Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100634

Counting Singularities on Rational Septic Curves

Mohammed A. Saleem *

Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt.

Received: 2 April 2016, Revised: 20 June 2016, Accepted: 25 June 2016
Published online: 1 Nov. 2016

Abstract: In this paper, we classify rational plane septic curves. Moreover, new examples and a complete list of rational irreducible projective plane curves of type (7,4,1) are given. Furthermore, we proved that such curves are transformable into a line by means of Cremona transformations.

Keywords: Rational curves, Septic curves, Singularities of plane curves.

1 Introduction

Throughout this paper, we denote by $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ the projective plane over the field of the complex numbers. Let $C \subset \mathbb{P}^2$ be a plane curve of degree d. The classification of plane algebraic curves for a given degree d is one of the classical and interesting problems in algebraic geometry. For curves $C \subset \mathbb{P}^2$ there is a very important geometric invariant associated to these curves which ic called the genus of C can be computed as (see for instant [1] page 614 or [2] page 222):

$$g = \frac{(d-1)(d-2)}{2} - \sum_{P \in \mathrm{Sing}(C)} \frac{m_P(m_P-1)}{2},$$

where $\mathrm{Sing}(C)$ is the set of all singular points P of the curve C including the infinitely near point of the point P and m_P denotes the multiplicity of $P \in C$. In fact, g plays a very important role in the problem of classification of algebraic curves. For example, plane algebraic curves are called rational curves when $g = 0$. In case $g = 1$, 2, C are called elliptic and hyperelliptic curves, respectively. Also, by the genus formula, we easily see that, the lines and the conics have no singular points and an irreducible cubic has at most one double point. Curves of degrees $d = 4$, 5, 6 and 7 are called quartic, quintic, sextic and septic curves, respectively. Some of these types of curves whose singular points are only cusps and with small degrees are classified by Yoshihara in [3,4,5]. In this paper, we focus on irreducible rational projective plane septic curves.

As a convension, we use the notation (d, ν, ι) for curves of degree d, maximal multiplicity of the singularities ν and $\iota = \iota(C) = \sum_{P \in \mathrm{Sing}(C)}(r_P - 1)$, where r_P is the number of the branches of C at the singular point $P \in C$. It is known that a cusp is aunibranch singular point, i.e., $r_P = 1$. In case $r_P \geq 2$, Saleem in [6], introduced the notion of the system of the multiplicity sequences of the branches of the curve C at P which explains after how many times of blowing ups of C at P the branches separate from each other.

All rational plane curves of type $(d, d-2)$ with multibranched singular points are classified by Sakai and Saleem in [7]. In [8], they generalized the results with Tono to plane curves of type $(d, d-2)$ with any genus. It turns out that still the answer of Matsuka and Sakai's conjectured in [9], is affirmative. As a generalization of these results, the following question arrises: Is any rational plane curve of type $(d, d-3)$ is transformable into a line by a Cremona transformation? Flenner and Zaidenberg in [10,11], and Fenske in [12] discussed and answered affirmatively the cuspidal case. To study the case for all rational plne curves of type $(d, d-3)$, there are many dificulties. In [6], the author give a list, but not complete, for rational plne curves of type $(d, d-3, 1)$. In this paper, we answer the question for some classes of rational plane curves of types $(7, 4, 1)$.

2 Quadratic Cremona Transformation

In this section, we give a tool to construct curve germs with one branch and two branches which we will use in this paper. Let $(x, y, z) \in \mathbb{P}^2$ be homogeneous coordinates. Sakai and Tono in [13] defined the (degenerate) quadratic

* Corresponding author e-mail: abuelhassan@yahoo.com

Cremona transformation $\varphi_c : (x,y,z) \longrightarrow (xy, y^2, x(z-cx))$ for $c \in \mathbb{C}$. The inverse of this transformation is $\varphi_c^{-1}(x,y,z) = (x^2, xy, yz + cx^2)$. By a suitable change of coordinates, we can set the two lines l and t such that $l : x = 0$, $t : y = 0$ and the points O, A and B have the coordinates $O = (0,0,1)$, $A = (1,0,c)$ and $B = (0,1,0)$. We remark that the base points of φ_c are O, A and the infinitely near point of O which corresponds to the direction of l and the base points of φ_c^{-1} are O, B and the infinitely near point of O which corresponds to the direction of t, (see also [10]).

