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On Weakly Regular Fuzzy Ordered Ternary Semigroups

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Abstract: Weakly regular ordered ternary semigroups are characterized in terms of fuzzy left, fuzzy right, fuzzy quasi-ideals and fuzzy bi-ideals, which generalizes and unifies results on weakly regular ordered semigroups and weakly regular ternary semigroups.

Keywords: ordered ternary semigroup, weakly regular, quasi-ideal, bi-ideal, fuzzy ideal

1 Introduction

Since it was introduced by Zadeh [30] in 1965, fuzzy set theory has entered many fields of mathematics, and fuzzy algebraic systems have been developed rapidly. Among others, fuzzy ideals in semigroups were introduced by Kuroki [16] and fuzzy bi-ideals, fuzzy quasi-ideals and fuzzy semiprime ideals of semigroups were studied ([16, 17, 18, 19, 20]). Regular semigroups were characterized in terms of fuzzy ideals (see [24]).

Fuzzy semigroups were generalized in two folds: fuzzy ordered semigroups and fuzzy ternary semigroups. Since ordered semigroups are useful for computer science, especially in theory of automata and formal language, fuzzy ordered semigroup has been extensively studied (see [6, 8, 9, 10, 11]). The notion of ternary semigroups was introduced by S. Banach, and he proved that a ternary semigroup does not necessarily reduce to an ordinary semigroups. Los [23] proved that any ternary semigroup however may be embedded in an ordinary semigroup. Several kinds of regularity of ordered semigroups ternary semigroups are characterized in terms of fuzzy ideals, for example, weakly regular (or quasi-regular) ordered semigroups [13, 14, 25, 27, 28] and regular ternary semigroups [3, 21].

Recently, ordered semigroups and ternary semigroups were extended to ordered ternary semigroups in [1, 3, 4, 5]. Ordered ternary semigroups provide a unified setting for ordered semigroups and ternary semigroups. It is natural to extend studies for regularity of ordered semigroups and ternary semigroups to ordered ternary semigroups. Lekkoksung and Jampachon in [22] proved that an ordered ternary semigroup S is right weakly regular if and only if every fuzzy right ideal of S is idempotent.

In this paper we find that unlike the case of semigroups, a regular ordered ternary semigroup is not necessarily weakly regular and we characterize weakly regular ordered ternary semigroups in terms of fuzzy ideals. Several results in [12, 13, 14, 22, 25, 27, 28, 29] are generalized and new results are found.

2 Preliminaries

By an ordered ternary groupoid $(S,\cdot,\leq)$ we mean a non-empty set S with a ternary operation denoted by juxtaposition and a partial ordering $\leq$ such that $x_1x_2x_3 \leq y_1y_2y_3$ for any x_i, $y_i \in S$ with $x_i \leq y_i$, $i = 1,2,3$, or equivalently $x_1 \leq x_2$ implies

$$x_1x_3x_4 \leq x_2x_3x_4,\ \ x_3x_1x_4 \leq x_3x_2x_4,\ \ x_3x_4x_1 \leq x_3x_4x_2$$

for all $x_1, x_2, x_3, x_4 \in S$. Additionally if $(S,\cdot)$ is a ternary semigroup we say that $(S,\cdot,\leq)$ is an ordered ternary semigroup.

A ternary semigroup can be viewed as an ordered ternary semigroup with the trivial ordering. An ordered semigroup can be viewed as an ordered ternary semigroup with the ternary operation induced by the binary operation in a natural way.

In what follows by subsets we means nonempty ones.

We write A^3 for AAA for any element or subset A of an ordered ternary groupoid.

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Let S be an ordered ternary groupoid. For $A \subseteq S$ write

$$(A] = \{t \in S \mid t \leq a \text{ for some } a \in A\}.$$

Then for any subsets A, B, C of S we have $A \subseteq (A] \subseteq ((A]]$, $(A] \subseteq (B]$ whenever $A \subseteq B$, and $((A](B](C]] = (ABC]$.

A subset A of an ordered ternary groupoid S is called a ternary subgroupoid of S if $(A] \subseteq A$ and $AAA \subseteq A$.

A subset A of an ordered ternary groupoid S is called left (resp. lateral, right) ideal of S, if $(A] = A$ and $SSA \subseteq A$ (resp. $SAS \subseteq A$, $ASS \subseteq A$). A left and right ideal is called a two-sided ideal. A left, lateral and right ideal is called an ideal.

Let S be an ordered ternary semigroup. A subset Q of S is called a quasi-ideal if $(Q] = Q$, $(QSS] \cap (SQS] \cap (SSQ] \subseteq Q$ and $(QSS] \cap (SSQSS] \cap (SSQ] \subseteq Q$. A subsemigroup B of S is called a bi-ideal if $(B] = B$ and $BSBSB \subseteq B$.

We denote by $L(a)$ (resp. $R(a)$, $M(a)$, $T(a)$, $I(a)$, $Q(a)$, $B(a)$) the left (resp. right, lateral, two-sided, ideal, quasi- and bi-) ideal of S generated by $a \in S$. Then we have

$$\begin{aligned}
L(a) &= (a \cup SSa], \\
M(a) &= (a \cup SaS \cup SSaSS], \\
R(a) &= (a \cup aSS], \\
T(a) &= (a \cup aSS \cup SSa \cup SSaSS], \\
I(a) &= (a \cup aSS \cup SaS \cup SSa \cup SSaSS], \\
Q(a) &= (a \cup aSS] \cap (a \cup SaS \cup SSaSS] \cap (a \cup SSa], \\
B(a) &= (a \cup a^3 \cup aSaSa].
\end{aligned}$$

Let S be an ordered ternary groupoid. By a fuzzy subset of S we mean a mapping $f : S \to [0,1]$. For $A \subseteq S$ the fuzzy subset f_A of S is the characteristic function of A defined as follows:

$$f_A : S \to [0,1] \mid x \to f_A(x) = \begin{cases} 1, \text{ if } x \in A; \\ 0, \text{ if } x \notin A. \end{cases}$$

Usually we write $f_S = 1$ and $f_\emptyset = 0$.

