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Kähler Graphs of Connected Product Type

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Abstract: Kähler graphs are compound graphs which consist of principal and auxiliary graphs. We show some natural commutative product operations of getting Kähler graphs whose principal graphs are connected. We study eigenvalues of Laplacians for Kähler graphs obtained by these operations.

Keywords: Kähler graphs, principal and auxiliary graphs, connected graphs, product operations, transition Laplacians, isospectral

1 Introduction

A graph $G = (V,E)$ consists of a set V of vertices and a set E of edges. Regarding graphs as 1-dimensional CW-complexes geometers consider them to be discrete models of Riemannian manifolds. Paths, which are chains of edges, on a graph correspond to geodesics on a Riemannian manifold. Being inspired by papers [6] and [8], the second author began to study Kähler manifolds from the Riemannian geometric point of view by make use of Kähler magnetic fields ([1,3] and their references). Also, he introduced in [2] the notion of Kähler graphs as discrete models of Riemannian manifolds admitting magnetic fields. A Kähler graph is a graph whose set of edges is divided into two disjoint subsets. We may say that a Kähler graph is a compound of two graphs. One is called the principal graph and the other the auxiliary graph. Geometrically, paths on the principal graph of a Kähler graph are regarded as geodesics, which are motions of charged particles without getting the influence of magnetic fields. In order to show the influence of magnetic fields, we use the auxiliary graph. We regard a p-step path in the principal graph followed by a q-step path in the auxiliary graph as a motion of a charged particle under the influence of a magnetic field of strength of Lorentz force q/p.

When we introduce new notions, it is needless to say that the most important thing is to construct many good examples. In [9] and [10], we gave examples of Kähler graphs; Cayley Kähler graphs, Kähler graphs obtained by the complement-filling operation, those obtained by product operations and so on. From the geometrical point of view, the simplest example of a Kähler manifold should be a complex line $\mathbb{C}$, which can be regarded as a plane $\mathbb{R}^2$ from the viewpoint of real Riemannian geometry. Thus we are interested in giving a model of a complex line whose principal graph is a model of a real plane. In this paper, keeping above in our mind we add five kinds of product operations. Being different from operations we gave in [10] these operations are commutative, and give Kähler graphs whose principal graphs are connected as 1-dimensional CW-complexes. As we obtain new Kähler graphs we study eigenvalues of their Laplacians to show their properties.

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2 Kähler graphs

A graph $G = (V,E)$ is a pair of a set V of vertices and a set E of edges. In this paper we suppose that each edge does not have its orientation. Also, we suppose that graphs do not have loops and multiple edges. Here, a loop is an edge joining a vertex and itself, and multiple edges are edges joining the same pair of vertices. We say two

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vertices $v, v' \in V$ to be adjacent to each other if there is an edge joining them. In this case we denote as $v \sim v'$.

We say a graph $G = (V,E)$ to be *Kähler* if the set E is divided into two subsets as $E = E^{(p)} \cup E^{(a)}$ ($E^{(p)} \cap E^{(a)} = \emptyset$) and if it satisfies that both the *principal graph* $G^{(p)} = (V, E^{(p)})$ and the *auxiliary graph* $G^{(a)} = (V, E^{(a)})$ do not have hairs. That is, for an arbitrary vertex $v \in V$ there exist at least two edges in $E^{(p)}$ emanating from v and exist at least two edges in $E^{(a)}$ emanating from v. We call an edge belonging to $E^{(p)}$ principal and call that belonging to $E^{(a)}$ auxiliary. Given two vertices v, v', we denote as $v \sim_p v'$ if they are adjacent to each other in $G^{(p)}$, and denote as $v \sim_a v'$ if they are adjacent to each other in $G^{(a)}$. The reason why we consider such compound graphs is that we intend to give a discrete model of a manifold with complex structure. Since graphs are 1-dimensional objects, it is not so easy to introduce an object of real 2-dimension. We therefore use such compound graphs.

The simplest example of a Kähler manifold is a complex Euclidean space. We are hence interested in considering its discrete model.

*Example 1.*We take a set of lattice $V = \{z = x + \sqrt{-1}y \in \mathbb{C} \mid x, y \in \mathbb{Z}\}$, where $\mathbb{Z}$ denotes the set of integers. For two vertices $z = x + \sqrt{-1}y$, $z' = x' + \sqrt{-1}y' \in V$ we define $z \sim_p z'$ if and only if $z' - z = \pm 1$ holds, and define $z \sim_a z'$ if and only if $z' - z = \pm\sqrt{-1}$ holds. With these adjacency rules we get a Kähler graph. We shall call this a Kähler graph of complex lattice.

