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Asian Options with Harmonic Average

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Abstract: The foreign exchange markets felt the necessity of using contracts written on harmonic averages. They are attractive because are cheaper than the contracts written on the arithmetic average, and make more financial sense than contracts written on geometric averages. The goal of this paper is to consider Asian options and future contracts on harmonic averages of stock values. Since the harmonic average of a set of lognormal random variables does not have an explicit representation, a close-form pricing formula for options and futures is missing. However, we obtain the value of a future contract expressed as an infinite series and provide an approximative formula for it. In the absence of a closed form formula for the value of a call, we obtain an approximation formula for the case when the stock volatility σ is small. This is done by using a variable reduction technique and applying a convolution with the heat kernel of the underlying operator.

Keywords: Asian options, harmonic average, Black-Scholes, put-call parity, option pricing

1 Introduction

Asian options are option contracts written on a certain type of average value of the stock over some given period of time. These types of options have the advantage of reducing the risk of market manipulation of the underlying stock near or at maturity (see Hull [18]). Since they are also cheaper than plain vanilla European options, they are desirable for time-dependent insurance contracts. There are several types of continuous averages on the stock S_t that can be considered:

Arithmetic average: $A_T = \frac{1}{T}\int_0^T S_t\,dt$,

Geometric average: $G_T = e^{\frac{1}{T}\int_0^T \ln S_t}\,dt$,

Harmonic average: $\mathscr{H}_T = \frac{T}{\int_0^T \frac{1}{S_t}\,dt}$.

It is well-known that the arithmetic average dominates the geometric average, which in turn dominates the harmonic average, i.e.,

$$A_T \geq G_T \geq \mathscr{H}_T.$$

This implies that if one considers call options on the aforementioned three types of averages, then the cheapest call option is on the harmonic mean and the most expensive corresponds to the arithmetic mean.

Asian contracts on arithmetic average had been first introduced by Boyle and Emanuel [4] in 1980, and since then there has been a great interest and scholarly effort dedicated to their study.

Close-form pricing formulas exist for Asian options on geometric continuous averages (Kemma and Vorst [19]). However, no exact pricing formulas have been found in the case of arithmetic average. Their pricing techniques are based either on numerical methods or on analytical approximations. The former contains approaches using binomial trees (Cho and Lee [9], Ritchken and Vijh [25], Tan and Vetsal [28]), numerical schemes (Alzairy et al. [1], Barraquand and Prudet [2], Dewynne and Wilmott [10]), fast Fourier transform (Caverhill and Clewlow [7]), or Monte-Carlo method ([13] Fu et al. [13]). The latter approach involves analytical and pseudo-analytical approximations involving Laplace transforms or lower bounds (Geman and Yor [14], Rogers and Shi [26], Levy and Turnbull [21], Turnbull and Wakeman [29], Milevsky and Posner [22], etc.) We also note that some relatively recent work provided new insights into the pricing problem: Linetsky [20] found a spectral expansion of the Asian option price

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involving Wittaker functions; Bayraktar and Xing [3] found an iterative numerical method for solving the corresponding partial integro-differential equation for Asian options; Hoogland and Neumann [17] used symmetry arguments to solve the pricing differential equation; Fouque and Han [12] extended the study to stochastic volatility models; Vecer and Xu [30] modeled the case of volatility jumps. The reader can also find more theoretical classical details in the books of Hull [18] and Wilmott [32]. For implementations with Mathematica the reader is referred to Shaw [27].

Even if in practice the underlying asset is the arithmetic average of the stock, it is always theoretically possible to consider Asian options on other types of averages, such as harmonic averages, which are apparently more mathematically challenging. This is the reason why the academic literature regarding harmonic averages is very limited. There are two types of methods available in the literature: one is using Monte-Carlo simulations and approximations methods, see Chen and Wan [8]; and another using a simplification to a partial differential equation in one spatial variable after a numeraire change and using the time reversal argument, see the recent paper of Vecer [31]. More details regarding contracts on a harmonic average can be found in Wystup [33].

Asian options on a harmonic average of the stock are attractive because of their reduced price (the call option on a harmonic average is cheaper than a call on the other two previously discussed averages); however, this type of options do not trade at the present time.

The present paper investigates Asian options on harmonic average, the main concern being pricing calls, puts and future contracts with the underlying asset being the continuous harmonic average of the stock. Since the harmonic average of a set of lognormal random variables does not have an explicit representation, a close-form pricing formula for options and futures is missing. However, we obtain the value of a future contract expressed as an infinite series and provide an approximative formula for it. In the absence of a closed form formula for the value of a call, we obtain an approximation formula for the case when the stock volatility σ is small. This is done by using a variable reduction technique and applying a convolution with the heat kernel of the underlying operator.

By the harmonic average of N numbers we understand the inverse of the arithmetic average of the inverses of the numbers. We shall use a continuous analog of this concept since the sampling is considered continuous. Given the non-linear behavior of the harmonic average, we are looking just for an approximate pricing formula for the case when the stock volatility is small (the semi-classical asymptotics as $\sigma \to 0$).

The main reason for which the literature on Asian options on harmonic average is rather limited is certainly related with the apparent lack of financial meaning of the harmonic mean. However, we shall present a few links with the field of finance.