Now, successive compositions of the quadratic Cremona transformations $\varphi = \varphi_{c_k} \circ \cdots \circ \varphi_{c_k}$ for $c_1, \ldots, c_k \in \mathbb{C}$ can be written as

$$\varphi^{-1}(x,y,z) = \left(x^{k+1}, x^k y, y^k z + \sum_{i=2}^{k+1} c_{k+2-i} x^i y^{k+1-i} \right).$$

Let $(C,P) \subset (\mathbb{C}^2, P)$ be a plane curve germ, where $P \in C$ is a singular point. We obtain the minimal embedded resolution of the singularity (C,P), by means of a sequence of blowing-ups $X_i \xrightarrow{\pi_i} X_{i-1}, i = 1,2,\ldots,k$, over P. Let $C^{(i)} \subset X_i$ be the strict (also called proper) transform of C in X_i and E is the exceptional divisor of the whole resolution. Hence, the total transform of C in X_k is a simple normal crossing (SNC) divisor $D = E + C^{(k)}$ as in the following diagram:

$$\begin{array}{ccccccccc} X_k & \xrightarrow{\pi_k} & X_{k-1} & \xrightarrow{\pi_{k-1}} & \cdots & \xrightarrow{\pi_2} & X_1 & \xrightarrow{\pi_1} & U \\ \cup & & \cup & & & & \cup & & \cup \\ C^{(k)} & \longrightarrow & C^{(k-1)} & \longrightarrow & \cdots & \longrightarrow & C^{(1)} & \longrightarrow & C = C^{(0)} \end{array}$$

where k is a finite positive integer.

In case $r_P = 1$, let m_i be the multiplicity of $C^{(i)}$ at P_i, where P_i is the infinitely near point of P on $C^{(i)}$. We define the *multiplicity sequence* of (C,P) to be $\underline{m}_P(C) = (m_0, m_1, ..., m_k)$, where $m_0 \geq m_1 \geq \ldots \geq m_k = 1$. We write (m_a) for the sequence $(\overbrace{m,...,m}^{a\text{-times}}, 1, 1)$. We understand that when $a = 0$, then $(m_0) = 1$.

In case $r_P = 1$, we recall the definition of *the system of the multiplicity sequences of* $P \in C$, (see [6,7] for more details).

Definition 1*The systems of the multiplicity sequences of a bibranched singular point are defined as follows:*

$$\underline{m}_P(\zeta_1, \zeta_2) = \left\{ \begin{pmatrix} m_{1,0} \\ m_{2,0} \end{pmatrix} \cdots \begin{pmatrix} m_{1,\rho} \\ m_{2,\rho} \end{pmatrix} \begin{matrix} m_{1,\rho+1}, m_{1,\rho+2}, \ldots, m_{1,s_1} \\ m_{2,\rho+1}, m_{2,\rho+2}, \ldots, m_{2,s_2} \end{matrix} \right\},$$

where the brackets mean that the germs go through the same infinitely near points of P and $\underline{m}_P(\zeta_i) = (m_{i,0}, m_{i,1}, ..., m_{i,s_i})$ are the multiplicity sequences of the branches $(\zeta_i, P), i = 1,2,$ of the germ (C,P).

Since we deal with curves of types $(7,4,1)$, we have the following Lemma.

Lemma 1*Let C be a rational plane septic curve and $P \in C$ be a unibranched or a bibranched singular point with multiplicity 4. Then, the system of the multiplicity sequences of P are divided into the types as in Table 1:*

Table 1: Deferent types of the systems of the multiplicity sequences of a singularity of multiplicity 4.

Number of branches	Number of tangent lines	System of the multiplicity sequences
1	1	$(4),\ (4,2_i),\ (4,3),\ i = 2,3.$
2	1	$\left\{\binom{2}{2}\binom{1}{1}_i\right\}, \left\{\binom{3}{1}\binom{1}{1}_i\right\}, \left\{\binom{2}{2}\binom{2}{1}\right\}, \left\{\binom{3}{1}\binom{2}{1}\right\}, i = 2,3.$
2	2	$\left\{\binom{2}{2}\begin{matrix}2_i\\2_j\end{matrix}\right\}, \left\{\binom{3}{1}^{2}\right\}, \left\{\binom{3}{1}^{3_i}\right\}, \left\{\binom{3}{1}^{(3_2,2)}\right\}, i,j = 0,1,2,3$

3 Main results

In this section, we construct some classes of rational plane curves of type $(7,4,1)$. We show that these curves are transformable into a line by using suitable Cremona transformations.

