}{2\rho}\parallel u-\bar{u}\parallel^2. \quad (26)$$

Since $\bar{u}$ be a solution of general variational inclusion (1), then

$$\langle -T\bar{u},g(v)-g(\bar{u})\rangle+\phi(g(v))-\phi(g(\bar{u}))\geq 0$$
$$\forall v\in\mathscr{H}:g(v)\in\mathscr{H}.$$

Taking $v=u$ in above inequality, we have

$$\langle -T\bar{u},g(u)-g(\bar{u})\rangle+\phi(g(u))-\phi(g(\bar{u}))\geq 0. \quad (27)$$

Combining (26) and (27), we get

$$\mathscr{G}_\rho(u)\geq\alpha\tau^2\parallel u-\bar{u}\parallel^2-\frac{\tau^2}{2\rho}\parallel u-\bar{u}\parallel^2$$
$$=\tau^2(\alpha-\frac{1}{2\rho})\parallel u-\bar{u}\parallel^2,$$

which implies

$$\parallel u-\bar{u}\parallel\leq\frac{1}{\tau\sqrt{\alpha-\frac{1}{2\rho}}}\sqrt{\mathscr{G}_\rho(u)}. \quad (28)$$

This is the required result.

We consider another merit function associated with the problem (1), which can be viewed as a difference of two regularized merit functions. Such type of merit functions were introduced and studied by many authors for solving variational inequalities and complementarity problems, see [10,17,18,26,27,28]. Here we define the D-merit function by a formal difference of the regularized merit function defined by (21) with different parameters. To this end, we consider the following function

$$\mathscr{D}_{\rho,\psi}(u)=\mathscr{G}_\rho-\mathscr{G}_\psi,\quad u\in\mathscr{H}$$

with parameters $\rho>\psi>0$.
Now, $\mathscr{D}$-gap function associated with the general variational inclusion (1) is given by

$$\mathscr{D}_{\rho,\psi}(u)\ =\ max_{v\in\mathscr{H}:g(v)\in\mathscr{H}}\{\langle Tu,g(v)-g(u)\rangle-\phi(g(v))$$
$$+\phi(g(u))-\frac{1}{2\rho}\parallel g(u)-g(v)\parallel^2$$
$$+\frac{1}{2\psi}\parallel g(u)-g(v)\parallel^2\},u\in\mathscr{H},g(u)\in\mathscr{H},$$
$$\rho>\psi>0. \quad (29)$$

The $\mathscr{D}$-gap function defined by (29) can be written as

$$\mathscr{D}_{\rho,\psi}(u)$$
$$=\langle Tu,J_\phi[g(u)+\rho Tu]-J_\phi[g(u)+\psi Tu]\rangle-\phi(J_\phi[g(u)+\rho Tu])$$
$$+\phi(J_\phi[g(u)+\psi Tu])-\frac{1}{2\rho}\parallel g(u)-J_\phi[g(u)+\rho Tu]\parallel^2$$
$$+\frac{1}{2\psi}\parallel g(u)-J_\phi[g(u)+\psi Tu]\parallel^2.$$

Further it can be written as

$$\mathscr{D}_{\rho,\psi}(u)$$
$$=\langle Tu,\mathscr{R}_\psi(u)-\mathscr{R}_\rho(u)\rangle-\phi(J_\phi[g(u)+\rho Tu])$$
$$+\phi(J_\phi[g(u)+\psi Tu])-\frac{1}{2\rho}\parallel\mathscr{R}_\rho(u)\parallel^2+\frac{1}{2\psi}\parallel\mathscr{R}_\psi(u\parallel^2 \quad (30)$$

We now show that the function $\mathscr{D}_{\rho,\psi}(u)$ defined by (29) is a merit function for the general variational inclusion (1).

Theorem 6. *Let $\mathscr{R}_\rho(u)$ is defined by (21), then for all $u \in \mathscr{H} : g(u) \in \mathscr{H}$ and $\rho > \psi > 0$, we have*

$$\frac{1}{2}(\frac{1}{\psi}-\frac{1}{\rho}) \parallel \mathscr{R}_\psi(u) \parallel^2 \leq \parallel \mathscr{D}_{\rho,\psi}(u) \parallel \leq \frac{1}{2}(\frac{1}{\psi}-\frac{1}{\rho}) \parallel \mathscr{R}_\rho(u) \parallel^2. \quad (31)$$

In particular, $D_{\rho,\psi}(u) = 0$, if and only if, $u \in \mathscr{H} : g(u) \in \mathscr{H}$ solves problem (1).