As pointed out in Vecer [31], the contracts written on the harmonic average of the underlying price are very popular in the foreign exchange markets, where they are used for protection purposes. The advantage of using harmonic averages is twofold: the corresponding calls are cheaper and the harmonic average is more stable than the arithmetic average (see Chen and Wan [8]). Moreover, there is a link between the options on harmonic average and the previously studied Australian option, see Vecer [31], Handley [15], Moreno and Navas [23], or Ewald et al. [11]. It is worth noting that there is more financial intuition related to the harmonic average rather than to the geometric average. This financial sense comes from the fact that the inverse of the harmonic average is the average of the inverse of the stock prices.

On the other hand, the harmonic mean has a well known application in electrical circuits: the total resistance of a set of parallel resistors is obtained by adding up the reciprocals of the individual resistance values, and then taking the reciprocal of their total. More precisely, if R_1 and R_2 are the resistances of two parallel resistors, then the total resistance is computed by the formula $R_T = \frac{1}{\frac{1}{R_1}+\frac{1}{R_2}} = \frac{1}{2}\mathscr{H}(R_1,R_2)$, which is half the harmonic mean of the value of resistors' resistances.

If we accept the hypothesis that the financial market is similar to an electrical circuit (whose elements have time-dependent parameters), then the flow of money corresponds to the electrical current, while the financial institutions correspond to elements of the electrical circuit. If one can meaningfully define the concept of market resistance of a financial institution, then the total market resistance of parallel financial institutions is computed using a harmonic average. By "parallel institutions" we mean institutions with the same source of inflow of money and the same out-flow pool of capital (this mimics the way in which two parallel resistors have common inflow and outflows of current).

A second observation that can sustain the comparison hypothesis between the financial market and an electrical circuit is given by the similarity of their governing laws: the motion of the electrical current in an electrical transmission line is described by the telegrapher's equations (introduced by Heaviside in 1880). These are similar to the Black-Scholes equation satisfied by the price of a derivative.

Last, but not least, finding a pricing formula for an Asian derivative on a harmonic average of stocks is a mathematically challenging problem, which deserves the study effort, regardless whether it can have a more or less immediate practical application.

The plan of the paper is as follows: After a brief introduction to the continuous harmonic average of stock values in Section 2, the next two sections deal with the Black-Scholes equation satisfied by derivatives on harmonic average and the boundary conditions associated with strike Asian options. Section 5 provides a pricing formula for forward contracts on harmonic averages, the pricing formula being expressed as an infinite series with time variable coefficients. The put-call parity is presented in Section 6. Section 7 provides a semi-classical asymptotics formula for the price of the call in the case when the stock volatility is small. The main result of this section is presented in Theorem 2. The discussion presented in Section 7 explains the limitations of the method in improving the accuracy.

2 The harmonic average of stocks

We denote by S_{t_i} the values of a stock evaluated discretely at the sampling dates t_i, $i = 1, \ldots, N$, with $t_i \in [0,T]$, where T stands for the maturity time of the option. The harmonic average of the stock values S_{t_i} is defined as

$$\mathscr{H}(t_1,\cdots,t_N) = \frac{N}{\sum_{i=1}^{N} \frac{1}{S_{t_i}}}.$$

If consider equidistant sampling dates $t_j = \frac{jT}{N}$, the continuously sampled harmonic average of the stock price between 0 and T is obtained by taking the limit $N \to \infty$ in the previous formula

$$\lim_{N\to\infty} \mathscr{H}(t_1,\cdots,t_N) = \lim_{N\to\infty} \frac{N}{\sum_{i=1}^{N} \frac{1}{S_{t_i}}} = \lim_{N\to\infty} \frac{T}{\sum_{i=1}^{N} \frac{1}{S_{t_i}} \frac{T}{N}} = \frac{T}{\int_0^T \frac{1}{S_\tau}\, d\tau}.$$

Therefore, it makes sense to define the continuously sampled harmonic average as

$$\mathscr{H}_t = \frac{t}{\int_0^t \frac{1}{S_\tau}\, d\tau}. \tag{1}$$

We can equivalently write $\mathscr{H}_t = \frac{t}{I_t}$, where $I_t = \int_0^t \frac{1}{S_\tau}\, d\tau$ satisfies

$$dI_t = \frac{1}{S_t}\, dt, \qquad I_0 = 0, \qquad d\Big(\frac{1}{I_t}\Big) = -\frac{1}{S_t I_t^2}\, dt.$$

In the above relations S_t denotes the stock price at time t having the constant return μ and volatility σ, i.e.

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t.$$

The process $\mathscr{H}_t$ satisfies the following stochastic differential equation

$$d\mathscr{H}_t = t\, d\Big(\frac{1}{I_t}\Big) + \frac{1}{I_t}\, dt + dt\, d\Big(\frac{1}{I_t}\Big) = \frac{1}{I_t}\Big(1 - \frac{t}{S_t I_t}\Big)\, dt = \frac{1}{t}\mathscr{H}_t\Big(1 - \frac{\mathscr{H}_t}{S_t}\Big)\, dt.$$

This can be written equivalently as the following integral equation:

$$\mathscr{H}_t = S_0 + \int_0^t \frac{1}{\tau}\mathscr{H}_\tau\Big(1 - \frac{\mathscr{H}_\tau}{S_\tau}\Big)\, d\tau.$$

The previous integral equation cannot be solved explicitly for $\mathscr{H}_t$. However, some straightforward properties of $\mathscr{H}_t$ are given in the following:

(1) $\mathscr{H}_t > 0, \qquad \forall t > 0;$

(2) $\mathscr{H}_0 = S_0 > 0;$

(3) $\frac{\mathscr{H}_t}{t}$ is a decreasing function of t.