We denote by $F(S)$ the set of all fuzzy subsets of S and define an ordered relation $\preceq$ on $F(S)$ as follows:

$$f \preceq g \text{ if and only if } f(x) \leq g(x) \text{ for all } x \in S.$$

It is easy to see that $(F(S), \preceq)$ is a poset with the least element 0 and the greatest element 1. For two fuzzy subset f, g of S we clearly have $f \wedge g = \min\{f, g\}$ and $f \vee g = \max\{f, g\}$.

For $a \in S$, we define

$$A_a = \{(x,y,z) \in S \times S \times S \mid a \leq xyz\}.$$

If S is an ordered ternary semigroup, then $A_x = \emptyset$ for all $(x,y,z) \in A_a$ if and only if $A_y = \emptyset$ for all $(x,y,z) \in A_a$, if and only if $A_z = \emptyset$ for all $(x,y,z) \in A_a$.

For three fuzzy subsets f, g, h of S, we define a fuzzy subset $f \circ g \circ h : S \to [0,1]$ as follows

$$a \to \begin{cases} \bigvee_{(x,y,z) \in A_a} \min\{f(x), g(y), h(z)\}, & \text{if } A_a \neq \emptyset; \\ 0, & \text{if } A_a = \emptyset. \end{cases}$$

One can easily see that the multiplication $\circ$ on $F(S)$ is well defined and $(F(S), \circ)$ is a ternary groupoid with the zero 0, that is,

$$\begin{aligned}
0 \circ f \circ g &= f \circ 0 \circ g = f \circ g \circ 0 \\
&= f \circ 0 \circ 0 = 0 \circ g \circ 0 = 0 \circ 0 \circ h = 0
\end{aligned}$$

for all $f, g, h \in F(S)$.

Clearly, for $f_i, g_i \in F(S)$, $i = 1,2,3$, we have $f_1 \circ f_2 \circ f_3 \preceq g_1 \circ g_2 \circ g_3$ if and only if $(f_1 \circ f_2 \circ f_3)(a) \leq (g_1 \circ g_2 \circ g_3)(a)$ for any $a \in S$ with $A_a \neq \emptyset$. Particularly, $f_1 \circ f_2 \circ f_3 = g_1 \circ g_2 \circ g_3$ if and only if $(f_1 \circ f_2 \circ f_3)(a) = (g_1 \circ g_2 \circ g_3)(a)$ for any $a \in S$ with $A_a \neq \emptyset$.

Definition 1.([1]) *A fuzzy subset f of an ordered ternary semigroup S is called a fuzzy subsemigroup if for all $x, y, z \in S$ we have*

(1)$x \leq y$ implies $f(x) \geq f(y)$;
(2)$f(xyz) \geq \min\{f(x), f(y), f(z)\}$.

Definition 2.([1]) *A fuzzy subset f of an ordered ternary groupoid S is called a fuzzy left (resp. lateral, right) ideal of S if for all $x, y, z \in S$ we have*

(1)$x \leq y$ implies $f(x) \geq f(y)$;
(2)$f(xyz) \geq f(z)$ (resp. $f(xyz) \geq f(y)$, $f(xyz) \geq f(x)$).

A fuzzy left and fuzzy right ideal is called a fuzzy two-sided ideal. A fuzzy left, fuzzy lateral and fuzzy right ideal is called a fuzzy ideal.

Lemma 1.*Let S be an ordered ternary semigroup and f be a fuzzy subset of S. Then $1 \circ 1 \circ f$ (resp. $f \circ 1 \circ 1$, $1 \circ f \circ 1$ and $1 \circ 1 \circ f \circ 1 \circ 1$) is a fuzzy left (resp. right, two-sided) ideal of S.*

*Proof.*By Lemma 4,

$$(1 \circ 1 \circ f)(xyz) \geq \min\{1(x), 1(y), f(z)\}$$

for any $x, y, z \in S$. It follows that

$$\begin{aligned}
(1 \circ 1 \circ f)(xyz) &\geq \bigvee_{x,y,z \in S} \min\{1(x), 1(y), f(z)\} \\
&\geq (1 \circ 1 \circ f)(z).
\end{aligned}$$

For $x, y \in S$ such that $x \leq y$ Lemma 4 yields $(1 \circ 1 \circ f)(y) \leq (1 \circ 1 \circ f)(x)$. Thus $1 \circ 1 \circ f$ is a fuzzy left ideal of S. The rest can be proved similarly.

Lemma 2.([21]) *Let S be an ordered ternary groupoid and f a fuzzy left (resp. right, lateral) ideal of S. Then $1 \circ 1 \circ f \preceq f$ (resp. $f \circ 1 \circ 1 \preceq f$, $1 \circ f \circ 1 \preceq f$).*

Definition 3.([21]) *Let S be an ordered ternary semigroup. A fuzzy subset f of S is called a fuzzy quasi-ideal of S if*

(1)$x \leq y$ implies $f(x) \geq f(y)$ for any $x, y \in S$;
(2)$(f \circ 1 \circ 1) \wedge (1 \circ f \circ 1) \wedge (1 \circ 1 \circ f) \preceq f$;
(3)$(f \circ 1 \circ 1) \wedge (1 \circ 1 \circ f \circ 1 \circ 1) \wedge (1 \circ 1 \circ f) \preceq f$.