Proof. From the definition of subdifferential, we have

$$\langle -Tu - \frac{1}{\rho}(u - J_\phi[g(u)+\rho Tu]), g(v) - J_\phi[g(u)+\rho Tu]\rangle$$
$$+\phi(g(v)) - g(J_\phi[g(u)+\rho Tu]),$$

substituting $g(v) = J_\phi[g(u)+\psi Tu]$ and using the definition of residuals in above inequality, we have

$$\langle Tu, \mathscr{R}_\psi(u) - \mathscr{R}_\rho(u)\rangle$$
$$\geq \langle \mathscr{R}_\rho(u), \mathscr{R}_\rho(u) - \mathscr{R}_\psi(u)\rangle - \phi(\mathscr{R}_\psi(u)) + \phi(\mathscr{R}_\rho(u)). \quad (32)$$

From (30) and (32), we have

$$D_{\rho,\psi}(u)$$
$$\geq \frac{1}{\rho}\langle \mathscr{R}_\rho(u), \mathscr{R}_\rho(u) - \mathscr{R}_\psi(u)\rangle - \frac{1}{2\rho}\|\mathscr{R}_\rho(u)\|^2 + \frac{1}{2\rho} \parallel \mathscr{R}_\psi(u) \parallel^2$$
$$= \frac{1}{2\psi}\|\mathscr{R}_\psi(u)\|^2 - \frac{1}{2\rho}\|\mathscr{R}_\psi(u)\|^2 - \frac{1}{2\rho}\|\mathscr{R}_\rho(u) - \mathscr{R}_\psi(u)\|^2$$
$$+\frac{1}{\rho}\langle \mathscr{R}_\rho(u) - \mathscr{R}_\psi(u), \mathscr{R}_\rho(u) - \mathscr{R}_\psi(u)\rangle$$
$$= \frac{1}{2}(\frac{1}{\psi}-\frac{1}{\rho})\|\mathscr{R}_\psi(u)\|^2 + \frac{1}{2\rho}\|\mathscr{R}_\rho(u) - \mathscr{R}_\psi(u)\|^2$$
$$\geq \frac{1}{2}(\frac{1}{\psi}-\frac{1}{\rho})\|\mathscr{R}_\psi(u)\|^2,$$

which implies the right most inequality of the required result, that is,

$$D_{\rho,\psi}(u) \geq \frac{1}{2}(\frac{1}{\psi}-\frac{1}{\rho})\|\mathscr{R}_\psi(u)\|^2. \quad (33)$$

In a similar way, for $\psi > 0$, by the definition of subdifferential, substituting $g(u) = J_\phi[g(u)+\psi Tu], g(v) = J_\phi[g(u)+\rho Tu]$, we have

$$\langle Tu, \mathscr{R}_\psi(u) - \mathscr{R}_\rho(u)\rangle$$
$$\leq \frac{1}{\psi}\langle \mathscr{R}_\psi(u), \mathscr{R}_\rho(u) - \mathscr{R}_\psi(u)\rangle - \phi(\mathscr{R}_\psi(u)) + \phi(\mathscr{R}_\rho(u)). \quad (34)$$

From (30) and (34), we have

$$D_{\rho,\psi}(u)$$
$$\leq \frac{1}{\psi}\langle \mathscr{R}_\psi(u), \mathscr{R}_\rho(u) - \mathscr{R}_\psi(u)\rangle - \frac{1}{2\rho}\|\mathscr{R}_\rho(u)\|^2 + \frac{1}{2\psi} \parallel \mathscr{R}_\psi(u) \parallel^2$$
$$= \frac{1}{2\psi}\|\mathscr{R}_\rho(u)\|^2 - \frac{1}{2\rho}\|\mathscr{R}_\rho(u)\|^2 + \frac{1}{2\psi}\|\mathscr{R}_\psi(u) - \mathscr{R}_\rho(u)\|^2$$
$$-\frac{1}{\psi}\langle \mathscr{R}_\psi(u) - \mathscr{R}_\rho(u), \mathscr{R}_\psi(u) - \mathscr{R}_\rho(u)\rangle$$
$$= \frac{1}{2}(\frac{1}{\psi}-\frac{1}{\rho})\|\mathscr{R}_\rho(u)\|^2 - \frac{1}{2\psi}\|\mathscr{R}_\psi(u) - \mathscr{R}_\rho(u)\|^2$$
$$\leq \frac{1}{2}(\frac{1}{\psi}-\frac{1}{\rho})\|\mathscr{R}_\rho(u)\|^2,$$

which implies the left most inequality of the required result, that is,

$$D_{\rho,\psi}(u) \leq \frac{1}{2}(\frac{1}{\psi}-\frac{1}{\rho})\|\mathscr{R}_\rho(u)\|^2. \quad (35)$$

Combining (33) and (37), we have

$$\frac{1}{2}(\frac{1}{\psi}-\frac{1}{\rho}) \parallel \mathscr{R}_\psi(u) \parallel^2 \leq \parallel \mathscr{D}_{\rho,\psi}(u) \parallel \leq \frac{1}{2}(\frac{1}{\psi}-\frac{1}{\rho}) \parallel \mathscr{R}_\rho(u) \parallel^2,$$

which is the required result. The last assertion follows from Theorem 1.