The condition (1) follows from the fact that $S_t > 0$. The initial condition given by (2) is obtained as an application of the L'Hospital rule

$$\mathscr{H}_0 = \lim_{t\searrow 0} \mathscr{H}_t = \lim_{t\searrow 0} \frac{t}{I_t} = S_0.$$

Taking the increment and using Ito's formula we obtain (3)

$$d\Big(\frac{\mathscr{H}_t}{t}\Big) = \frac{t\, d\mathscr{H}_t - \mathscr{H}_t\, dt}{t^2} = -\frac{1}{t^2}\frac{\mathscr{H}_t}{S_t} dt < 0.$$

It is worthy to note that the random variable $\mathscr{H}_t$ is neither normally nor lognormally distributed, and an exact expression for the moments $\mathbb{E}[\mathscr{H}_t]$ and $\mathbb{E}[\mathscr{H}_t^2]$ is difficult to obtain in a closed form.

3 The Black-Scholes equation

If consider an Asian option, whose value $V(S_t, t, \mathscr{H}_t)$ depends on the variables $S_t, \mathscr{H}_t, t$, then a standard use of the non-arbitrage argument and Ito's formula leads to the following Black-Scholes-type equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{1}{t}\mathscr{H}_t\Big(1 - \frac{\mathscr{H}_t}{S_t}\Big)\frac{\partial V}{\partial \mathscr{H}_t} = rV.$$

Unfortunately, this equation is more complicated than a standard Black-Scholes equation. This is the reason why we consider another approach. Since $\mathscr{H}_t$ is the quotient of t and I_t, we shall consider the value of an Asian derivative as $V(S_t, t, I_t)$, depending on the variables S_t, I_t, t. By standard methods we can show that its associated Black-Scholes-type equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + rS_t\frac{\partial V}{\partial S_t} + \frac{1}{S_t}\frac{\partial V}{\partial I_t} = rV. \tag{2}$$

We are interested in solving the equation (2) subject to the following call or put boundary conditions, where the strike is replaced by the harmonic average

$$\text{Call payoff:} \quad C(S_T, T) = \max\Big(S_T - \mathscr{H}_T, 0\Big), \tag{3}$$

$$\text{Put payoff:} \quad P(S_T, T) = \max\Big(\mathscr{H}_T - S_T, 0\Big), \tag{4}$$

with

$$\mathscr{H}_T = \frac{T}{\int_0^T \frac{1}{S(\tau)}\, d\tau} = \frac{T}{I_T}.$$

Since we can write

$$C(S_T,T) = S_T \max\Big(1-\frac{T}{S_T I_T},0\Big), \quad (5)$$

$$P(S_T,T) = S_T \max\Big(\frac{T}{S_T I_T}-1,0\Big), \quad (6)$$

then both payoffs are of type $S_T\Lambda(R_T,T)$, with $R_T = S_T I_T$. Then it makes sense to look for solutions for the Black-Scholes equation (2) of the same type, i.e., $V(S_t,t,H_t) = S_t W(R_t,t)$, with $R_t = S_t I_t$. This will reduce the variables in equation (2). Using the chain rule we have

$$\frac{\partial V}{\partial t} = S_t\frac{\partial W}{\partial t}, \qquad \frac{\partial V}{\partial I_t} = S_t^2\frac{\partial W}{\partial R_t}$$
$$\frac{\partial V}{\partial S_t} = W(t,R) + S_t I_t\frac{\partial W}{\partial R_t}$$
$$\frac{\partial^2 V}{\partial S_t^2} = 2I_t\frac{\partial W}{\partial R_t} + S_t I_t^2\frac{\partial^2 W}{\partial R_t^2}.$$

Substituting in (2) yields[1]

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 R_t^2\frac{\partial^2 W}{\partial R_t^2} + \Big(1+(r+\sigma^2)R_t\Big)\frac{\partial W}{\partial R_t} = 0. \quad (7)$$

The advantage of this equation is that it depends only on two variables, t and R_t. The final boundary conditions (5)–(6) become

$$W_C(R_T,T) = \max\Big(1-\frac{T}{R_T},0\Big), \quad (8)$$

$$W_P(R_T,T) = \max\Big(\frac{T}{R_T}-1,0\Big). \quad (9)$$

The formal condition corresponds to a call and the latter to a put.