Definition 4.([21]) *Let S be an ordered ternary semigroup. A fuzzy subsemigroup f of S is called a fuzzy bi-ideal of S if for any $p,q,x,y,z \in S$*

(1) $x \leq y$ implies $f(x) \geq f(y)$;
(2) $f(xpyqz) \geq \min\{f(x), f(y), f(z)\}$.

Let S be an ordered ternary semigroup. It is proved in [1] that a subset A of S is a left (resp. right, lateral, two-sided, quasi-, bi-) ideal of S if and only if the characteristic function f_A of A is a fuzzy left (resp. right, lateral, two-sided, quasi-, bi-) ideal of S.

Lemma 3.([7]) *In an ordered ternary semigroup, (fuzzy) one-sided ideals of S are (fuzzy) quasi-ideals, and (fuzzy) quasi-ideals are (fuzzy) bi-ideals.*

3 The fuzzy sets of ordered ternary semigroups

The following theorem was proved in [11] for ordered semigroups, and for ordered ternary semigroups it has been implicitly used by several authors. However, we can not find a proof in the literature. Here we present a proof, which is certainly nontrivial.

Theorem 1. *If S be an ordered ternary groupoid, then $(F(S), \circ, \preceq)$ is an ordered ternary groupoid. If S is additionally an ordered ternary semigroup, then $(F(S), \circ, \preceq)$ is also an ordered ternary semigroup.*

Proof. To prove $(F(S), \circ, \preceq)$ is an ordered ternary groupoid, it is sufficient to prove $f_1 \circ f_2 \circ f_3 \preceq g_1 \circ g_2 \circ g_3$ for all $f_i, g_i \in F(S)$ with $f_i \preceq g_i$ for $i = 1,2,3$. In fact, for $a \in S$ with $A_a \neq \emptyset$ we have

$$\begin{aligned}(f_1 \circ f_2 \circ f_3)(a) &= \bigvee_{(x,y,z)\in A_a} \min\{f_1(x), f_2(y), f_3(z)\} \\ &\leq \bigvee_{(x,y,z)\in A_a} \min\{g_1(x), g_2(y), g_3(z)\} \\ &= (g_1 \circ g_2 \circ g_3)(a).\end{aligned}$$

Thus $f_1 \circ f_2 \circ f_3 \preceq g_1 \circ g_2 \circ g_3$. Now suppose that S is an ordered ternary semigroup. To prove $(F(S), \circ, \preceq)$ is also an ordered ternary semigroup, it is enough to check the associative law:

$$\begin{aligned}(f_1 \circ f_2 \circ f_3) \circ f_4 \circ f_5 &= f_1 \circ (f_2 \circ f_3 \circ f_4) \circ f_5 \\ &= f_1 \circ f_2 \circ (f_3 \circ f_4 \circ f_5)\end{aligned}$$

for all $f_1, f_2, f_3, f_4, f_5 \in F(S)$. We only verify the first equality. Indeed, for $a \in S$ with $A_a \neq \emptyset$,

$$\begin{aligned}&[(f_1 \circ f_2 \circ f_3) \circ f_4 \circ f_5](a) \\ &\quad = \bigvee_{(x,y,z)\in A_a} (f_1 \circ f_2 \circ f_3)(x) \wedge f_4(y) \wedge f_5(z),\end{aligned}$$

$$\begin{aligned}&[f_1 \circ (f_2 \circ f_3 \circ f_4) \circ f_5](a) \\ &\quad = \bigvee_{(p,u,z)\in A_a} f_1(p) \wedge (f_2 \circ f_3 \circ f_4)(u) \wedge f_5(z).\end{aligned}$$

We assume, without loss of generality, that $A_x \neq \emptyset$ for some $(x,y,z) \in A_a$ and $A_u \neq \emptyset$ for some $(p,u,z) \in A_a$, since $A_x = \emptyset$ for any $(x,y,z) \in A_a$ if and only if $A_u = \emptyset$ for any $(p,u,z) \in A_a$. Then

$$\begin{aligned}&[(f_1 \circ f_2 \circ f_3) \circ f_4 \circ f_5](a) \\ &= \bigvee_{(x,y,z)\in A_a} (f_1 \circ f_2 \circ f_3)(x) \wedge f_4(y) \wedge f_5(z) \\ &= \bigvee_{(x,y,z)\in A_a} \bigvee_{(p,q,r)\in A_x} f_1(p) \wedge f_2(q) \wedge f_3(r) \wedge f_4(y) \wedge f_5(z) \\ &= \bigvee_{(p,q,r,y,z)\in A_1} f_1(p) \wedge f_2(q) \wedge f_3(r) \wedge f_4(y) \wedge f_5(z),\end{aligned}$$

where $A_1 = \{(p,q,r,y,z) \mid (p,q,r) \in A_x \text{ and } (x,y,z) \in A_a \text{ for some } x \in S\}$. Similarly,

$$\begin{aligned}&[f_1 \circ (f_2 \circ f_3 \circ f_4) \circ f_5](a) \\ &\quad = \bigvee_{(p,q,r,y,z)\in A_2} f_1(p) \wedge f_2(q) \circ f_3(r) \circ f_4(y) \wedge f_5(z),\end{aligned}$$

where $A_2 = \{(p,q,r,y,z) \mid (q,r,y) \in A_u \text{ and } (p,u,z) \in A_a \text{ for some } u \in S\}$. It suffices to prove $A_1 = A_2$. Suppose that $(p,q,r,y,z) \in A_1$. Then $x \leq pqr$ and $a \leq xyz$ for some $x \in S$. Taking $u = qry$, we have $a \leq xyz \leq pqryz = puz$ and $u \leq qry$. It follows that $(q,r,y) \in A_u$ and $(p,u,z) \in A_a$. Thus $(p,q,r,y,z) \in A_2$, and so $A_1 \subseteq A_2$. Similarly, $A_2 \subseteq A_1$. Therefore, $A_1 = A_2$, as desired.