We see that a Kähler graph of complex lattice consists of horizontal lines for the principal graph and vertical lines for the auxiliary graph. In other words, it is a "product" of a principal graph of real lattice and an auxiliary graph of real lattice. It is known that we have four typical ways of product operations of graphs; Cartesian product, strong product, semi-tensor product and lexicographic product. Viewing a Kähler graph of complex lattice we defined four kinds of Kähler graphs of product type in [10] in the following manner. Let $G = (V,E)$, $H = (W,F)$ be two graphs. We define Kähler graphs of Cartesian product type $G\widehat{\Box}H$, of strong product type $G\widehat{\boxtimes}H$, of semi-tensor product type $G\widehat{\otimes}H$ and of lexicographic product type $G \triangleright H$ as follows.

i)Their sets of versices are the product $V \times W$:
ii)For two vertices $(v,w), (v',w') \in V \times W$, we define $(v,w) \sim_p (v',w')$ if and only if $v \sim v'$ in G and $w = w'$:
iii)For two vertices $(v,w), (v',w') \in V \times W$, we define $(v,w) \sim_a (v',w')$ if and only if they satisfy the following conditions:
  a)$v = v'$ and $w \sim w'$ in H for $G\widehat{\Box}H$,
  b)either $v = v'$ and $w \sim w'$ in H, or $v \sim v'$ in G and $w \sim w'$ in H for $G\widehat{\boxtimes}H$,
  c)$v \sim v'$ in G and $w \sim w'$ in H for $G\widehat{\otimes}H$,
  d)$w \sim w'$ in H for $G \triangleright H$.

These Kähler graphs have common principal graphs, and they are not connected as 1-dimensional CW-complexes. Moreover, these product operations are not commutative, that is $G\widehat{\Box}H \neq H\widehat{\Box}G$, for example.

From physical point of view, we may say that a Kähler graph of complex lattice shows motions of charged particles which move to the horizontal direction. On a graph $G = (V,E)$, a chain $\gamma = (v_0, \dots, v_n) \in V \times \cdots \times V$ of n-edges (i.e. $v_{i-1} \sim v_i$ for $i = 1, \dots, n$) is said to be an n-step path. We say a path γ contains a backtracking if there is i_0 $(1 \le i_0 \le n-1)$ with $v_{i_0-1} = v_{i_0+1}$. Paths without backtrackings are considered to correspond to geodesics on a Riemannian manifold. Coming back to a Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$, we take a pair (p,q) of relatively prime positive integers. A $(p+q)$-step path $\gamma = (v_0, \dots, v_{p+q})$ without backtrackings on G is said to be a *(p,q)-primitive bicolored path* if it satisfies $v_{i-1} \sim_p v_i$ for $i = 1, \dots, p$ and $v_{i-1} \sim_a v_i$ for $i = p+1, \dots, p+q$. An $m(p+q)$-step path $\gamma = (v_0, \dots, v_{m(p+q)})$ without backtrackings is said to be a (p,q)-bicolored path if each $(p+q)$-step subpath $\gamma_j = (v_{(j-1)(p+q)}, \dots, v_{j(p+q)})$, $j = 1, \dots, m$, is a (p,q)-primitive bicolored path. As we mentioned in §1, we consider (p,q)-bicolored paths show motions of charged particles under a magnetic field of strength q/p. Every p-step path on a principal graph is bended and turns to a (p,q)-bicolored path under the influence of this magnetic field. In this sense we should say that a Kähler graph of complex lattice is a model of a "bundle of beams" of electrons. But on a complex Euclidean line we have many geodesics of different directions. In order to consider motions of charged particles which move to the horizontal and the vertical directions at the same time, we need to give a Kähler graph which have both horizontal and vertical principal edges. For this sake we give other product operations.

Given two graphs $G = (V,E)$, $H = (W,F)$ we define their Kähler graphs $G \boxplus H$, $G \boxdot H$, $G \Diamond H$ of Cartesian-tensor product type, of Cartesian-complement product type and of Cartesian-lexicographical product type as follows:

i)Their sets of vertices are the product $V \times W$;
ii)two vertices $(v,w), (v',w') \in V \times W$ are $(v,w) \sim_p (v',w')$ if and only if either $v = v'$ and $w \sim w'$ in H or $w = w'$ and $v \sim v'$ in G;
iii)two vertices $(v,w), (v',w') \in V \times W$ are $(v,w) \sim_a (v',w')$ if and only if the following conditions hold;
  a)$v \sim v'$ in G and $w \sim w'$ in H for Cartesian-tensor product type,
  b)either $v \neq v'$, $v \not\sim v'$ in G and $w \sim w'$ in H, or $w \neq w'$, $w \not\sim w'$ in H and $v \sim v'$ in G for Cartesian-complement product type,
  c)either $v \neq v'$ and $w \sim w'$ in H, or $w \neq w'$ and $v \sim v'$ in G for Cartesian-lexicographical product type.

Here, when we make a Kähler graph of Cartesian -complement product type, we suppose one of the following conditions:

i)for each vertex $v \in V$ there is $v' \in V$ with $v' \neq v$ and $v' \not\sim v$ in G;
ii)for each vertex $w \in W$ there is $w' \in W$ with $w' \neq w$ and $w' \not\sim w$ in H.