Definition 2*Let $Sing(C) = \{P_1, P_2, \ldots, P_s\}$ be the set of all the singular points on the rational plane curve C. The collection of the systems of the multiplicity sequences of C at the points P_i is called the numerical data of C and is written as* $\text{Data}(C) = [\underline{m}_{p_1}(C), \underline{m}_{p_2}(C), \ldots, \underline{m}_{p_s}(C)]$.

Our result is written in the following theorem.

Theorem 1*Let C be a rational plane curve of type $(7,4,1)$. Let P be the singularites with the maximal multiplicity 4. Then,* Data(C) *are classified (up to projective equivalent) as in Tables* 2 *and* 3*:*

Table 2: P is a unibranched singular point (cusp)

Class I (P is a unibranched singular point (cusp)			
No.	Data(C)	No.	Data(C)
1	$(4),\ (3_2),\ \binom{2}{1}$	18	$(4,2_2),\ (2_2),\ (2_2),\ \binom{2}{1}$
2	$(4),\ (3),\ \binom{2}{1}_2$	19	$(4,2_3),\ (3),\ \binom{2}{1}$
3	$(4),\ \binom{2}{1}_3$	20	$(4,2_3),\ (3,2),\ (2),\ \binom{1}{1}$
4	$(4,3),\ (3),\ \binom{2}{1}$	21	$(4,2_3),\ (2),\ (2),\ \left\{\binom{2}{1}^{(2)}\right\}$
5	$(4,3),\ \binom{2}{1}_2$	22	$(4,2_3),\ \binom{2}{1}_2$
6	$(4,2_2),\ (3),\ \left\{\binom{2}{1}^{(2)}\right\}$	23	$(4,2_3),\ (2_2),\ \left\{\binom{2}{1}\binom{1}{1}\right\}$
7	$(4,2_2),\ \left\{\binom{2}{1}\binom{2}{1}^{(2)}\right\}$	24	$(4,2_3),\ (3,2),\ \binom{1}{1}_2$
8	$(4,2_2),\ \left\{\binom{2}{1}\binom{2}{1}\binom{1}{1}\right\}$	25	$(4,2_3),\ (3),\ \binom{1}{1}_3$
9	$(4,2_2),\ (3),\ \left\{\binom{2}{1}\binom{1}{1}\right\}$	26	$(4,2_3),\ (2_2),\ \binom{1}{1}_4$
10	$(4,2_2),\ \left\{\binom{2}{1}_2\binom{1}{1}\right\}$	27	$(4,2_3),\ (2),\ \binom{1}{1}_5$
11	$(4,2_2),\ (3_2),\ \binom{1}{1}$	28	$(4,2_3),\ \binom{1}{1}_6$
12	$(4,2_2),\ (2),\ \binom{2}{1}_2$	29	$(4,2_4),\ (3,2),\ \binom{1}{1}$
13	$(4,2_2),\ (3,2),\ \binom{2}{1}$	30	$(4,2_4),\ (2_2),\ \binom{2}{1}$
14	$(4,2_2),\ (3,2),\ \binom{1}{1}_3$	31	$(4,2_4),\ (2),\ \left\{\binom{2}{1}\binom{1}{1}\right\}$
15	$(4,2_2),\ (3),\ \binom{1}{1}_4$	32	$(4,2_4),\ \left\{\binom{2}{1}^{(2_2)}\right\}$
16	$(4,2_2),\ (2_2),\ \left\{\binom{2}{1}^{(2_2)}\right\}$	33	$(4,2_6),\ \binom{2}{1}$
17	$(4,2_2),\ \left\{\binom{2}{1}^{(2_4)}\right\}$		