Theorem 1 is so basic that we will freely use it without explicit mention.

Lemma 4. *Let S be an ordered ternary semigroup. Then for any $f_i, g_i \in F(S)$ and $x_i \in S$, $i = 1,2,\ldots,2n+1$, we have*

(1) $f_1 \circ f_2 \circ \cdots \circ f_{2n+1} \preceq g_1 \circ g_2 \circ \cdots \circ g_{2n+1}$ if and only if $(f_1 \circ f_2 \circ \cdots \circ f_{2n+1})(a) \leq (g_1 \circ g_2 \circ \cdots \circ g_{2n+1})(a)$ for any $a \in S$ with $A_a \neq \emptyset$;
(2) $(f_1 \circ \cdots \circ f_{2n+1})(a) \geq \min\{f_1(x_1), \ldots, f_{2n+1}(x_{2n+1})\}$ for any $a \in S$ with $a \leq x_1x_2\cdots x_{2n+1}$;
(3) $(f_1 \circ f_2 \circ \cdots \circ f_{2n+1})(a) \geq (f_1 \circ f_2 \circ \cdots \circ f_{2n+1})(b)$ for any $a,b \in S$ with $a \leq b$.

Proof. (1) It follows immediately from the definition of the ternary operation $\circ$ by noting that there is nothing to prove when A_a is empty.

(2) We only check the case $n = 1$. The general case can be proved by induction on n. If $a \leq xyz$, then $(x,y,z) \in A_a$, and so

$$\begin{aligned}(f_1 \circ f_2 \circ f_3)(a) &= \bigvee_{(x,y,z)\in A_a} \min\{f_1(x), f_2(y), f_3(z)\} \\ &\geq \min\{f_1(x), f_2(y), f_3(z)\}.\end{aligned}$$

(3) We only prove the case $n=1$. The general case can be proved by induction on n. Given $a,b\in S$ with $a\leq b$, note that $A_b\subseteq A_a$. If $A_b=\emptyset$ then $(f_1\circ f_2\circ f_3)(b)=0\leq (f_1\circ f_2\circ f_3)(a)$. If $A_b\neq\emptyset$ then $A_a\neq\emptyset$ and so

$$\begin{aligned}(f_1\circ f_2\circ f_3)(b) &= \bigvee_{(x,y,z)\in A_b}\min\{f_1(x),f_2(y),f_3(z)\}\\ &\leq \bigvee_{(x,y,z)\in A_a}\min\{f_1(x),f_2(y),f_3(z)\}\\ &=(f_1\circ f_2\circ f_3)(a),\end{aligned}$$

as desired.

Lemma 5. *Let S be an ordered ternary semigroup. Then*

$$f_A\circ f_B\circ f_C=f_{(ABC]}$$

for any subsets A,B,C of S.

Proof. For $p\in(ABC]$, $f_{(ABC]}(p)=1$. Since $p\leq abc$ for some $a\in A$, $b\in B$ and $c\in C$. Then by Lemma 4,

$$1\geq (f_A\circ f_B\circ f_C)(p)\geq \min\{f_A(a),f_B(b),f_C(c)\}\geq 1.$$

Therefore, $(f_A\circ f_B\circ f_C)(p)=f_{(ABC]}(p)$. For $p\notin(ABC]$, $f_{(ABC]}(p)=0$. If $A_p=\emptyset$, then $(f_A\circ f_B\circ f_C)(p)=0$ and $(f_A\circ f_B\circ f_C)(p)=f_{(ABC]}(p)$. If $A_p\neq\emptyset$, then $xyz\notin ABC$ for any $(x,y,z)\in A_p$, which implies that $x\notin A$ or $y\notin B$ or $z\notin C$, and so $\min\{f_A(x),f_B(y),f_C(z)\}=0$ and

$$\begin{aligned}(f_A\circ f_B\circ f_C)(p) &= \bigvee_{(x,y,z)\in A_p}\min\{f_A(x),f_B(y),f_C(z)\}\\ &=0\\ &=f_{(ABC]}(p).\end{aligned}$$

Lemma 5 was proved for ordered semigroups in [12].

In the rest of this section we suppose S is an ordered semigroup. A binary operation $\circ$ over $F(S)$ has been defined, which we denote by $*$ to avoid causing confusion. It was proved that $(F(S),*,\preceq)$ is an ordered semigroup in [11]. For any $a\in S$, let

$$B_a=\{(x,y)\in S\times S\mid a\leq xy\}.$$

For $f,g\in F(S)$, $f*g$ is defined as follows

$$f*g:S\to[0,1]\mid a\to\begin{cases}\bigvee_{(x,y)\in B_a} f(x)\wedge g(y), & \text{if } B_a\neq\emptyset;\\ 0, & \text{if } B_a=\emptyset.\end{cases}$$

The following lemma looks like a trivial fact, but its proof is nontrivial.