Clearly, these product operations are commutative. It is also clear that principal graphs of Kähler graphs made by these operations are connected. For a vertex v of an ordinary graph $G = (V,E)$ we denote by $d_G(v)$ the cardinality of the set $\{v' \in V \mid v' \sim v\}$ and call it the degree at v. For a Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ we put $d_G^{(p)}(v) = d_{G^{(p)}}(v)$ and $d_G^{(a)}(v) = d_{G^{(a)}}(v)$ and call them the principal and the auxiliary degrees at v. When G and H are of finite degree, then $G \boxplus H$ is also of finite degree, and when G and H are finite graphs, then $G \boxdot H$ and $G \Diamond H$ are also finite graphs. At $(v,w) \in V \times W$ we have

$$d^{(p)}_{G\boxplus H}(v,w) = d^{(p)}_{G\boxdot H}(v,w) = d^{(p)}_{G\Diamond H}(v,w) = d_G(v) + d_H(w),$$
$$d^{(a)}_{G\boxplus H}(v,w) = d_G(v) d_H(w),$$
$$d^{(a)}_{G\boxdot H}(v,w) = (\sharp V - d_G(v) - 1) d_H(w) + (\sharp W - d_H(w) - 1) d_G(v),$$
$$d^{(a)}_{G\Diamond H}(v,w) = (\sharp V - 1) d_H(w) + (\sharp W - 1) d_G(v).$$

*Example 2.*For two integers n, n' we define $n \sim n'$ if and only if $n' = n \pm 1$, and call the graph $(\mathbb{Z}, E)$ given by this adjacency rule a graph of real lattice.

When G and H are graphs of real lattice, then principal edges and auxiliary edges emanating from a vertex of their Kähler graphs $G \boxplus H$, $G \boxdot H$, $G \Diamond H$ of Cartesian-tensor product type, of Cartesian-complement product type and of Cartesian-lexicographical product type are like Figures 1, 2 and 3, respectively. We note that $G \boxdot H$ and $G \Diamond H$ are not locally finite in this case. The Kähler graph $G \boxplus H$ is another candidate of the model of a complex Euclidean line.

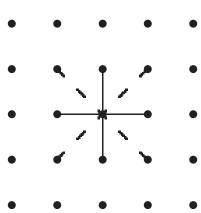

Fig. 1: edges at a vertex of $G \boxplus H$

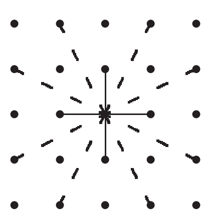

Fig. 2: edges at a vertex of $G \boxdot H$

We here explain the meaning of bicolored paths a bit more. Since graphs are 1-dimensional objects, we can not determine the direction of the action of the Lorentz force. We therefore treat (p,q)-bicolored paths probabilistically. For a (p,q)-primitive bicolored path $\gamma = (v_0, \dots, v_{p+q})$, we

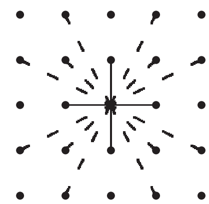

Fig. 3: edges at a vertex of $G \Diamond H$

define its probabilistic weight $\omega(\gamma)$ by

$$\omega(\gamma) = \left\{d_G^{(a)}(v_p)\right\}^{-1} \left\{d_G^{(a)}(v_{p+1}) - 1\right\}^{-1} \times \cdots \times \left\{d_G^{(a)}(v_{p+q-1}) - 1\right\}^{-1}.$$

For a (p,q)-bicolored path $\gamma = \gamma_1 \cdots \gamma_m$, which is an m-chain of (p,q)-primitive bicolored paths, we set $\omega(\gamma) = \omega(\gamma_1) \times \cdots \times \omega(\gamma_m)$. Let (p,q) be a pair of positive integers which satisfies the condition that either it consists of relatively prime integers or $q = 0$. For a Kähler graph $G = (E^{(p)} \cup E^{(a)})$ we define its (p,q)-derived (oriented) graph $G^{(p,q)} = (V, E^{(p,q)})$ so that v is adjacent to v' if and only if there is a (p,q)-primitive bicolored path γ with origin $o(\gamma) = v$ and terminus $t(\gamma) = v'$. In the case $q = 0$ we consider p-step paths alternatively. Derived graphs may have loops and multiple edges. When G satisfies that for each distinct pair (v, v') of vertices the set of probabilistic weights of (p,q)-primitive bicolored paths of v to v' coincides with that of v' to v by taking account of their multiplicities, we make a reduction and get a non-oriented graph $\widetilde{G}^{(p,q)} = (V, \widetilde{E}^{(p,q)}, \mathfrak{m})$ with weight of edges. That is, we define $v \sim v'$ in $\widetilde{G}^{(p,q)}$ if and only if there is a (p,q)-primitive bicolored paths of v to v', and set $\mathfrak{m}\big((v,v')\big)$ to be the sum of probabilistic weights of such paths. We call this the reduced (p,q)-derived graph. When $p = 1$, $q = 0$, we can always reduce the $(1,0)$-derived graph. It is the original principal graph with its degree function.

*Example 3.*When G and H are graphs of real lattice, the reduced $(1,0)$-derived graph and the reduced $(1,1)$-derived graph of $G \boxplus H$ are like Figures 4 and 5. As the reduced $(1,1)$-derived graph of $G \widehat{\boxdot} H$ is like Figure 7, one can see the difference.

