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A Note on q-Analogue of Boole Polynomials

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Abstract: In this paper, we consider the q-extensions of Boole polynomials. From those polynomials, we derive some new and interesting properties and identities related to special polynomials.

Keywords: q-Boole number, q-Boole polynomial, q-Euler number, q-Euler polynomial

1 Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = 1/p$. The space of continuous functions on $\mathbb{Z}_p$ is denoted by $C(\mathbb{Z}_p)$. Let q be an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-1/p-1}$. The q-number of x is defined by $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-1)^x,$$

where $[x]_{-q} = \dfrac{1-(-q)^x}{1+q}$ (see $[1-9]$). (1.1)

From (1.1), we note that

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l),$$

where $f_n(x) = f(x+n), (n \geq 1)$ (see [4]). (1.2)

In particular, for n=1,

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0). \tag{1.3}$$

As is well known, the Boole polynomials are defined by the generating function to be

$$\sum_{n=0}^{\infty} Bl_n(x|\lambda)\frac{t^n}{n!} = \frac{1}{1+(1+t)^{\lambda}}(1+t)^x, \text{ (see } [2,12]). \tag{1.4}$$

When $\lambda = 1, 2Bl_n(x|1) = Ch_n(x)$ are Changhee polynomials which are defined by

$$\frac{2}{t+2}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!} \text{ (see } [2,3,13,14]). \tag{1.5}$$

The Euler polynomials of order α are defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!}, \text{ (see } [2,11]). \tag{1.6}$$

When $x = 0, E_n^{(\alpha)} = E_n^{(\alpha)}(0)$ are called the Euler numbers of order α.

In particular, for $\alpha = 1, E_n(x) = E_n^{(1)}(x)$ are called the ordinary Euler polynomials.

The Stirling number of the first kind is given by the generating function to be

$$\log(1+t)^m = m! \sum_{l=m}^{\infty} S_1(l,m)\frac{t^l}{l!}, (m \geq 0), \tag{1.7}$$

and the Stirling number of the second kind is defined by the generating function to be

$$(e^t-1)^m = m! \sum_{l=m}^{\infty} S_2(l,m)\frac{t^l}{l!}, \text{ (see } [11,12]). \tag{1.8}$$

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In this paper, we consider the q-extensions of Boole polynomials. From those polynomials, we derive new and interesting properties and identities related to special polynomials.

2 q-analogue of Boole polynomials

In this section, we assume that $t \in \mathbb{C}_p$ with $|t|_p < p^{\frac{-1}{p-1}}$ and $\lambda \in \mathbb{Z}_p$ with $\lambda \neq 0$. From (1.3), we note that

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+t)^{x+\lambda y}d\mu_{-q}(y) &= \frac{1+q}{1+q(1+t)^{\lambda}}(1+t)^x \\ &= \sum_{n=0}^{\infty}[2]_q Bl_{n,q}(x|\lambda)\frac{t^n}{n!},\end{aligned} \tag{2.1}$$

where $Bl_{n,q}(x|\lambda)$ are the q-Boole polynomials which are defined by

$$\frac{1}{1+q(1+t)^{\lambda}}(1+t)^x = \sum_{n=0}^{\infty} Bl_{n,q}(x|\lambda)\frac{t^n}{n!}. \tag{2.2}$$

From (2.1), we can derive the following equation :

$$\int_{\mathbb{Z}_p}\binom{x+\lambda y}{n}d\mu_{-q}(y) = \frac{[2]_q}{n!}Bl_{n,q}(x|\lambda). \tag{2.3}$$

When $x=0, Bl_{n,q}(\lambda) = Bl_{n,q}(0|\lambda)$ are called the q-Boole numbers.
Now, we observe that

$$\begin{aligned}(1+t)^{x+\lambda y} &= e^{(x+\lambda y)\log(1+t)} \\ &= \sum_{m=0}^{\infty}\frac{(x+\lambda y)^m}{m!}\left(\log(1+t)\right)^m \\ &= \sum_{m=0}^{\infty}\frac{(x+\lambda y)^m}{m!}m!\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}(x+\lambda y)^m S_1(n,m)\right\}\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

The q-Euler polynomials are defined by the generating function to be

$$\frac{[2]_q}{qe^t+1}e^{xt} = \sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}. \tag{2.5}$$

Note that $\lim_{q\to 1}E_{n,q}(x) = E_n(x)$.
When $x=0, E_{n,q} = E_{n,q}(0)$ are called the q-Euler numbers.
By (1.3), we easily get

$$\begin{aligned}\int_{\mathbb{Z}_p}e^{(x+y)t}d\mu_{-q}(y) &= \frac{[2]_q}{qe^t+1}e^{xt} \\ &= \sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

Thus, by (2.6), we get

$$\int_{\mathbb{Z}_p}(x+y)^n d\mu_{-q}(y) = E_{n,q}(x), (n \geq 0). \tag{2.7}$$

From (2.1), (2.4) and (2.7), we have

$$\begin{aligned}&\int_{\mathbb{Z}_p}(1+t)^{x+\lambda y}d\mu_{-q}(y) \\ &= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\int_{\mathbb{Z}_p}(x+\lambda y)^m d\mu_{-q}(y)S_1(n,m)\right\}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\lambda^m E_{m,q}\left(\frac{x}{\lambda}\right)S_1(n,m)\right\}\frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

Therefore, by (2.1), (2.3) and (2.8), we obtain the following theorem.