Finally, we derive error bound for general variational inclusion (1).

Theorem 7. *Let $\bar{u} \in \mathscr{H} : g(\bar{u}) \in \mathscr{H}$ be a solution of (1). If the operator T is strongly g-antimonotone with constant $\alpha > 0$ and g is nonexpanding with constant $\tau > 0$, then*

$$\parallel u - \bar{u} \parallel^2$$
$$\leq \frac{1}{\tau\sqrt{\alpha + \frac{1}{2}(\frac{1}{\psi}-\frac{1}{\rho})}}\sqrt{\mathscr{D}_{\rho,\psi}(u)}, \forall u \in \mathscr{H} : g(u) \in \mathscr{H},$$
$$\alpha > \frac{1}{2}(\frac{1}{\rho}-\frac{1}{\psi}). \quad (36)$$

Proof. From (29), it can be written as,

$$\mathscr{D}_{\rho,\psi}(u)$$
$$\geq \langle Tu, g(\bar{u}) - g(u)\rangle - \phi(g(\bar{u})) + \phi(g(u))$$
$$- \frac{1}{2\rho} \parallel g(u) - g(\bar{u}) \parallel^2$$
$$+\frac{1}{2\psi} \parallel g(u) - g(\bar{u}) \parallel^2,$$

using the strongly g-antimonotonicity of operator T with constant $\alpha > 0$, in above inequality, we have

$$\mathscr{D}_{\rho,\psi}(u)$$
$$\geq \langle T\bar{u}, g(\bar{u}) - g(u)\rangle + \alpha \parallel g(u) - g(\bar{u}) \parallel^2$$
$$-\phi(g(\bar{u})) + \phi(g(u))$$
$$-\frac{1}{2\rho} \parallel g(u) - g(\bar{u}) \parallel^2 + \frac{1}{2\psi} \parallel g(u) - g(\bar{u}) \parallel^2. \quad (37)$$

Since $\bar{u} \in \mathscr{H} : g(\bar{u}) \in \mathscr{H}$ be the solution of general variational inclusion (1), we have

$$\langle -T\bar{u}, g(v) - g(\bar{u})\rangle + \phi(g(v)) - \phi(g(\bar{u})) \geq 0,$$
$$\forall v \in \mathscr{H} : g(v) \in \mathscr{H},$$

substituting $v = u$ in above inequality, we have

$$\langle -T\bar{u}, g(u) - g(\bar{u})\rangle + \phi(g(u)) - \phi(g(\bar{u})) \geq 0. \quad (38)$$

From (37) and (38), we have

$$\mathscr{D}_{\rho,\psi}(u)$$
$$\geq \alpha \parallel g(u) - g(\bar{u}) \parallel^2$$
$$-\frac{1}{2\rho} \parallel g(u) - g(\bar{u}) \parallel^2$$
$$+\frac{1}{2\psi} \parallel g(u) - g(\bar{u}) \parallel^2.$$

Using nonexpandicity of g with constant $\tau > 0$, we have

$$\mathscr{D}_{\rho,\psi}(u)$$
$$\geq \tau^2(\alpha + \frac{1}{2}(\frac{1}{\psi}-\frac{1}{\rho})) \parallel u - \bar{u} \parallel^2,$$

which implies

$$\| u - \bar{u} \| \leq \frac{1}{\tau\sqrt{\alpha + \frac{1}{2}(\frac{1}{\psi} - \frac{1}{\rho})}} \sqrt{\mathscr{D}_{\rho,\psi}(u)}. \quad (39)$$

Which completes the proof.

4 conclusion

In this paper we have investigated some merit functions associated with general variational inclusion. We have shown that general variational inclusions are equivalent to fixed point problem. We have used this fixed point formulation to introduce some merit functions such as normal residue vector, regularized merit function and $\mathscr{D}$-merit function. We have shown that the normal residue vector is non-differentiable, while regularized and $\mathscr{D}$-merit functions are differentiable. Using these merit functions, we have derived new error bounds. We have shown that error bounds derived by regularized and $\mathscr{D}$-merit function do not required Lipschitz continuity of operators T, g and ϕ. One can use these error bounds to develop some new iterative methods for solving the variational inclusions and related variational inequalities. This is an interesting problem for future research.

Acknowledgement

The authors are grateful to Dr. S. M. Junaid Zaidi (H.I., S.I.), Rector, COMSATS Institute of Information Technology, Pakistan for providing the excellent research facilities.

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