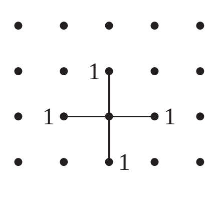

Fig. 4: $(1,0)$-derived edges at a vertex of $G \boxplus H$

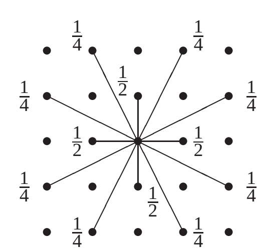

Fig. 5: $(1,1)$-derived edges at a vertex of $G \boxplus H$

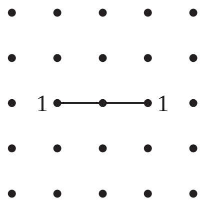

Fig. 6: $(1,0)$-derived edges at a vertex of $G\widehat{\square}H$

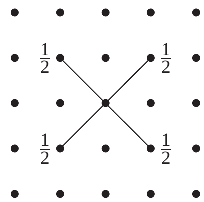

Fig. 7: $(1,1)$-derived edges at a vertex of $G\widehat{\square}H$

We give two more commutative product operations obtaining connected principal graphs. Let $G=(V,E)$, $H=(W,F)$ be ordinary graphs satisfying the same condition as we posed when we made Kähler graphs of Cartesian-complement product type. We define their Kähler graph $G \divideontimes H$ of strong-complement product type as follows:

i)the set of vertices is the product $V\times W$;
ii)two vertices $(v,w),(v',w')\in V\times W$ are $(v,w)\sim_p (v',w')$ if and only if one of the following conditions holds;
ii-a)$v\sim v'$ in G and $w=w'$,
ii-b)$v=v'$ and $w\sim w'$ in H,
ii-c)$v\sim v'$ in G and $w\sim w'$ in H,
iii)two vertices $(v,w),(v',w')\in V\times W$ are $(v,w)\sim_a (v',w')$ if and only if either $w\sim w'$ in H, $v\neq v'$ and $v\not\sim v'$ in G, or $v\sim v'$ in G, $w\neq w'$ and $w\not\sim w'$ in H.

Given two graphs $G=(V,E)$, $H=(W,F)$ we define their Kähler graph $G\spadesuit H$ of complement-tensor product type as follows:

i)the set of vertices is the product $V\times W$;
ii)two vertices $(v,w),(v',w')\in V\times W$ are $(v,w)\sim_p (v',w')$ if and only if they satisfy one of the following conditions;
a)$v\sim v'$ in G and $w\not\sim w'$,
b)$v\not\sim v'$ in G and $w\sim w'$ in H;
iii)two vertices $(v,w),(v',w')\in V\times W$ are $(v,w)\sim_a (v',w')$ if and only if $v\sim v'$ in G and $w\sim w'$ in H.

It is clear that these operations are commutative. When G and H are finite graphs, then $G\divideontimes H$ and $G\spadesuit H$ are also finite graphs. At $(v,w)\in V\times W$ we have

$$\begin{aligned}
d^{(p)}_{G\divideontimes H}(v,w) &= d_G(v)+d_H(w)+d_G(v)d_H(w),\\
d^{(a)}_{G\divideontimes H}(v,w) &= d_G(v)(\sharp W-d_H(w)-1)\\
&\quad +(\sharp V-d_G(v)-1)d_H(w),\\
d^{(p)}_{G\spadesuit H}(v,w) &= d_G(v)(\sharp W-d_H(w))+d_H(w)(\sharp V-d_G(v)),\\
d^{(a)}_{G\spadesuit H}(v,w) &= d_G(v)+d_H(w).
\end{aligned}$$

*Example 4.*When G and H are graphs of real lattice, then principal edges and auxiliary edges emanating from a vertex of their Kähler graphs $G\divideontimes H$, $G\spadesuit H$ of strong-complement product type and of complement-tensor product type are like Figures 8 and 9. In this case $G\divideontimes H$ and $G\spadesuit H$ are not locally finite.

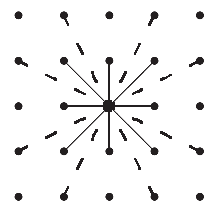

Fig. 8: edges at a vertex of $G\divideontimes H$

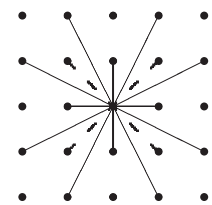

Fig. 9: edges at a vertex of $G\spadesuit H$

3 Eigenvalues of Laplacians

In order to show properties of graphs, to study their eigenvalues are one of ways (c.f. [7]). In this section we study eigenvalues of Laplacians for our Kähler graphs of product type. For a finite graph $G=(V,E)$, the adjacency operator $\mathscr{A}_G$ and the transition operator $\mathscr{P}_G$ acting on the set of functions of V are given by

$$\mathscr{A}_G f(v)=\sum_{v':v'\sim v} f(v'),\qquad \mathscr{P}_G f(v)=d_G(v)^{-1}\mathscr{A}_G f(v),$$

respectively. The combinatorial Laplacian $\Delta_{\mathscr{A}_G}$ and the transitional Laplacian $\Delta_{\mathscr{P}_G}$ are defined by $\Delta_{\mathscr{A}_G} f = d_G(v)f(v)-\mathscr{A}_G f(v)$ and $\Delta_{\mathscr{P}_G}=I-\mathscr{P}_G$. When the graph is regular, that is, its degree is a constant function, we have $\Delta_{\mathscr{A}_G}=d_G\Delta_{\mathscr{P}_G}$.