Table 3: *P* is a bibranched singular point with two tangent lines

Class II (P is a bibranched singular point with two coincide tangent lines)		Class III (P is a bibranched singular point with two different tangent lines)	
No.	Data(C)	No.	Data(C)
1	$\left\{\binom{2}{2}\binom{1}{1}\right\}, (3_2), (2_2)$	1	$\binom{2}{2}, (3_3)$
2	$\left\{\binom{2}{2}\binom{1}{1}\right\}, (3_2), (2), (2)$	2	$\binom{2}{2}, (3_2), (3)$
3	$\left\{\binom{2}{2}\binom{1}{1}\right\}, (3_2,2), (2)$	3	$\binom{2}{2}, (3_2,2), (2_2)$
4	$\left\{\binom{2}{2}\binom{1}{1}\right\}, (3,2), (3,2)$	4	$\left\{\binom{2}{2}\substack{(2)\\(2)}\right\}, (3_2), (2)$
5	$\left\{\binom{2}{2}\binom{1}{1}_2\right\}, (3_2), (2)$	5	$\left\{\binom{2}{2}\substack{(2)\\(2)}\right\}, (3_2,2)$
6	$\left\{\binom{2}{2}\binom{1}{1}_2\right\}, (3_2,2)$	6	$\left\{\binom{2}{2}^{(2_2)}\right\}, (3_2), (2)$
7	$\left\{\binom{2}{2}\binom{1}{1}_2\right\}, (3,2), (3)$	7	$\left\{\binom{2}{2}^{(2_2)}\right\}, (3_2,2)$
8	$\left\{\binom{2}{2}\binom{1}{1}_3\right\}, (3_2)$	8	$\left\{\binom{2}{2}\substack{(2_2)\\(2)}\right\}, (3_2)$
9	$\left\{\binom{2}{2}\binom{1}{1}_3\right\}, (3), (3)$	9	$\left\{\binom{2}{2}\substack{(2_2)\\(2)}\right\}, (3), (3)$
10	$\left\{\binom{2}{2}\binom{1}{1}_3\right\}, (3,2), (2_2)$	10	$\left\{\binom{2}{2}^{(2_3)}\right\}, (3_2)$
11	$\left\{\binom{2}{2}\binom{1}{1}_3\right\}, (3,2), (2), (2)$	11	$\left\{\binom{2}{2}^{(2_3)}\right\}, (3), (3)$
12	$\left\{\binom{2}{2}\binom{1}{1}_4\right\}, (3,2), (2)$	12	$\binom{3}{1}, (3_3)$
13	$\left\{\binom{2}{2}\binom{1}{1}_4\right\}, (3), (2_2)$	13	$\binom{3}{1}, (3_2), (3)$
14	$\left\{\binom{2}{2}\binom{1}{1}_5\right\}, (3), (2)$	14	$\binom{3}{1}, (3_2,2), (2_2)$
15	$\left\{\binom{2}{2}\binom{1}{1}_5\right\}, (3,2)$	15	$\left\{\binom{3}{1}^{(2)}\right\}, (3_2), (2_2)$
16	$\left\{\binom{2}{2}\binom{1}{1}_5\right\}, (3), (2)$	16	$\left\{\binom{3}{1}^{(2)}\right\}, (3_2,2), (2)$
17	$\left\{\binom{2}{2}\binom{1}{1}_6\right\}, (3)$	17	$\left\{\binom{3}{1}^{(2)}\right\}, (3,2), (3,2)$
18	$\left\{\binom{2}{2}\binom{1}{1}_6\right\}, (2), (2_2)$	18	$\left\{\binom{3}{1}^{(3_3)}\right\}$
19	$\left\{\binom{2}{2}\binom{1}{1}_6\right\}, (2_3)$	19	$\left\{\binom{3}{1}^{(3_2)}\right\}, (3)$
20	$\left\{\binom{2}{2}\binom{2}{1}\right\}, (3), (3)$	20	$\left\{\binom{3}{1}^{(3_2)}\right\}, (2_3)$
21	$\left\{\binom{2}{2}\binom{2}{1}\right\}, (3_2)$	21	$\left\{\binom{3}{1}^{(3_2)}\right\}, (2_2), (2)$
22	$\left\{\binom{2}{2}\binom{2}{1}\right\}, (3), (2_3)$	22	$\left\{\binom{3}{1}^{(3)}\right\}, (3_2)$
23	$\left\{\binom{2}{2}\binom{2}{1}\right\}, (3,2), (2_2)$	23	$\left\{\binom{3}{1}^{(3)}\right\}, (3), (3)$
24	$\left\{\binom{3}{1}\binom{1}{1}\right\}, (3_2), (2_2)$	24	$\left\{\binom{3}{1}^{(3)}\right\}, (3,2), (2_2)$
25	$\left\{\binom{3}{1}\binom{2}{1}\right\}, (3_2)$	25	$\left\{\binom{3}{1}^{(3,2)}\right\}, (3,2), (2)$
		26	$\left\{\binom{3}{1}^{(3_2,2)}\right\}, (2_2)$