4 Boundary conditions

In order to have uniqueness for the equation (7) with final boundary conditions (8)–(9), we shall impose some boundary conditions on the solution $W(R,t)$ as $R\to 0$ and $R\to\infty$. When $R\to\infty$, from (8)–(9) yields

$$W_C(\infty,T) = 1, \qquad W_P(\infty,T) = 0.$$

In order to investigate the behavior as $R\to 0$, we first find the stochastic equation followed by the process R_t.

$$dR_t = d(S_t I_t) = S_t\,dI_t + I_t\,dS_t + \underbrace{dI_t\,dS_t}_{=0}$$
$$= dt + \mu R_t\,dt + \sigma R_t\,dW_t.$$

[1] The same reduction technique applied for continuous arithmetic average leads to the similar equation

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 R_t^2\frac{\partial^2 W}{\partial R_t^2} + (1-rR_t)\frac{\partial W}{\partial R_t} = 0.$$

Hence R_t satisfies

$$dR_t = (1+\mu R_t)dt + \sigma R_t\,dW_t, \qquad R_0 = 0. \quad (10)$$

Since the coefficients do not increase faster than a linear function, the aforementioned equation has only one solution R_t (see Øksendal [24], p. 68). The probability density of R_t is given by the heat kernel of the generator operator of R_t

$$L = \frac{1}{2}\sigma^2 x^2\frac{d^2}{dx^2} + (1+\mu x)\frac{d}{dx}. \quad (11)$$

Finding the heat kernel of (11) is as difficult as solving the reduced Black-Scholes equation (7). Integrating in the aforementioned equation and taking the expectation operator yields the integral equation

$$\mathbb{E}(R_t) = \int_0^t (1+\mu E(R_s))\,ds,$$

with the solution $\mathbb{E}(R_t) = \frac{e^{\mu t}-1}{\mu}$, so $\mathbb{E}(R_t)\to\infty$, as $t\to\infty$.

Making $R\to 0$ in (7) yields the boundary condition

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial R} = 0 \quad \text{at} \quad R=0,$$

which says that $\Delta_W + \theta_W = 0$ at $R=0$ (where Δ_W and θ_W stand for the delta and theta of the option).

The unique solution of the equation

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 R_t^2\frac{\partial^2 W}{\partial R_t^2} + \Big(1+(r+\sigma^2)R_t\Big)\frac{\partial W}{\partial R_t} = 0$$
$$W(R,T) = \max\Big(1-\frac{T}{R},0\Big)$$
$$W(\infty,t) = 1$$
$$\frac{\partial W}{\partial t}_{|R=0} = -\frac{\partial W}{\partial R}_{|R=0}$$

will lead to a call option, while

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 R_t^2\frac{\partial^2 W}{\partial R_t^2} + \Big(1+(r+\sigma^2)R_t\Big)\frac{\partial W}{\partial R_t} = 0$$
$$W(R,T) = \max\Big(\frac{T}{R}-1,0\Big)$$
$$W(\infty,t) = 0$$
$$\frac{\partial W}{\partial t}_{|R=0} = -\frac{\partial W}{\partial R}_{|R=0}$$

corresponds to a put.

5 Pricing a Forward Contract

A forward contract on the harmonic average is a derivative satisfying equation (2) that pays at maturity

$F_T = S_T - \mathscr{H}_T$. Its value $F(t,S_t,I_t)$ satisfies the final condition problem

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial S_t^2} + rS_t \frac{\partial F}{\partial S_t} + \frac{1}{S_t}\frac{\partial F}{\partial I_t} = rF \tag{12}$$

$$F(T,S_T,I_T) = S_T - \frac{T}{I_T}. \tag{13}$$

Since the payoff can be written as $S_T - \frac{T}{I_T} = S_T\left(1 - \frac{T}{R_T}\right)$, with $R_T = S_T I_T$, it makes sense to look for a solution of the form

$$F(t,S_t,I_t) = S_t Y(t,R_t), \tag{14}$$

where $Y(t,R)$ satisfies

$$\frac{\partial Y}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 Y}{\partial R^2} + \left(1 + (1+\sigma^2)R\right)\frac{\partial Y}{\partial R} = 0 \tag{15}$$

$$Y(T,R) = 1 - \frac{T}{R}. \tag{16}$$

We look for a solution in the form of a Laurent series in R with coefficients functions of t

$$Y(t,R) = a_0(t) + \sum_{j\geq 1} a_j(t)R^j + \sum_{j\geq 1} a_{\_j}(t)R^{-j}, \tag{17}$$

Substituting (17) into equation (15) and equating the coefficients of similar powers of R yields the following infinite system of differential equations

$$\begin{aligned}
a_0'(t) + a_1(t) &= 0\\
a_1'(t) + 2a_2 + (\sigma^2 + r)a_1(t) &= 0\\
a_2'(t) + 3a_3(t) + 2(\sigma^2 + r)a_2(t) + \sigma^2 a_2(t) &= 0\\
\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots & \ \cdots\\
a_{_1}'(t) - (\sigma^2 + r)a_{_1}(t) + \sigma^2 a_{_1}(t) &= 0\\
a_{_2}'(t) - a_{_1}(t) - 2(\sigma^2 + r^2)a_{_2}(t) + 3\sigma^2 a_{_2}(t) &= 0\\
\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots & \ \cdots
\end{aligned}$$

The payoff condition implies the following final conditions

$$a_0(T) = 1,\ a_{\_1}(T) = -T,\ a_j(T) = a_{\_j}(T) = 0,\ \forall j \neq -1, 0.$$

We start solving the system from the equation in solely $a_{\_1}(t)$ and obtain

$$a_{\_1}(t) = -Te^{-r(T-t)}.$$

Substituting in the next equation we obtain the linear equation for $a_{\_2}(t)$

$$a_{\_2}(t) + (\sigma^2 - 2r)a_{\_2}(t) = -Te^{-r(T-t)}, \qquad a_{\_2}(T) = 0.$$

with the solution

$$a_{\_2}(t) = \frac{T}{\sigma^2 - r}\left(e^{(\sigma^2-2r)(T-t)} - e^{-r(T-t)}\right).$$