In [10], we defined corresponding Laplacians for a Kähler graph $G=(V,E^{(p)}\cup E^{(a)})$. Let (p,q) be a pair of relatively prime positive integers. We define the (p,q)-adjacency operator $\mathscr{A}^{(p,q)}_G : C(V)\to C(V)$ and the (p,q)-probabilistic transition operator $\mathscr{Q}^{(p,q)}_G : C(V)\to C(V)$ acting on the set $C(V)$ of all complex-valued functions on V by

$$\mathscr{A}^{(p,q)}_G f(v)=\sum_{\gamma\in\mathfrak{P}_{p,q}(v;G)}\omega_G(\gamma)f\bigl(t(\gamma)\bigr),$$

$$\mathscr{Q}^{(p,q)}_G f(v)=\frac{1}{\sum_{\gamma\in\mathfrak{P}_{p,q}(v;G)}\omega_G(\gamma)}\sum_{\gamma\in\mathfrak{P}_{p,q}(v;G)}\omega_G(\gamma)f\bigl(t(\gamma)\bigr),$$

where $\mathfrak{P}_{p,q}(v;G)$ denotes the set of all (p,q)-primitive bicolored paths with origin v. We denote by $d^{(p)}_{G^{(p)}}(v)$ the cardinality of the set of all p-step paths on the principal graph $G^{(p)}$ with origin v, and define $D^{(p)}_G : C(V)\to C(V)$ by $D^{(p)}_G f(v)=d^{(p)}_{G^{(p)}}(v)f(v)$. The operators

$$\Delta^{(p,q)}_{\mathscr{A}_G}=D^{(p)}_G-\mathscr{A}^{(p,q)}_G,\quad \Delta^{(p,q)}_{\mathscr{Q}_G}=I-\mathscr{Q}^{(p,q)}_G$$

acting on $C(V)$ are called the (p,q)-combinatorial Laplacian and the (p,q)-transitional Laplacian, respectively. When the principal graph is regular, we see $\mathscr{A}_G^{(p,q)} = d_{G^{(p)}}^{(p)} \mathscr{P}_G^{(p,q)}$, hence find that $\Delta_{\mathscr{A}_G}^{(p,q)} = d_{G^{(p)}}^{(p)} \Delta_{\mathscr{P}_G}^{(p,q)}$. We should note that (p,q)-Laplacians are not selfadjoint, in general (see [4]).

When $p=q=1$, we find that $\mathscr{A}_G^{(1,1)} = \mathscr{A}_{G^{(p)}} \mathscr{P}_{G^{(a)}}$ and $\mathscr{Q}_G^{(1,1)} = \mathscr{P}_{G^{(p)}} \mathscr{P}_{G^{(a)}}$. Similarly we can decompose the (p,q)-adjacency operator and the (p,q)-probabilistic transition operator into compositions of operators for the principal graph and for the auxiliary graph (see [11]). In this paper we treat Kähler graphs of product type made by regular graphs. We call a Kähler graph regular if both of its principal and its auxiliary graphs are regular. By making use of decompositions of the adjacency and the probabilistic transition operators, we showed the following in [11].

Proposition 3.1. Let $G=(V,E^{(p)}\cup E^{(a)})$ be a finite regular Kähler graph. Suppose its adjacency operators $\mathscr{A}_{G^{(p)}}, \mathscr{A}_{G^{(a)}}$ of the principal and the auxiliary graphs are commutative. If the eigenvalues of $\mathscr{A}_{G^{(p)}}$ and $\mathscr{A}_{G^{(p)}}$ are λ_k and ρ_k $(k=1,\dots,\sharp V)$, then the eigenvalues of $\Delta_{\mathscr{P}_G^{(p,q)}}$ are of the form

$$1-\frac{F_p\big(\lambda_k;d_G^{(p)}\big)F_q\big(\rho_k;d_G^{(a)}\big)}{d_G^{(p)}\big(d_G^{(p)}-1\big)^{p-1}d_G^{(a)}\big(d_G^{(a)}-1\big)^{q-1}} \qquad (k=1,\dots,\sharp V).$$

Here, a sequence of polynomials $\{F_n(t;d)\}_{n=1}^{\infty}$ is inductively given by the relation

$$\begin{cases} F_{n+1}(t;d) = tF_n(t;d)-(d-1)F_{n-1}(t;d) & (n\geq 2),\\ F_0(t;d)=1, \quad F_1(t;d)=t, \quad F_2(t;d)=t^2-d. \end{cases}$$

By use of this result we are enough to check that our Kähler graphs of product type have commutative adjacency operators and to calculate their eigenvalues.