Remark 1*Rational plane curves of type $(7,4)$, $\iota = 0$, are classified in **[12]** as follows:*

Class	Data(C)
(1)	$[(4),(3_3)]$
(2)	$[(4,3),(3_2)]$
(3)	$[(4,2_3),(3_2)]$
(4)	$[(4,2_2),(3_2,2)]$
(5)	$[(4,2_2),(3_2),(2)]$

By applying a suitable quadratic Cremona transformations, we give a construction of cuspidal rational plane sextic curves. By a suitable change of coordinates, we set the two lines l and t and the points O, A and B as follows: $l : x = 0$, $t : y = 0$, $O = (0,0,1)$, $A = (1,0,c)$ and $B = (0,1,0)$. In what follows, Applying φ_c, we construct the curve C' from the curve C, where C' is the strict transform of C via φ_c.

As a technique for choosing the initial curves C with a specific Data(C), we apply the inverse of a suitable quadratic Cremona transformations. These initial curves with given data are neither fixed nor unique (see [6], §4.2 for more details).

1.[Class (1)] We begin with the sextic curve C with $\mathrm{Data}(C) = \left[(3_2),(3),\binom{1}{1}\right]$. We choose two lines l and t such that $l \cdot C = 3O + 3B$ and $t \cdot C = 4O + 2A$, $A = P$. We find that $P' = A_1 + A_2$, with multiplicity sequence $\underline{m}_{A_1} = \underline{m}_{A_1} = (1)$, $B' = B$ with $\underline{m}_B = (3_3)$ and $O' = O$ with $\underline{m}_{O'} = (4)$.

2.[Class (2)] We start with the quartic curve C with $\mathrm{Data}(C) = [(3)]$. We choose the lines l and t such that $l \cdot C = 4B$ and $t \cdot C = 3P + A$, $A \neq P$. We see that $B' = B$ with $\underline{m}_{B'} = (4,3)$ and $P' = O$ with $\underline{m}_O = (3_2)$.

3.[Class (3)] In this case we begin with the quartic curve C with $\mathrm{Data}(C) = [(2_2),\ (2)]$. We choose the lines l and t such that $l \cdot C = 4B$ and $t \cdot C = 3P + A$, $A \neq P$. We see that $B' = B$ with $\underline{m}_{B'} = (4,2_2)$ and $P' = O$ with $\underline{m}_O = (3_2,2)$.

4.[Class (4)] We start with the quartic curve C with $\mathrm{Data}(C) = [(2_3)]$. We choose the lines l and t such that $l \cdot C = 4B$ and $t \cdot C = 3P + A$, $A \neq P$. We see that $B' = B$ with $\underline{m}_{B'} = (4,2_3)$ and $P' = O$ with $\underline{m}_O = (3_2)$.

5.[Class (5)] We begin with the quartic curve C with $\mathrm{Data}(C) = [(2_2),\ (2)]$. We choose the lines l and t such that $l \cdot C = 4B$ and $t \cdot C = 3P + A$, $A \neq P$. We see that $B' = B$ with $\underline{m}_{B'} = (4,2_2)$ and $P' = O$ with $\underline{m}_O = (3_2)$.

4 Construction

In this section, we construct some of the curves in the Tables in Theorem 1 by using suitable Cremona transformations. The other curves can be constructed in the same manner. By suitable changing of coordinates, we may assume that $l : x = 0$, $t : y = 0$, $O = (0,0,1)$, $A = (1,0,c)$ and $B = (0,1,0)$. In what follows, Applying $\varphi_c : (x,y,z) \longrightarrow (xy, y^2, x(z-cx))$ for $c \in \mathbb{C}$, we construct the curve C' from the curve C, where C' is the strict transform of C via φ_c. We infere that to construcat most of the curves here, we may use curves as initial curves, but, these initial curves with given data are neither fixed nor unique (see [6], §4.2 for more details).