In order to solve for $a_0(t)$, we choose the simplest function satisfying $a_0(T) = 1$, which is $a_0(t) = 1$. Solving recursively for $a_j(t)$, $j \geq 1$, we obtain $a_1(t) = a_2(t) = \cdots = 0$. Therefore (17) becomes

$$\begin{aligned}
&Y(t,R) = 1 + \sum_{j\geq 1} a_{_j}(t)R^{-j}\\
&\approx 1 - e^{-r(T-t)}\frac{T}{R}\\
&+ \frac{T}{\sigma^2 - r}\left(e^{(\sigma^2-2r)(T-t)} - e^{-r(T-t)}\right)\frac{1}{R^2}.
\end{aligned} \tag{18}$$

Substituting in (14) provides the approximate value of the forward contract $F_t = F(t,S_t,I_t)$

$$\begin{aligned}
F(t,S_t,I_t) &= S_t Y(t,I_t S_t) = S_t\Big(1 + \sum_{j\geq 1} a_{_j}(t)(S_t I_t)^{-j}\Big)\\
&\approx S_t - e^{-r(T-t)}\frac{T}{I_t}\\
&\quad + \frac{T}{\sigma^2 - r}\left(e^{(\sigma^2-2r)(T-t)} - e^{-r(T-t)}\right)\frac{1}{S_t I_t^2}.
\end{aligned}$$

This can be also written in terms of harmonic average $\mathscr{H}_t = t/I_t$ as

$$\begin{aligned}
F_t &\approx S_t - e^{-r(T-t)}\mathscr{H}_t\frac{T}{t}\\
&\quad + \frac{1}{S_t}\frac{\mathscr{H}_t^2}{t^2}T\left(e^{(\sigma^2-2r)(T-t)} - e^{-r(T-t)}\right).
\end{aligned} \tag{19}$$

We can also approach solving the aforementioned system using the exponential of a matrix. We recall that the exponential of an $n \times n$ matrix A is the $n \times n$ matrix defined by $e^A = \sum_{k\geq 0}\frac{1}{k!}A^k = \mathbb{I}_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots$. In order to apply the aforemention method, the ODE system for the coefficients $a_j(t)$, $j \geq 1$, is written in the following matrix form

$$\frac{d}{dt}\begin{pmatrix} a_0\\ a_1\\ a_2\\ \vdots \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots\\ 0 & -(r+\sigma^2) & -2 & 0 & \cdots\\ 0 & 0 & -(3\sigma^2+2r)-3 & 0 & \cdots\\ . & . & . & . & . \end{pmatrix}\begin{pmatrix} a_0\\ a_1\\ a_2\\ \vdots \end{pmatrix},$$

or equivalently, as $X(t)' = AX(t)$, with the final condition $X(T) = \begin{pmatrix} 1\\ 0\\ 0\\ \vdots \end{pmatrix}$. Since $X(t) = \begin{pmatrix} 1\\ 0\\ 0\\ \vdots \end{pmatrix}$ obviously verifies the system, by the uniqueness theorem of linear ODE systems, $X(t)$ is the unique solution of the system. This implies

$$a_1(t) = 1,\ a_2(t) = a_3(t) = \cdots = 0.$$

The ODE system for the coefficients $a_{\_j}(t)$, $j \geq 1$, can be written in the matrix form

$$\frac{d}{dt}\begin{pmatrix} a_{\_1}(t)\\ a_{\_2}(t)\\ \vdots \end{pmatrix} = \begin{pmatrix} r & 0 & 0 & 0 & \cdots\\ 1 & 2r-\sigma^2 & 0 & 0 & \cdots\\ 0 & . & . & . & \cdots \end{pmatrix}\begin{pmatrix} a_{\_1}(t)\\ a_{\_2}(t)\\ \vdots \end{pmatrix},$$

or equivalently $Y(t)' = BY(t)$, with the final condition

$$Y(T) = \begin{pmatrix} -T \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

The solution is given by

$$Y(t) = e^{B(T-t)}X(T) = -T\,\mathrm{col}^1(e^{-A(T-t)}),$$

where $\mathrm{col}^1(M)$ denotes the first column of matrix M.

6 The put-call parity

A put-call parity for the Asian strike options on harmonic average can be developed as in the following. Since we have

$$W_C(R_T,T) - W_P(R_T,T) = 1 - \frac{T}{R_T},$$

using a non-arbitrage opportunity argument yields

$$W_C(R_t,t) - W_P(R_t,t) = Y_t, \qquad 0 \le t \le T \tag{20}$$

where Y_t is a security that satisfies equation (7), paying at maturity $Y_T = 1 - \frac{T}{R_T}$, and having the value computed in the previous section, see formula (18).