Theorem 2. *Let S be an ordered semigroup. If S is viewed as an ordered ternary semigroup in the natural way, then the ternary operation $\circ$ of $F(S)$ is induced by the binary operation $*$ of $F(S)$.*

Proof. Suppose $a\in S$ is such that $A_a=\emptyset$. Then $(f\circ g\circ h)(a)=0$. If $B_a=\emptyset$, then $(f*g*h)(a)=0$; if $B_a\neq\emptyset$, then $B_x=\emptyset$ for all $(x,y)\in B_a$ and so

$$(f*g*h)(a)=\bigvee_{(x,y)\in B_a}(f*g)(x)\wedge h(y)=0.$$

Thus $(f*g*h)(a)=f\circ g\circ h)(a)$. Suppose $a\in S$ is such that $A_a\neq\emptyset$. Then $B_a\neq\emptyset$ and $B_x\neq\emptyset$ for some $(x,y)\in B_a$. Thus

$$\begin{aligned}(f*g*h)(a) &= \bigvee_{(x,y)\in B_a}(f*g)(x)\wedge h(y)\\ &=\bigvee_{(x,y)\in B_a}\bigvee_{(p,q)\in B_x} f(p)\wedge g(q)\wedge h(y)\\ &=\bigvee_{(p,q,y)\in A} f(p)\wedge g(q)\wedge h(y),\end{aligned}$$

where $A=\{(p,q,y)\mid (p,q)\in B_x$ and $(x,y)\in B_a$ for some $x\in S\}$. Clearly, $A\subseteq A_a$. Conversely, for $(p,q,y)\in A_a$, set $x=pq$. Then $a\leq pqy\leq xy$ and $x\leq pq$, and so $(p,q)\in B_x$ and $(x,y)\in B_a$. Thus $(p,q,y)\in A$. Therefore, $A=A_a$ and so $(f*g*h)(a)=(f\circ g\circ h)(a)$. Consequently, $f*g*h=f\circ g\circ h$.

We now remark that if $(S^3]=S$, then left ideals, right ideals and ideals of the ordered semigroup S are the same as left ideals, right ideals and two-sided ideals of the ordered ternary semigroup S, respectively. The same conclusions hold for fuzzy ideals.

The last theorem and remark will ensure that our discussion on weakly regular ordered ternary semigroups includes weakly regular ordered semigroups as a special case.

4 Characterizations of left weakly regular ordered ternary semigroup in terms of fuzzy ideals

An ordered ternary semigroup S is called regular if $a\in(aSa]$ for any $a\in S$. This is a natural analogue of regularity of semigroups in the setting of ordered ternary semigroups.

An ordered semigroup S is called left weakly regular (or left quasi-regular ([14])) if $a\in(SaSa]$ for all $a\in S$. A regular ordered semigroup is weakly regular.

Definition 5. *([22]) An ordered ternary semigroup S is called left weakly regular if its every element a is left weakly regular in the sense that $a\in((SSa)^3]$. Right weak regularity is defined similarly. If S is both left and right weakly regular then it is called weakly regular.*

It is easy to see that ordered ternary semigroup S is left weakly regular if and only if $A\subseteq((SSA)^3]$, where A is any subset (resp. left ideal, quasi-ideal and bi-ideal).

If an ordered ternary semigroup S is commutative and weakly regular, then S is regular. In general, either of regularity and weak regularity of ordered ternary semigroups do not imply another. Motivated by [2] we have the following example.

*Example 1.*Let T be the multiplicative semigroup of the direct product $\mathbf{R}\times\mathbf{R}$ of the real number field $\mathbf{R}$, and let $S=T\times T$. For any $(a_1,a_2),(b_1,b_2),(c_1,c_2)\in S$ we define

$$(a_1,a_2)(b_1,b_2)(c_1,c_2)=(a_1b_2c_1,a_2b_1c_2).$$

Then S is a regular ternary semigroup but not left weakly regular. Actually, it is a routine matter to verify that S is a ternary semigroup and is regular. Now take $a=(e_1,e_2)\in S$, where $e_1=(1,0)$ and $e_2=(0,1)$. Then for any $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in S$, we have

$$\begin{aligned}(x_1,x_2)&(e_1,e_2)(y_1,y_2)(z_1,z_2)(e_1,e_2)\\&=(x_1e_2y_1z_2e_1,x_2e_1y_2z_1e_2)=(0,0).\end{aligned}$$

Thus a is not left weakly regular.

It is easy to see that an ordered semigroup S is left weakly regular if and only if it is left weakly regular as an ordered ternary semigroup.

Let S be an ordered ternary semigroup. An element a of S is called left (resp. right) regular if $a\in(Saa]$ (resp. $a\in(aaS]$).

Lemma 6.*A left (resp. right) regular element a of an ordered ternary semigroup is left (resp. right) weakly regular.*

*Proof.*Let S be an ordered ternary semigroup and a be a left regular element of S. Then $a\le xaa$ for some $x\in S$. It follows that $a\le xaa\le xa(xaa)\le x(xaa)(xaa)=(xxa)(axa)a\le(xxa)(axa)(xaa)$. Thus $a\in((SSa)^3]$.

Lemma 7.*Let S be an ordered ternary semigroup. For $a\in S$ the following are equivalent:*

(1)a is left weakly regular;
(2)$a\in(SaSSa]$;
(3)$a\in(ST(a)L(a)]$.

*Proof.*Since $((SSa)^3]\subseteq(SaSSa]\subseteq(ST(a)L(a)]$, it suffices to prove $(3)\Rightarrow(1)$. To do this, note that

$$\begin{aligned}(ST(a)L(a)]&=(S(a\cup aSS\cup SSa\cup SSaSS)(a\cup SSa)]\\&\subseteq(Saa\cup SaSSa].\end{aligned}$$

Thus $a\in(Saa]$ or $a\in(SaSSa]$. If $a\in(Saa]$ then a is left weakly regular by Lemma 6. If $a\in(SaSSa]$ then $a\le xayza$ for some $x,y,z\in S$. So

$$a\le xayza=x(xayza)yza=(xxa)(yza)(yza),$$

which implies that $a\in((SSa)^3]$. Hence a is left weakly regular.