1.[Class I, $No.8$:] We begin with the smooth cubic curve C. We choose two lines l and t such that $l \cdot C = 3B$ and $t \cdot C = 2P + A$, $A \neq P$. We find that $P' = O$ with multiplicity sequence $\underline{m}_{P'} = (2_2)$,and $B' = B$ with $\underline{m}_{B'} = \left\{\binom{2}{1}\binom{1}{1}\right\}$. Again, we apply a suitable Cremona transformations on the strict trnsform C' of the curve C.We choose the two lines l and t such that $l \cdot C = 2O + 3B'$ and $t \cdot C = 4O + A$. We find that $O'' = O$ with multiplicity sequence $\underline{m}_O = (4,2_2)$,and $B'' = B$ with $\underline{m}_B = \left\{\binom{2}{1}_2\binom{1}{1}\right\}$.

2.[Class I, $No.12$:] We begin with the cuspidal cubic curve C. We choose two lines l and t such that $l \cdot C = 2B + S$ and $t \cdot C = 2P + A$, $A \neq P$. We find that $P' = O$ with multiplicity sequence $\underline{m}_{P'} = (2_2)$,and $B' = S' = B$ with $\underline{m}_{B'} = \binom{2}{1}$. Again, we apply a suitable Cremona transformations on the strict trnsform C' of the curve C.We choose the two lines l and t such that $l\cdot C = 2O+3B'$ and $t\cdot C = 4O+A$. We find that $O'' = O$ with multiplicity sequence $\underline{m}_O = (4,2_2)$,and $B'' = B$ with $\underline{m}_B = \binom{2}{1}_2$.

3.[Class I, $No.16$:] We use the quintic curve C with Data$(C) = [(2_2),(2_2),(2_2)]$ as an initial curve. We choose two lines l and t such that $l\cdot C = 2O+2R+S$ and $t\cdot C = 4O+A$. By applying quadratic Cremona transformation, we get $O^{'} = O$ with multiplicity sequence $\underline{m}_O = (4,2_2)$,and $R' = S' = B$ with $\underline{m}_B = \left\{\binom{2}{1}^{(2_2)}\right\}$.

4.[Class I, $No.20$:] We start with the quartic curve C with Data$(C) = \left[(2),(2),\binom{1}{1}\right]$. We choose two lines l and t such that $l\cdot C = 4O$ and $t\cdot C = O+2P+A, A\neq P$. We find that $P^{'} = O$ with multiplicity sequence $\underline{m}_{P'} = (2_3)$,and $O^{'} = B$ with $\underline{m}_B = (2)$. Again, we apply a suitable Cremona transformations on the strict trnsform C' of the curve C.We choose the two lines l and t such that $l\cdot C = 2O+3B$ and $t\cdot C = 4O+A$. We find that O'' $= O$ with multiplicity sequence $\underline{m}_O = (4,2_3)$,and $B'' = B$ with $\underline{m}_B = (3,2)$.

5.[Class I, $No.23$:] We begin with the quartic curve C with Data$(C) = [(2),(2_2)]$. We choose two lines l and t such that $l\cdot C = 3O+B$ and $t\cdot C = O+2P+A, A\neq P$. We find that $P^{'} = O$ with multiplicity sequence $\underline{m}_{P'} = (2_3)$,and $O' = B' = B$ with $\underline{m}_B = \binom{1}{1}$. Again, we apply a suitable Cremona transformations on the strict trnsform C' of the curve C.We choose the two lines l and t such that $l\cdot C = 2O+3B$ and $t\cdot C = 4O+A$. We find that O'' $= O$ with multiplicity sequence $\underline{m}_O = (4,2_3)$,and $B'' = B$ with $\underline{m}_B = \left\{\binom{2}{1}\binom{1}{1}\right\}$.

6.[Class I, $No.32$:] We use the quintic curve C with Data$(C) = [(2_4),(2_2)]$ as an initial curve. We choose two lines l and t such that $l\cdot C = 2O+2R+S$ and $t\cdot C = 4O+A$. By applying quadratic Cremona transformation, we get $O^{'} = O$ with multiplicity sequence $\underline{m}_O = (4,2_4)$,and $R' = S' = B$ with $\underline{m}_B = \left\{\binom{2}{1}^{(2_2)}\right\}$.

7.[Class I, $No.33$:] We use the unicuspidal quintic curve C as an initial curve. We choose two lines l and t such that $l\cdot C = 2O+2R+S$ and $t\cdot C = 4O+A$. By applying quadratic Cremona transformation, we get $O^{'} = O$ with multiplicity sequence $\underline{m}_O = (4,2_6)$,and $R' = S' = B$ with $\underline{m}_B = \binom{2}{1}$.