Theorem 3.1. Let $G=(V,E)$ and $H=(W,F)$ be finite regular graphs whose eigenvalues of transitional Laplacians $\Delta_{\mathscr{P}_G}$ and $\Delta_{\mathscr{P}_H}$ are μ_i $(1\leq i\leq m(=\sharp V))$ and ν_α $(1\leq i\leq n(=\sharp W))$, respectively. Then the adjacency operators $\mathscr{A}_{(G\boxplus H)^{(p)}}, \mathscr{A}_{(G\boxplus H)^{(a)}}$ of the principal and the auxiliary graphs of the Kähler graph $G\boxplus H$ of Cartesian-tensor product type are commutative. Their eigenvalues are given as

$$d_G(1-\mu_i)+d_H(1-\nu_\alpha), \quad d_Gd_H(1-\mu_i)(1-\nu_\alpha) \qquad (1\leq i\leq m, 1\leq\alpha\leq n),$$

respectively.

Proof. By taking the basis $\{\delta_v\}_{v\in V}$, $\{\delta_w\}_{w\in W}$ of characteristic functions of V and W, we represent the adjacency operators $\mathscr{A}_G, \mathscr{A}_H$ by the adjacency matrices $A_G=(a_{ij}^G), A_H=(a_{\alpha\beta}^H)$. Then the adjacency matrices $A_{(G\boxplus H)^{(p)}}=\big(a_{(i,\alpha),(j,\beta)}^{(p)}\big)$, $A_{(G\boxplus H)^{(a)}}=\big(a_{(i,\alpha),(j,\beta)}^{(a)}\big)$ of the principal graph and the auxiliary graph of $G\boxplus H$ are given as

$$a_{(i,\alpha),(j,\beta)}^{(p)} = a_{ij}^G\delta_{\alpha\beta}+\delta_{ij}a_{\alpha\beta}^H, \qquad a_{(i,\alpha),(j,\beta)}^{(a)} = a_{ij}^G a_{\alpha\beta}^H,$$

respectively. Thus we find

$$A_{(G\boxplus H)^{(p)}}A_{(G\boxplus H)^{(a)}} = A_{(G\boxplus H)^{(a)}}A_{(G\boxplus H)^{(p)}}.$$

For $f\in C(V)$ satisfying $\Delta_{\mathscr{P}_G}f=\mu f$ and $g\in C(W)$ satisfying $\Delta_{\mathscr{P}_H}g=\nu g$, we define a function $\varphi_{f,g}\in C(V\times W)$ by $\varphi_{f,g}(v,w)=f(v)g(w)$. Since we have $\mathscr{A}_G f = d_G(1-\mu)f$, $\mathscr{A}_H g = d_H(1-\nu)g$, we get our result by direct computation. □

Theorem 3.2. Let $G=(V,E)$ and $H=(W,F)$ be finite connected regular graphs whose eigenvalues of $\Delta_{\mathscr{P}_G}$ and $\Delta_{\mathscr{P}_H}$ are μ_i $(1\leq i\leq m(=\sharp V))$ and ν_α $(1\leq i\leq n(=\sharp W))$ with $\mu_1=\nu_1=0$. Then the adjacency operators $\mathscr{A}_{(G\boxdot H)^{(p)}}, \mathscr{A}_{(G\boxdot H)^{(a)}}$ of the principal and the auxiliary graphs of the Kähler graph $G\boxdot H$ of Cartesian -complement product type are commutative. Their eigenvalues are given as

$$d_G(1-\mu_i)+d_H(1-\nu_\alpha) \qquad (1\leq i\leq m, 1\leq\alpha\leq n)$$

and

$$\begin{aligned} & d_G(n-d_H-1)+d_H(m-d_G-1),\\ & d_H(1-\nu_\alpha)(m-2d_G-1)-d_G,\\ & d_G(1-\mu_i)(n-2d_H-1)-d_H,\\ & -2d_Gd_H(1-\mu_i)(1-\nu_\alpha)-d_G(1-\mu_i)-d_H(1-\nu_\alpha)\\ & \qquad (2\leq i\leq m, 2\leq\alpha\leq n), \end{aligned}$$

respectively.

Proof. Let $G^c=(V,E^c)$ denote the complement graph of $G=(V,E)$ which is defined so that two distinct vertices are adjacent to each other in G^c if and only if they are not adjacent to each other in G. Similarly we denote by H^c the complement graph of H. We denote by $A_{G^c}=\big(a_{ij}^{G^c}\big)$ and $A_{H^c}=\big(a_{ij}^{H^c}\big)$ the adjacency matrices of G^c and H^c, respectively. They satisfy $A_{G^c}=M-I-A_G$ and $A_{H^c}=M-I-A_H$, where M denotes the matrix all of whose entries are 1 and I denotes the identity matrix.