7 Pricing the Call

In the virtue of the put-call parity it suffices to price only the call. Making the substitutions

$$\tau = T - t, \quad y = (\ln R)/\sigma, \quad U(\tau, y) = W(R,t)$$

we obtain

$$\sigma R \frac{\partial W}{\partial R} = \frac{\partial U}{\partial y}$$
$$\frac{1}{2}\sigma^2 R^2 \frac{\partial^2 W}{\partial R^2} = \frac{1}{2}\frac{\partial^2 U}{\partial y^2} - \frac{\sigma}{2}\frac{\partial U}{\partial y}.$$

Therefore, the final condition problem for W, which is given by (7 –8), becomes the following initial problem for $U(\tau,y)$

$$\frac{\partial}{\partial \tau}U = \frac{1}{2}\frac{\partial^2}{\partial y^2}U + \frac{1}{\sigma}\Big(e^{-\sigma y} + r + \frac{\sigma^2}{2}\Big)\frac{\partial}{\partial y}U \tag{21}$$
$$U(0,y) = \max(1 - Te^{-\sigma y}, 0). \tag{22}$$

The problem (21)–(22) is almost impossible to be solved explicitly. However, there are good chances to find an approximation of the solution in the case of small volatility σ. Since for $\sigma \to 0$ we have

$$\frac{1}{\sigma}\Big(e^{-\sigma y} + r + \frac{\sigma^2}{2}\Big) = \frac{1+r}{\sigma} - y + O(\sigma), \tag{23}$$

the leading term, for σ small, is $\frac{1+r}{\sigma} - y$. Then the equation (21) can be approximated by

$$\frac{\partial}{\partial \tau}U = \frac{1}{2}\frac{\partial^2}{\partial y^2}U + \Big(\frac{1+r}{\sigma} - y\Big)\frac{\partial}{\partial y}U.$$

In order to simplify the equation, we substitute $x = y - \frac{1+r}{\sigma}$, $u = \tau/2$ and $Z(u,x) = U(\tau,y)$ to obtain the following simplified version of the problem $(21-22)$

$$\frac{\partial}{\partial u}Z = \Big(\frac{\partial^2}{\partial x^2} - 2x\frac{\partial}{\partial x}\Big)Z \tag{24}$$
$$Z(0,x) = \max(1 - Te^{-(\sigma x + 1 + r)}, 0) := \phi(x) \tag{25}$$

Let $K(x_0,x;u)$ denote the heat kernel of the operator $L = \frac{\partial^2}{\partial x^2} - 2x\frac{\partial}{\partial x}$, i.e. $K(x_0,x;u)$ verifies the equation (24) and satisfies the limit condition $\lim_{u\searrow 0} K(x_0,x;u) = \delta_{x_0}$, where δ_{x_0} denotes the Dirac distribution.

Then the solution of the initial problem (24)–(25) can be written as

$$\begin{aligned} Z(u,x) &= \int_{-\infty}^{\infty} K(x,y;u)\phi(y)\,dy \\ &= \int_{(\ln T - r - 1)/\sigma}^{\infty} K(x,y;u)\Big(1 - Te^{-(\sigma y + 1 + r)}\Big)\,dy \\ &= \int_{(\ln T - r - 1)/\sigma}^{\infty} K(x,y;u)\,dy \\ &\quad - Te^{-(1+r)}\int_{(\ln T - r - 1)/\sigma}^{\infty} e^{-\sigma y}K(x,y;u)\,dy \\ &= I_1(u,x) - Te^{-(1+r)}I_2(u,x). \end{aligned} \tag{26}$$

Therefore, it suffices to find a formula for the heat kernel $K(x_0,x;u)$ and then to compute integrals $I_1(u,x)$ and $I_2(u,x)$. According to Theorem 10.28 of Calin and Chang [5], p. 223, we have:

Theorem 1. *Let $a \in \mathbb{R}$. The heat kernel, $K = K(x_0,x;u)$, of the operator $\partial_x^2 - ax\partial_x$ is given by*

$$K = \frac{1}{\sqrt{4\pi u}}\sqrt{\frac{au}{\sinh(au)}}\, e^{-\frac{1}{4u}\frac{au}{\sinh(au)}\left[(x^2 + x_0^2)\cosh(au) - 2xx_0\right]},$$

with $u > 0$.

Choosing $a = 2$, the above formula becomes

$$K(x,y;u) = \frac{1}{\sqrt{2\pi\sinh(2u)}}\, e^{-\frac{1}{2\sinh(2u)}\left[(x^2+y^2)\cosh(2u) - 2xy\right]}. \tag{27}$$

Using (27) we shall be able to compute explicitly the integrals $I_1(u,x)$ and $I_2(u,x)$ contained in the expression (26). This will be done using standard techniques involving Gaussian integrals and completion to a square. We shall start with the computation of the integral $I_1(u,x)$.