Lemma 8.*Let S be an ordered ternary semigroup and $a\in S$. If $a\in(SaS]$, then*

$$M(a)=T(a)=I(a)=(SaS]=(SSaSS].$$

*Proof.*Since $(SaS]$ is a two-sided ideal, we have $(SSa]\cup(aSS]\subseteq(SaS]$ and so $(SSaSS]\subseteq(SaS]$. By $a\in(SaS]$ we have $(SaS]\subseteq(SSaSS]$. Thus $(SaS]=(SSaSS]$. Now by the definitions we can see that $M(a)=T(a)=I(a)=(SaS]$.

Lemma 9.*Let S be an ordered ternary semigroup. If S is left weakly regular then $a\in(SSa]\subseteq(SaS]=(SSaSS]$ for any $a\in S$. Moreover, a similar conclusion holds if S is right weakly regular.*

*Proof.*Let S be a left weakly regular and $a\in S$. Then $a\in(SaSSa]\subseteq(SaS]$, whence $(SaS]=(SSaSS]$ by Lemma 8 and $(SSa]\subseteq(SaS]$ since $(SaS]$ is a left ideal.

*Remark.*Lemma 8 and Lemma 9 show that lateral ideals, two-sided ideals and ideals are the same in left weakly regular ordered ternary semigroups.

Recall that a subset A of an ordered ternary semigroup is called idempotent if $A=(A^3]$.

A fuzzy subset f of an ordered ternary semigroup is called idempotent if $f\circ f\circ f=f$.

Lemma 10.([22]) *An ordered ternary semigroup S is left (resp. right) weakly regular if and only if every fuzzy left (resp. right) ideal f is idempotent.*

Lemma 11.*Let S be an ordered ternary semigroups and $\mathscr{A}$ be a collection of subsets of S containing all left ideals of S. Then the following are equivalent:*

(1)S is left weakly regular;
(2)$J\cap K\cap L\subseteq(JKL]$ for any left ideals J,K,L;
(3)$L\cap T\cap A\subseteq(LTA]$ for any left ideal L and two-sided ideal T, and for any $A\in\mathscr{A}$;
(4)$T\cap R\cap A\subseteq(TRA]$ for any right ideal R and two-sided ideal T, and for any $A\in\mathscr{A}$;
(5)$T\cap A\cap L\subseteq(TAL]$ for any left ideal L and two-sided ideal T, and for any $A\in\mathscr{A}$;
(6)$T\cap A\subseteq(STA]$ for any two-sided ideal T and for any $A\in\mathscr{A}$;
(7)$R\cap A\subseteq(SRA]$ for any right ideal R and for any $A\in\mathscr{A}$;
(8)$A\cap L\subseteq(SAL]$ for any left ideals L and for any $A\in\mathscr{A}$.

Proof.$(1)\Leftrightarrow(2)$ was proved in [22].

To complete the remainder of the proof, we note that (6) through (8) are special cases $T=S$ of (3) through (5), respectively. To prove (1) implying (3) through (8), it is enough to prove (1) implying (3) through (5) for the case where $\mathscr{A}$ is the collection of all subsets of S.

$(1)\Rightarrow(3)$ For any $a\in L\cap T\cap A$, we have

$$a\in((SSa)(SSaSS)a]\subset(LTA].$$

$(1)\Rightarrow(4)$ For any $a\in T\cap R\cap A$, we have

$$a\in((SSaSS)(aSS)a]\subset(TRA].$$

$(1) \Rightarrow (5)$ For any $a \in T \cap A \cap L$, we have

$$a \in ((SSaSS)a(SSa)] \subset (TAL].$$

To prove (3)-(8) implying (1), it is enough to prove (6) implying (1) for the case where $\mathscr{A}$ is the collection of all left ideals of S, since (6) in this case is the weakest one among all conditions.

$(6) \Rightarrow (1)$ For any $a \in S$, we have $a \in T(a) \cap L(a) \subseteq (ST(a)L(a)]$, from which it follows that a is left weakly regular by Lemma 7.

Lemma 12. *Let S be an ordered ternary semigroup. If S is left weakly regular then*

$$f \preceq 1 \circ 1 \circ f \preceq 1 \circ f \circ 1 = 1 \circ 1 \circ f \circ 1 \circ 1$$

for any fuzzy subset f of S. In addition, a similar conclusion holds if S is right weakly regular.

Proof. For $a \in S$, we have $a \le s_1 a s_2 s_3 a$ for some $s_1, s_2, s_3 \in S$ by Lemma 9. Thus by Lemma 4,

$$(1 \circ 1 \circ f)(a) \ge \min\{1(s_1 a s_2), 1(s_3), f(a)\} = f(a),$$

whence $f \preceq 1 \circ 1 \circ f$. Applying Lemma 10 to the fuzzy left ideal $1 \circ 1 \circ f$ we get $1 \circ 1 \circ f = (1 \circ 1 \circ f \circ 1 \circ 1) \circ f \circ (1 \circ 1 \circ f) \preceq 1 \circ f \circ 1$. Note that $1 \circ f \circ 1$ is also a fuzzy left ideal by Lemma 1 and it is idempotent by Lemma 10 . Thus $1 \circ f \circ 1 = (1 \circ f \circ 1) \circ 1 \circ f \circ 1 \circ (1 \circ f \circ 1) \preceq 1 \circ 1 \circ f \circ 1 \circ 1$, from which it follows that $1 \circ 1 \circ f \circ 1 \circ 1 \preceq 1 \circ 1 \circ 1 \circ f \circ 1 \circ 1 \circ 1 \preceq 1 \circ f \circ 1$. Hence $1 \circ f \circ 1 = 1 \circ 1 \circ f \circ 1 \circ 1$.