By use of the same notations as in the proof of Theorem 3.1, the adjacency matrices $A_{(G\boxdot H)^{(p)}}, A_{(G\boxdot H)^{(a)}}$ are given as

$$a_{(i,\alpha),(j,\beta)}^{(p)} = a_{ij}^G\delta_{\alpha\beta}+\delta_{ij}a_{\alpha\beta}^H, \qquad a_{(i,\alpha),(j,\beta)}^{(a)} = a_{ij}^G a_{\alpha\beta}^{H^c}+a_{ij}^{G^c}a_{\alpha\beta}^H,$$

respectively. As we have $a_{ij}^{G^c}=1-\delta_{ij}-a_{ij}^G$, $a_{\alpha\beta}^{H^c}=1-\delta_{\alpha\beta}-a_{\alpha\beta}^H$, we find that

$$A_{(G\boxdot H)^{(p)}}A_{(G\boxdot H)^{(a)}} = A_{(G\boxdot H)^{(a)}}A_{(G\boxdot H)^{(p)}}.$$

Since G is connected, if we take $f_i \in C(V)$ with $\Delta_{\mathscr{P}_G} f_i = \mu_i f_i$, we have $\mathscr{M} f_1 = \sharp V f_1$ and $\mathscr{M} f_i = 0$ for $i \geq 2$ because f_1 is a constant function and f_i $(i \geq 2)$ is orthogonal to f_1. Here $\mathscr{M}$ is the operator corresponding to the matrix M. Since H is also connected, we have similar properties on eigenfunctions g_α for $\Delta_{\mathscr{P}_H}$. Therefore, by taking functions φ_{f_i,g_α} $(i = 1,\dots,m,$ $\alpha = 1,\dots,n)$ we get our result by direct computations. □

Theorem 3.3. Let $G = (V,E)$ and $H = (W,F)$ be finite connected regular graphs whose eigenvalues of $\Delta_{\mathscr{P}_G}$ and $\Delta_{\mathscr{P}_H}$ are μ_i $(1 \leq i \leq m(=\sharp V))$ and ν_α $(1 \leq i \leq n(=\sharp W))$ with $\mu_1 = \nu_1 = 0$. Then the adjacency operators $\mathscr{A}_{(G\Diamond H)^{(p)}}$, $\mathscr{A}_{(G\Diamond H)^{(a)}}$ of the principal and the auxiliary graphs of the Kähler graph $G\Diamond H$ of Cartesian-lexicographical product type are commutative. Their eigenvalues are given as

$$d_G(1-\mu_i) + d_H(1-\nu_\alpha) \qquad (1 \leq i \leq m,\, 1 \leq \alpha \leq n)$$

and

$$\begin{aligned}&d_G(n-1) + d_H(m-1),\\ &d_H(1-\nu_\alpha)(m-1) - d_G,\\ &d_G(1-\mu_i)(n-1) - d_H,\\ &-d_G(1-\mu_i) - d_H(1-\nu_\alpha)\\ &\qquad\qquad (2 \leq i \leq m,\, 2 \leq \alpha \leq n),\end{aligned}$$

respectively.

Proof. By use of the same notations as in the proof of Theorem 3.1, the adjacency matrices $A_{(G\Diamond H)^{(p)}}$, $A_{(G\Diamond H)^{(a)}}$ are given as

$$\begin{aligned}a^{(p)}_{(i,\alpha),(j,\beta)} &= a^G_{ij}\delta_{\alpha\beta} + \delta_{ij}a^H_{\alpha\beta},\\ a^{(a)}_{(i,\alpha),(j,\beta)} &= a^G_{ij}(1-\delta_{\alpha\beta}) + (1-\delta_{ij})a^H_{\alpha\beta},\end{aligned}$$

respectively. Thus we find

$$A_{(G\Diamond H)^{(p)}}A_{(G\Diamond H)^{(a)}} = A_{(G\Diamond H)^{(a)}}A_{(G\Diamond H)^{(p)}}.$$

By taking functions φ_{f_i,g_α} $(i=1,\dots,m,\ \alpha=1,\dots,n)$ with eigenfunctions f_i for $\Delta_{\mathscr{P}_G}$ and g_α for $\Delta_{\mathscr{P}_H}$, we get our result by direct computations. □

Theorem 3.4. Let $G = (V,E)$ and $H = (W,F)$ be finite connected regular graphs whose eigenvalues of $\Delta_{\mathscr{P}_G}$ and $\Delta_{\mathscr{P}_H}$ are μ_i $(1 \leq i \leq m(=\sharp V))$ and ν_α $(1 \leq i \leq n(=\sharp W))$ with $\mu_1 = \nu_1 = 0$. Then the adjacency operators $\mathscr{A}_{(G\circledast H)^{(p)}}$, $\mathscr{A}_{(G\circledast H)^{(a)}}$ of the principal and the auxiliary graphs of the Kähler graph $G \circledast H$ of strong-complement product type are commutative. Their eigenvalues are given as

$$\begin{aligned}&d_G(1-\mu_i) + d_H(1-\nu_\alpha) + d_Gd_H(1-\mu_i)(1-\nu_\alpha)\\ &\qquad (1 \leq i \leq m,\, 1 \leq \alpha \leq n)\end{aligned}$$

and

$$\begin{aligned}&d_G(n-d_H-1) + d_H(m-d_G-1),\\ &d_H(1-\nu_\alpha)(m-2d_G-1) - d_G,\\ &d_G(1-\mu_i)(n-2d_H-1) - d_H,\\ &-2d_Gd_H(1-\mu_i)(1-\nu_\alpha) - d_G(1-\mu_i) - d_H(1-\nu_\alpha)\\ &\qquad\qquad (2 \leq i \leq m,\, 2 \leq \alpha \leq n),\end{aligned}$$

respectively.