$$\begin{aligned}
I_1(u,x) &= \int_{(\ln T-r-1)/\sigma}^{\infty} K(x,y;u)\,dy \\
&= \frac{e^{-\frac{x^2}{2}\coth(2u)}}{\sqrt{2\pi\sinh(2u)}}\times \\
&\int_{(\ln T-r-1)/\sigma}^{\infty} e^{-\frac{1}{2}\coth(2u)\left(y^2-\frac{2x}{\cosh(2u)}y\right)}\,dy \\
&= \frac{e^{-\frac{x^2}{2}\coth(2u)}e^{\frac{1}{2}\coth(2u)\frac{x^2}{\cosh^2(2u)}}}{\sqrt{2\pi\sinh(2u)}}\times \\
&\int_{(\ln T-r-1)/\sigma}^{\infty} e^{-\frac{1}{2}\coth(2u)\left(y-\frac{x}{\cosh(2u)}\right)^2}\,dy \\
&= \frac{e^{-\frac{x^2}{2}\tanh(2u)}}{\sqrt{2\pi\sinh(2u)}}\times \\
&\int_{\sqrt{\coth(2u)}\left(\frac{\ln T-r-1}{\sigma}-\frac{x}{\cosh(2u)}\right)}^{\infty} e^{-\frac{1}{2}z^2}\frac{1}{\sqrt{\coth(2u)}}\,dz \\
&= \frac{1}{\sqrt{\cosh(2u)}}e^{-\frac{x^2}{2}\tanh(2u)}\frac{1}{\sqrt{2\pi}}\times \\
&\int_{\sqrt{\coth(2u)}\left(\frac{\ln T-r-1}{\sigma}-\frac{x}{\cosh(2u)}\right)}^{\infty} e^{-\frac{1}{2}z^2}\,dz \\
&= \frac{1}{\sqrt{\cosh(2u)}}e^{-\frac{x^2}{2}\tanh(2u)}\times \\
&\mathbf{N}\left(\sqrt{\coth(2u)}\left(\frac{x}{\cosh(2u)}-\frac{1}{\sigma}(\ln T-r-1)\right)\right),
\end{aligned}$$

where $\mathbf{N}(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{1}{2}z^2}\,dz$ denotes the probability function of a standard normal random variable. Hence

$$I_1(u,x)=\frac{1}{\sqrt{\cosh(2u)}}e^{-\frac{x^2}{2}\tanh(2u)}\mathbf{N}(\delta_1), \tag{28}$$

where

$$\delta_1=\sqrt{\coth(2u)}\left(\frac{x}{\cosh(2u)}-\frac{\ln T-r-1}{\sigma}\right). \tag{29}$$

We shall compute next the integral $I_2(u,x)$.

$$\begin{aligned}
I_2(u,x) &= \int_{(\ln T-r-1)/\sigma}^{\infty} e^{-\sigma y}K(x,y;u)\,dy \\
&= \frac{e^{-\frac{x^2}{2}\coth(2u)}}{\sqrt{2\pi\sinh(2u)}}\times \\
&\int_{(\ln T-r-1)/\sigma}^{\infty} e^{-\frac{1}{2}\coth(2u)\left(y^2-\frac{2x}{\cosh(2u)}y\right)-\sigma y}\,dy \\
&= \frac{e^{-\frac{x^2}{2}\coth(2u)}}{\sqrt{2\pi\sinh(2u)}}\times \\
&\int_{(\ln T-r-1)/\sigma}^{\infty} e^{-\frac{1}{2}\coth(2u)\left(y^2-\frac{2(x-\sigma\sinh(2u))}{\cosh(2u)}y\right)}\,dy \\
&= \frac{e^{-\frac{x^2}{2}\coth(2u)}}{\sqrt{2\pi\sinh(2u)}}e^{\frac{1}{2}\coth(2u)\frac{(x-\sigma\sinh(2u))^2}{\cosh^2(2u)}}\times \\
&\int_{(\ln T-r-1)/\sigma}^{\infty} e^{-\frac{1}{2}\coth(2u)\left(y-\frac{(x-\sigma\sinh(2u))}{\cosh(2u)}\right)^2}\,dy \\
&= \frac{e^{-\frac{x^2}{2}\coth(2u)}e^{\frac{(x-\sigma\sinh(2u))^2}{\sinh(4u)}}}{\sqrt{2\pi\sinh(2u)\coth(2u)}}\times \\
&\int_{\sqrt{\coth(2u)}\left(\frac{\ln T-r-1}{\sigma}-\frac{x-\sigma\sinh(2u)}{\cosh(2u)}\right)}^{\infty} e^{-\frac{1}{2}z^2}\,dz \\
&= \frac{e^{\frac{(x-\sigma\sinh(2u))^2}{\sinh(4u)}-\frac{x^2}{2}\coth(2u)}}{\sqrt{\cosh(2u)}}\times \\
&\mathbf{N}\left(\sqrt{\coth(2u)}\left(\frac{x-\sigma\sinh(2u)}{\cosh(2u)}-\frac{\ln T-r-1}{\sigma}\right)\right).
\end{aligned}$$

Hence

$$I_2(u,x)=\frac{1}{\sqrt{\cosh(2u)}}e^{\frac{(x-\sigma\sinh(2u))^2}{\sinh(4u)}-\frac{x^2}{2}\coth(2u)}\mathbf{N}(\delta_2), \tag{30}$$

where

$$\delta_2=\sqrt{\coth(2u)}\left(\frac{x-\sigma\sinh(2u)}{\cosh(2u)}-\frac{\ln T-r-1}{\sigma}\right). \tag{31}$$

Working back through the previous substitutions

$$\begin{aligned}
x &= y-\frac{1+r}{\sigma}, & y &= \frac{\ln R}{\sigma}, & R &= IS, \\
u &= \frac{\tau}{2}, & \tau &= T-t, & I &= \frac{t}{\mathscr{H}},
\end{aligned}$$

and using that the price of the call is given by

$$C=V(S,t,\mathscr{H})=SW(R,t)=SU(\tau,y)=SZ(u,x),$$

we conclude with the following result.