Theorem 3. *Let S be an ordered ternary semigroup and $\mathscr{U}$ be a collection of fuzzy subsets of S containing all fuzzy left ideals of S. Then the following are equivalent.*

(1) S is left weakly regular;
(2) $f \wedge f' \wedge f'' \preceq f \circ f' \circ f''$ for any fuzzy left ideals f, f', f'';
(3) $f \wedge g \wedge u \preceq f \circ g \circ u$ for any fuzzy left ideal f and fuzzy two-sided ideal g, and for any $u \in \mathscr{U}$;
(4) $g \wedge h \wedge u \preceq g \circ h \circ u$ for any fuzzy two-sided ideal g and fuzzy right ideal h, and for any $u \in \mathscr{U}$;
(5) $g \wedge u \wedge f \preceq g \circ u \circ f$ for any fuzzy two-sided ideal g and fuzzy left ideal f, and for any $u \in \mathscr{U}$;
(6) $g \wedge u \preceq 1 \circ g \circ u$ for any fuzzy two-sided ideal g and for any $u \in \mathscr{U}$;
(7) $h \wedge u \preceq 1 \circ h \circ u$ for any fuzzy right ideal h and for any $u \in \mathscr{U}$;
(8) $u \wedge f \preceq 1 \circ u \circ f$ for any fuzzy left ideals f and for any $u \in \mathscr{U}$.

Proof. $(1) \Leftrightarrow (2)$ are proved in [22].

To prove (1) implying (3) through (8), it suffices to consider the case where $\mathscr{U}$ is the fuzzy subsets of S.

$(1) \Rightarrow (3)$ For $a \in S$ there exist $s_1, s_2, s_3, t_1, t_2, t_3 \in S$ such that $a \le (s_1 t_1 a)(s_2 t_2 a s_3 t_3)a$. Thus by Lemma 4 and the definitions of fuzzy ideals

$$\begin{aligned}(f \circ g \circ u)(a) &\ge \min\{f(s_1 t_1 a), g(s_2 t_2 a s_3 t_3), u(a)\} \\ &\ge \min(f(a), g(a), u(a)) = (f \wedge g \wedge u)(a).\end{aligned}$$

Hence we have $f \wedge g \wedge u \le f \circ g \circ u$.

$(1) \Rightarrow (4)$ and $(1) \Rightarrow (5)$ can be proved similarly, and (6) through (8) are special cases of (3) through (5), respectively. (3)-(8) $\Rightarrow$ (1) follow by considering the characteristic functions and using Lemma 11.

Remark. If S is an ordered semigroup. Then every condition in Theorem 3 implies $u = 1 \circ u = 1 \circ 1 \circ u$ for any fuzzy left ideal u of S. Thus Theorem 3 generalizes corresponding results in [13,26,27,28,29].

Corollary 1. *An ordered ternary semigroup S is left weakly regular if and only if $g \wedge f = 1 \circ g \circ f$ for any fuzzy two-sided ideal g and fuzzy left ideal f.*

5 Characterizations of weakly regular ordered ternary semigroups in terms of fuzzy ideals

Lemma 13. *Let S be a weakly regular ordered ternary semigroup. Then a subset Q of S is a quasi-ideal if and only if $Q = (SSQ] \cap (QSS]$.*

Proof. The sufficiency is clear. Now we prove the necessity. Suppose that Q is a quasi-ideal of S. By Lemma 9, we have $Q \subseteq (SSQ]$ by left weak regularity of S and $Q \subseteq (QSS]$ by right weak regularity of S. Thus $Q \subseteq (SSQ] \cap (QSS]$. On the other hand by Lemma 9 again we have $(SSQ] \cap (QSS] = (SSQ] \cap (SSQSS] \cap (QSS] \subseteq Q$. Hence $Q = (SSQ] \cap (QSS]$.

Theorem 4. *Let S be a weakly regular ordered ternary semigroup. Then a fuzzy subset f of S is a fuzzy quasi-ideal of S if and only if $f = (1 \circ 1 \circ f) \wedge (f \circ 1 \circ 1)$.*

Proof. The sufficiency is clear. Now we prove the necessity. Let f be a fuzzy quasi-ideal of S. By Lemma 1, $1 \circ 1 \circ f$ is a fuzzy left ideal of S and $f \circ 1 \circ 1$ is a fuzzy right ideal of S. By Lemma 10 , $1 \circ 1 \circ f = (1 \circ 1 \circ f \circ 1 \circ 1) \circ f \circ (1 \circ 1 \circ f) \preceq 1 \circ f \circ 1$. It follows that $(1 \circ 1 \circ f) \wedge (f \circ 1 \circ 1) = (1 \circ 1 \circ f) \wedge (1 \circ f \circ 1) \wedge (f \circ 1 \circ 1) \preceq f$. On the other hand, for any a in S, there exist s_i, t_i in S for $i = 1, 2, 3$ such that $a \le (s_1 t_1 a)(s_2 t_2 a)(s_3 t_3 a)$. By Lemma 4, $(1 \circ 1 \circ f)(a) \ge 1(s_1 t_1 a) \wedge 1(s_2 t_2 a s_3 t_3) \wedge f(a) = f(a)$. Hence $f \preceq 1 \circ 1 \circ f$. Similarly we can prove that $f \preceq f \circ 1 \circ 1$. Thus $f \preceq (1 \circ 1 \circ f) \wedge (f \circ 1 \circ 1)$. Hence the result is proved.