8.[Class II, $No.8$:] We begin with the smooth cubic curve C. We choose two lines l and t such that $l\cdot C = 3B$ and $t\cdot C = 2P+A, A\neq P$. We find that $P^{'}$ $= O$ with multiplicity sequence $\underline{m}_{P'} = \binom{1}{1}_3$,and $B' = B$ with $\underline{m}_{B'} = (3)$. Again, we apply a suitable Cremona transformations on the strict trnsform C' of the curve C.We choose the two lines l and t such that $l\cdot C = 2O+3B'$ and $t\cdot C = 4O+A$. We find that O'' $= O$ with multiplicity sequence $\underline{m}_O = \left\{\binom{2}{2}\binom{1}{1}_3\right\}$,and $B'' = B$ with $\underline{m}_B = (3_2)$.

9.[Class II, $No.10$:] In this case,we begin with the quartic curve C with Data$(C) = \left[(2_2),\binom{1}{1}\right]$. We choose two lines l and t such that $l\cdot C = 4O$ and $t\cdot C = O+2P+A$, $A\neq P$. We find that $P^{'} = O$ with multiplicity sequence $\underline{m}_{P'} = \binom{1}{1}_3$,and $O' = B$ with $\underline{m}_B = (2)$. Again, we apply a suitable Cremona transformations on the strict trnsform C' of the curve C.We choose the two lines l and t such that $l\cdot C = 2O+3B$ and $t\cdot C = 4O+A$. We find that O'' $= O$ with multiplicity sequence $\underline{m}_O = \left\{\binom{2}{2}\binom{1}{1}_3\right\}$,and $B'' = B$ with $\underline{m}_B = (3,2)$.

10.[Class II, $No.15$:] We start with the quartic curve C with the tacnode $\binom{1}{1}_3$. We choose two lines l and t such that $l\cdot C = 4O$ and $t\cdot C = O+2P+A$, $A\neq P$. We find that $P^{'} = O$ with multiplicity sequence $\underline{m}_{P'} = \binom{1}{1}_5$,and $O' = B$ with $\underline{m}_B = (2)$. Again, we apply a suitable Cremona transformations on the strict trnsform C' of the curve C.We choose the two lines l and t such that $l\cdot C = 2O+3B$ and $t\cdot C = 4O+A$. We find that O'' $= O$ with multiplicity sequence $\underline{m}_O = \left\{\binom{2}{2}\binom{1}{1}_5\right\}$,and $B'' = B$ with $\underline{m}_B = (3,2)$.

11.[Class II, $No.24$:] We begin with the quintic curve C with Data$(C) = [(3),(2_2),(2)]$. We choose two lines l and t such that $l\cdot C = 2O+3R$ and $t\cdot C = 3O+P+A$. We find that $P^{'} = O^{'} = O$ with multiplicity sequence $\underline{m}_O = \left\{\binom{3}{1}\binom{1}{1}\right\}$,and $R' = B$ with $\underline{m}_B = (3_2)$.

12.[Class II, $No.25$:] We begin with the quartic curve C with Data$(C) = \binom{2}{1}$. We choose two lines l and t such that $l\cdot C = 4R$ and $t\cdot C = 3P+A$. We find that $P^{'} = O$ with multiplicity sequence $\underline{m}_O = (3_2)$,and $R' = B$ with $\underline{m}_B = \left\{\binom{3}{1}\binom{2}{1}\right\}$.

13.[Class III, $No.1$:] We start with the quartic curve C with Data$(C) = [(3)]$. We choose two lines l and t such that $l\cdot C = 2S+2R$ and $t\cdot C = 3P+A$. We find that $P^{'}$ $= O$ with multiplicity sequence $\underline{m}_{P'} = (3_3)$,and $R' =$ $S' = B$ with $\underline{m}_B = \binom{2}{2}$.

14.[Class III, $No.8$:] In this case ,we start with the quartic curve C with Data$(C) = [(2),(2_2)]$. We choose two lines l and t such that $l\cdot C = 2S+2R$ and $t\cdot C = 3P+A$. We find that $P^{'} = O$ with multiplicity sequence $\underline{m}_O = (3_2)$,and $R' = S' = B$ with $\underline{m}_B = \left\{\binom{2}{2}\begin{smallmatrix}(2_2)\\(2)\end{smallmatrix}\right\}$.