Proof. By use of the same notations as in the proof of Theorem 3.2, the adjacency matrices $A_{(G\circledast H)^{(p)}}$, $A_{(G\circledast H)^{(a)}}$ are given as

$$\begin{aligned}a^{(p)}_{(i,\alpha),(j,\beta)} &= a^G_{ij}\delta_{\alpha\beta} + \delta_{ij}a^H_{\alpha\beta} + a^G_{ij}a^H_{\alpha\beta},\\ a^{(a)}_{(i,\alpha),(j,\beta)} &= a^G_{ij}a^{H^c}_{\alpha\beta} + a^{G^c}_{ij}a^H_{\alpha\beta},\end{aligned}$$

respectively. We hence find that

$$A_{(G\circledast H)^{(p)}}A_{(G\circledast H)^{(a)}} = A_{(G\circledast H)^{(a)}}A_{(G\circledast H)^{(p)}}.$$

By taking functions φ_{f_i,g_α} $(i=1,\dots,m,\ \alpha=1,\dots,n)$ with eigenfunctions f_i for $\Delta_{\mathscr{P}_G}$ and g_α for $\Delta_{\mathscr{P}_H}$, we get our result by direct computations. □

Theorem 3.5. Let $G = (V,E)$ and $H = (W,F)$ be finite connected regular graphs whose eigenvalues of $\Delta_{\mathscr{P}_G}$ and $\Delta_{\mathscr{P}_H}$ are μ_i $(1 \leq i \leq m(=\sharp V))$ and ν_α $(1 \leq i \leq n(=\sharp W))$ with $\mu_1 = \nu_1 = 0$. Then the adjacency operators $\mathscr{A}_{(G\spadesuit H)^{(p)}}$, $\mathscr{A}_{(G\spadesuit H)^{(a)}}$ of the principal and the auxiliary graphs of the Kähler graph $G\spadesuit H$ of complement-tensor product type are commutative. Their eigenvalues are given as

$$\begin{aligned}&d_G(n-d_H) + d_H(m-d_G),\\ &d_H(1-\nu_\alpha)(m-2d_G),\\ &d_G(1-\mu_i)(n-2d_H),\\ &-2d_Gd_H(1-\mu_i)(1-\nu_\alpha)\\ &\qquad\qquad (2 \leq i \leq m,\, 2 \leq \alpha \leq n)\end{aligned}$$

and

$$d_Gd_H(1-\mu_i)(1-\nu_\alpha) \qquad (1 \leq i \leq m,\, 1 \leq \alpha \leq n),$$

respectively.

Proof. By use of the same notations as in the proof of Theorem 3.2, the adjacency matrices $A_{(G\spadesuit H)^{(p)}}$, $A_{(G\spadesuit H)^{(a)}}$ are given as

$$\begin{aligned}a^{(p)}_{(i,\alpha),(j,\beta)} &= a^G_{ij}(a^{H^c}_{\alpha\beta} + \delta_{\alpha\beta}) + (a^{G^c}_{ij} + \delta_{ij})a^H_{\alpha\beta},\\ a^{(a)}_{(i,\alpha),(j,\beta)} &= a^G_{ij}a^H_{ab},\end{aligned}$$

respectively. Hence we have

$$A_{(G\spadesuit H)^{(p)}}A_{(G\spadesuit H)^{(a)}} = A_{(G\spadesuit H)^{(a)}}A_{(G\spadesuit H)^{(p)}}.$$

By taking functions φ_{f_i,g_α} $(i = 1,\ldots,m,\ \alpha = 1,\ldots,n)$ with eigenfunctions f_i for $\Delta_{\mathscr{P}_G}$ and g_α for $\Delta_{\mathscr{P}_H}$, we get our conclusion by direct computations. □

Two finite ordinary graphs are said to be combinatorially isospectral (resp. transitionary isospectral) if their combinatorial Laplacians (resp. transitional Laplacians) have the same eigenvalues by taking account of their multiplicities. Correspondingly, we say two Kähler graphs are (p,q)-combinatorially isospectral (resp. (p,q)-transitionary isospectral) if their principal graphs are combinatorially isospectral and their (p,q)-combinatorial Laplacians (resp. $(p.q)$-transitional Laplacians) have the same eigenvalues by taking account of their multiplicities. When two Kähler graphs are regular, they are (p,q)-combinatorially isospectral if and only if they are (p,q)-transitionary isospectral. Hence we just say that they are (p,q)-isospectral. It is known that there are many pair of isospectral regular ordinary connected graphs having the same degrees (see [5]). Our theorems guarantee the following.

Corollary. If we take two pairs G_1, G_2 and H_1, H_2 of isospectral finite connected regular graphs having the same degrees, then we get five pairs of Kähler graphs of product types which are (p,q)-isospectral for every pair (p,q) of relatively prime positive integers.

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