Remark. Theorem 4 shows that every fuzzy quasi-ideal is the intersection of a fuzzy left ideal and a fuzzy right ideal in a weakly regular ordered ternary semigroup. Particularly, that is true for ordinary ideals as shown in Lemma 13.

Lemma 14. *Let S be an ordered ternary semigroup. Then the following are equivalent.*

(1) S is weakly regular;
(2) For any quasi-ideal Q we have

$$Q = ((SSQ)^3] \cap ((QSS)^3];$$

(3)For every quasi-ideal Q we have

$$Q = ((QSS)^3] \cap ((SQS)^3] \cap ((SSQ)^3]$$
$$= ((QSS)^3] \cap ((SSQSS)^3] \cap ((SSQ)^3].$$

Proof.(1) ⇒ (2) By Lemma 13 and Lemma 10 we have $Q = (SSQ] \cap (QSS] = ((SSQ)^3] \cap ((QSS)^3]$ for any quasi-ideal Q.

(2) ⇒ (3) By Lemma 9,

$$(QSS] \cup (SSQ] \subseteq (SQS] = (SSQSS].$$

Then by (2),

$$((QSS)^3] \cap ((SQS)^3] \cap ((SSQ)^3] = ((QSS)^3] \cap ((SSQ)^3] = Q$$

and

$$((QSS)^3] \cap ((SSQSS)^3] \cap ((SSQ)^3]$$
$$= ((QSS)^3] \cap ((SSQ)^3] = Q.$$

(3) ⇒ (1) Let L be a left ideal of S. Then L is a quasi-ideal of S by Lemma 3, and so $L = ((LSS)^3] \cap ((SLS)^3] \cap ((SSL)^3] \subseteq ((SSL)^3] \subseteq (L^3] \subseteq (L] = L$, implying $L = (L^3]$. Hence S is left weakly regular by Lemma 10 . Similarly we can prove that S is right weakly regular.

Theorem 5.*An ordered ternary semigroup S is weakly regular if and only if $f = (1 \circ 1 \circ f)^3 \wedge (f \circ 1 \circ 1)^3$ for any fuzzy quasi-ideal f of S.*

Proof.(⇒) Let f be a fuzzy quasi-ideal of S. By Lemma 1 $1 \circ 1 \circ f$ is a fuzzy left ideal of S and $f \circ 1 \circ 1$ is a fuzzy right ideal of S. By Lemma 10, $1 \circ 1 \circ f$ and $f \circ 1 \circ 1$ are idempotent. Hence by Theorem 4, $f = (1 \circ 1 \circ f) \wedge (f \circ 1 \circ 1) = (1 \circ 1 \circ f)^3 \wedge (f \circ 1 \circ 1)^3$.

(⇐) Let f be a fuzzy left ideal of S. By Lemma 3, f is a fuzzy quasi-ideal. So by the supposition we have $f = (f \circ 1 \circ 1)^3 \wedge (1 \circ 1 \circ f)^3 \preceq (1 \circ 1 \circ f)^3 \preceq f \circ f \circ f \preceq f$. Hence $f = f \circ f \circ f$. By Lemma 10 S is left weakly regular. Similarly S is right weakly regular.

Theorem 6.*An ordered ternary semigroup S is weakly regular if and only if*

$$f = (f \circ 1 \circ 1)^3 \wedge (1 \circ f \circ 1)^3 \wedge (1 \circ 1 \circ f)^3$$
$$= (f \circ 1 \circ 1)^3 \wedge (1 \circ 1 \circ f \circ 1 \circ 1)^3 \wedge (1 \circ 1 \circ f)^3$$

for every fuzzy quasi-ideal f of S.

*Proof.*The sufficiency follows by considering characteristic functions and using Lemma 14. We now prove the necessity. By Lemma 12,

$$(f \circ 1 \circ 1) \vee (1 \circ 1 \circ f) \preceq 1 \circ f \circ 1 = 1 \circ 1 \circ f \circ 1 \circ 1.$$

It follows from Theorem 5 that $(f \circ 1 \circ 1)^3 \wedge (1 \circ f \circ 1)^3 \wedge (1 \circ 1 \circ f)^3 = (f \circ 1 \circ 1)^3 \wedge (1 \circ 1 \circ f)^3 = f$ and $(f \circ 1 \circ 1)^3 \wedge (1 \circ 1 \circ f \circ 1 \circ 1)^3 \wedge (1 \circ 1 \circ f)^3 = (f \circ 1 \circ 1)^3 \wedge (1 \circ 1 \circ f)^3 = f$.

*Remark.*Let S be an ordered semigroup and f be a fuzzy left ideal of S. Then $f \circ f \circ f \preceq f \circ f \preceq f$. Thus $f = f \circ f \circ f$ if and only if $f = f \circ f$, and so Lemma 10 generalizes corresponding results on ordered semigroups ([15,29]).

6 Conclusion

Ternary ordered semigroups are a unified extension of ordered semigroups and ternary semigroups. Like the case of ordered semigroups, the fuzzy subsets of an ordered ternary semigroup is proved to be an ordered ternary semigroup. Relationship between the fuzzy sets of weakly regular ordered semigroups and its ordered ternary semigroups is established. Technique of fuzzification is applied to ordered ternary semigroups to characterize weak regularity, which generalizes and unifies results of ordered semigroups and ternary semigroups. Unlike the case of ordered semigroups, either of regularity and weak regularity for ordered ternary semigroups does not imply another.

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