15.[Class III, $No.15$:] We begin with the quartic curve C with Data$(C) = [(2),(2_2)]$. We choose two lines l and t such that $l\cdot C = 3S+R$ and $t\cdot C = 3P+A$. We find that $P^{'} = O$ with multiplicity sequence $\underline{m}_O = (3_2)$,and $R' = S' = B$ with $\underline{m}_B = \left\{\binom{3}{1}^{(2)}\right\}$.

16.[Class III, $No.16$:] We begin with the quartic curve C with three simple cusps. We choose two lines l and t

such that $l \cdot C = 3S + R$ and $t \cdot C = 3P + A$. We find that $P^{'} = O$ with multiplicity sequence $\underline{m}_O = (3_2, 2)$,and $R' = S' = B$ with $\underline{m}_B = \left\{ \binom{3}{1}^{(2)} \right\}$.

17.[Class III, $No.22$:] We use the unicuspidal quartic curve C. We choose two lines l and t such that $l \cdot C = 3S + R$ and $t \cdot C = 3P + A$. We find that $P^{'} = O$ with multiplicity sequence $\underline{m}_O = (3_2)$,and $R' = S' = B$ with $\underline{m}_B = \left\{ \binom{3}{1}^{(3)} \right\}$.

18.[Class III, $No.25$:] We use the quintic curve C with $\text{Data}(C) = [(3,2),(2_2)]$. We choose two lines l and t such that $l \cdot C = O + 3S + R$ and $t \cdot C = 3O + 2A$. We find that $O^{'} = O$ with multiplicity sequence $\underline{m}_O = (3,2)$,$A^{'} = A$ with $\underline{m}_A = (2)$,and $R' = S' = B$ with $\underline{m}_B = \left\{ \binom{3}{1}(3,2) \right\}$.

Acknowledgement

The author is grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] E. Brieskorn and H. Knörrer: *Plane Algebraic Curves*. Birkhäuser,Basel, 1986.
[2] Theo de Jong and Gerhard Pfister: *Local Analytic Geometry*. Vieweg, 2000.
[3] H. Yoshihara: *On plane rational curves.* Proc. Japan Acad. **55** (A), 152-155, (1979).
[4] H. Yoshihara: *Rational curves with one cusps.* I Proc. Amer. Math. Soc. **89** (1983), 24-26. II ibid. **100**, 405-406, (1987).
[5] H. Yoshihara: *Plane curves whose singular points are cusps.* Proc. Amer. Math. Soc. **103**, 737-740, (1988).
[6] M. Saleem: *On the classification of rational plane curves of types* (d,m). Lambert Acad. Publishing, 2011, ISBN 978-3-8443-9988-2.
[7] F. Sakai and M. Saleem: *Rational plane curves of type* $(d,d-2)$. Saitama Math. J, **22**, 11-34, (2004).
[8] F. Sakai, M. Saleem and K. Tono: *Hyperelliptic plane curves of type* $(d,d-2)$. Contributions to Algebra and Geometry, **51**, No. 1, 31-44, (2010).
[9] T. Matsuoka and F. Sakai: *The degree of rational cuspidal plane curves*. Math. Ann., **285**, 233-247, (1989).
[10] H. Flenner and M. Zaidenberg: *On a class of rational cuspidal plane curves.* Manuscripta Math., **89**, 439-460, (1996).
[11] H. Flenner and M. Zaidenberg: *Rational cuspidal plane curves of type* $(d,d-3)$. Math. Nach., **210**, 93-110, (2000).
[12] T. Fenske: *Rational 1- and 2-cuspidal plane curves.* Beiträge Alg. Geometrie, **40**, No. 2, 309-329, (1999).
[13] F. Sakai and K. Tono: *Rational cuspidal curves of type* $(d,d-2)$ *with one or two cusps*. Osaka J. Math., **37**, 504-415, (2000).

Mohammed A. Saleem has got Ph. D. degree from Saitama University, Japan 2004 in the field of Algebraic Geometry. He returned to Japan again in the period 2005-2007 as a JSPS Fellow. Recently, he is a lecturer in Mathematics department, Faculty of Science, Sohag University. His research interests are in the areas of algebraic geometry and algebra especially in plane curves singularities and Weierstrass points. He has published research articles in reputed international journals of mathematical and engineering sciences. He is referee and editor of mathematical journals.