Theorem 2. *For σ small, the price of a strike call option on a harmonic average at time t is approximated by*

$$C = S_t Z\left(\frac{T-t}{2}, \frac{1}{\sigma}\left(\ln\left(\frac{tS_t}{\mathscr{H}_t}\right) - r - 1\right)\right), \tag{32}$$

where

$$Z(u,x) = I_1(u,x) - Te^{-(1+r)} I_2(u,x),$$

with

$$I_1(u,x) = \frac{1}{\sqrt{\cosh(2u)}} e^{-\frac{x^2}{2}\tanh(2u)} \mathbf{N}(\delta_1)$$
$$I_2(u,x) = \frac{1}{\sqrt{\cosh(2u)}} e^{\frac{(x-\sigma\sinh(2u))^2}{\sinh(4u)} - \frac{x^2}{2}\coth(2u)} \mathbf{N}(\delta_2),$$

and

$$\delta_1 = \sqrt{\coth(2u)}\left(\frac{x}{\cosh(2u)} - \frac{\ln T - r - 1}{\sigma}\right),$$
$$\delta_2 = \sqrt{\coth(2u)}\left(\frac{x - \sigma\sinh(2u)}{\cosh(2u)} - \frac{\ln T - r - 1}{\sigma}\right).$$

8 Discussion

A natural question is whether we can obtain a better approximation for the price of the call following the same lines as before. Can the approximation (23) be made more accurate while all the previous computations can be still carried out in an explicit form? If we go further with one more degree of accuracy in (23), we obtain

$$\frac{1}{\sigma}\left(e^{-\sigma y} + r + \frac{\sigma^2}{2}\right) = \frac{\sigma}{2}y^2 - y + \left(\frac{1+r}{\sigma} + \frac{\sigma}{2}\right) + O(\sigma^2),\ \sigma \to 0. \tag{33}$$

It is worth noting that the expression on the right is quadratic in y, while in the case of (23) the expression is just linear in y. This makes an essential difference in computation, which will be explained next. In order to carry out the computation, we need an analog of Theorem 1 to provide the heat kernel of the operator $\partial_x^2 - ax^2\partial_x$. Unfortunately, an explicit formula for the heat kernel of this operator does not exist. Next we shall explain briefly why this is the case.

This type of operator has been analyzed for instance in Calin and Chang [5], p. 223. It is known that the heat kernel of a differential operator, $K(x,y;t)$, measured between x and y in time t can be regarded as the amount of heat starting at $t=0$ from x and reaching y in time t. It makes sense to assume that the heat moves along certain curves of minimum resistance, which can be described as geodesics on the associated manifold. The principal symbol of the operator, which in our case is $H(\xi,x) = \xi^2 + ax^2\xi$, is considered as a Hamiltonian function defined on the cotangent bundle of $\mathbb{R}_x$. The bicharacteristics curves are solutions of the Hamiltonian system $\dot{x} = H_\xi$, $\dot{\xi} = H_x$. Their x-projection defines the geodesic used for the computation of the heat kernel. A straightforward computation of the ODE implied by the Hamiltonian system, which is satisfied by the geodesic $x(s)$, is

$$\ddot{x}(s) = 2a^2x^3(s)$$
$$x(0) = x$$
$$x(t) = y.$$

One difficulty of our problem is that this ODE has infinitely many solutions $x_n(s)$, $n \geq 1$, even for $|x-y|$ small. This means that the expression of the heat kernel will involve an infinite sum of contributions along each geodesic $x_n(s)$. It is known that each contribution along $x_n(s)$ involves the amount $e^{-\frac{1}{s}S_n(x,y;t)}$, where $S_n(x,y;t)$ is the action along the geodesic $x_n(s)$ from x to y in time t. The action $S_n(x,y;t)$ satisfies the Hamilton-Jacobi equation $\partial_t S_n + H(x,\nabla S_n) = 0$, a nonlinear equation that is usually hard to be solved explicitly, fact that brings another difficulty to the problem. Furthermore, even harder is to compute the weights of the contributions $e^{-\frac{1}{s}S_n(x,y;t)}$. These are denoted by $V_n(x,y;t)$ and satisfy certain transport equations, that describe how the volume element evolves along the geodesic flow starting at x. If all the above difficulties are overcome, the explicit solution of the heat kernel would be

$$K(x,y;t) = \sum_{n\geq 1} V_n(x,y;t)e^{-\frac{1}{s}S_n(x,y;t)}.$$

The coefficients V_n and exponents S_n can be computed numerically if needed; however, no explicit formulas can be worked out for them. The present paper deals only with the approximation of the heat kernel with only the main term, i.e., $K(x,y;t) \approx V_1(x,y;t)e^{-\frac{1}{s}S_1(x,y;t)}$. The "volume element" $V_1(x,y;t)$ is also called the van Vleck determinant and is obtained from the determinant of the matrix $\partial_x\partial_y S(x,y,t)$, see for instance Calin et al. [6]. Since from equation (24) we have $\phi(y) \geq 0$, then

$$\int_\infty^\infty K(x,y;u)\phi(y)\,dy > \int_\infty^\infty V_1(x,y;t)e^{-\frac{1}{s}S_1(x,y;t)}\phi(y)\,dy,$$

and hence the foregoing approximation will lead to a lower limit for the price of the call on harmonic average.

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