Theorem 1*For $n \geq 0$, we have*

$$Bl_{n,q}(x|\lambda) = \frac{1}{[2]_q}\sum_{m=0}^{n}\lambda^m E_{m,q}\left(\frac{x}{\lambda}\right)S_1(n,m),$$

and

$$\int_{\mathbb{Z}_p}\binom{x+\lambda y}{n}d\mu_{-q}(y) = \frac{[2]_q}{n!}Bl_{n,q}(x|\lambda).$$

From (2.3), we note that

$$Bl_{n,q}(x|\lambda) = \frac{1}{[2]_q}\int_{\mathbb{Z}_p}(x+\lambda y)_n d\mu_{-q}(y).$$

When $\lambda = 1$, we have

$$Bl_{n,q}(x|1) = \frac{1}{[2]_q}\int_{\mathbb{Z}_p}(x+y)_n d\mu_{-q}(y). \tag{2.9}$$

As is known, q-Changhee polynomials are defined by the generating function to be

$$\frac{[2]_q}{[2]_q+qt}(1+t)^x = \sum_{n=0}^{\infty}Ch_{n,q}(x)\frac{t^n}{n!}. \tag{2.10}$$

Thus, by (2.10), we get

$$\int_{\mathbb{Z}_p}(1+t)^{x+y}d\mu_{-q}(y) = \frac{[2]_q}{[2]_q+qt}(1+t)^x = \sum_{n=0}^{\infty}Ch_{n,q}(x)\frac{t^n}{n!}. \tag{2.11}$$

From (2.11), we have

$$\int_{\mathbb{Z}_p}(x+y)_n d\mu_{-q}(y) = Ch_{n,q}(x), \tag{2.12}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$.

By (2.9) and (2.12), we get

$$Bl_{n,q}(x|1) = \frac{1}{[2]_q}Ch_{n,q}(x). \tag{2.13}$$

By replacing t by e^t-1 in (2.2), we see that

$$\begin{aligned}\frac{[2]_q}{qe^{\lambda t}+1}e^{xt} &= [2]_q\sum_{n=0}^{\infty}Bl_{n,q}(x|\lambda)\frac{1}{n!}(e^t-1)^n\\ &= [2]_q\sum_{n=0}^{\infty}Bl_{n,q}(x|\lambda)\sum_{m=n}^{\infty}S_2(m,n)\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{m}[2]_qBl_{n,q}(x|\lambda)S_2(m,n)\frac{t^m}{m!},\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}\frac{[2]_q}{qe^{\lambda t}+1}e^{xt} &= \frac{[2]_q}{qe^{\lambda t}+1}e^{\left(\frac{x}{\lambda}\right)\lambda t}\\ &= \sum_{m=0}^{\infty}E_{m,q}\left(\frac{x}{\lambda}\right)\lambda^m\frac{t^m}{m!}.\end{aligned} \tag{2.15}$$

Therefore, by (2.14) and (2.15), we obtain the following theorem.

Theorem 2 *For $m\geq 0$, we have*

$$\sum_{n=0}^{m}Bl_{n,q}(x|\lambda)S_2(m,n)=\frac{1}{[2]_q}E_{m,q}\left(\frac{x}{\lambda}\right)\lambda^m.$$

Let us define the q-Boole numbers of the first kind with order $k(\in\mathbb{N})$ as follows :

$$\begin{aligned}&[2]_q^kBl_{n,q}^{(k)}(\lambda)\\ &=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda(x_1+\cdots+x_k))_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k),(n\geq 0).\end{aligned} \tag{2.16}$$

Thus, by (2.16), we see that

$$\begin{aligned}&[2]_q^k\sum_{n=0}^{\infty}Bl_{n,q}^{(k)}(\lambda)\frac{t^n}{n!}\\ &=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}\binom{\lambda(x_1+\cdots+x_k)}{n}t^n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\ &=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda(x_1+\cdots+x_k)}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\ &=\left(\frac{1+q}{1+q(1+t)^{\lambda}}\right)^k\\ &=[2]_q^k\sum_{n=0}^{\infty}\left(\sum_{l_1+\cdots+l_k=n}\binom{n}{l_1,\cdots,l_k}Bl_{l_1,q}\cdots Bl_{l_k,q}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.17}$$

Therefore, by (2.17), we obtain the following corollary.

Corollary 3 *For $n\geq 0$, we have*

$$Bl_{n,q}^{(k)}=\sum_{l_1+\cdots+l_k=n}\binom{n}{l_1,\cdots,l_k}Bl_{l_1,q}\cdots Bl_{l_k,q}.$$

The q-Euler polynomials of order k are defined by the generating function to be

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{(x_1+\cdots+x_k+x)t}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\ &=\left(\frac{[2]_q}{qe^t+1}\right)^k e^{xt}=\sum_{n=0}^{\infty}E_{n,q}^{(k)}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.18}$$

Thus, by (2.18), we get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_k+x)^n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)=E_{n,q}^{(k)}(x).$$

When $x=0, E_{n,q}^{(k)}=E_{n,q}^{(k)}(0)$ are called the q-Euler numbers of order k.
From (2.16), we note that

$$\begin{aligned}&[2]_q^kBl_{n,q}^{(k)}(\lambda)\\ &=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda(x_1+\cdots+x_k))_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\ &=\sum_{l=0}^{n}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\lambda^l(x_1+\cdots+x_k)^l d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\ &=\sum_{l=0}^{n}S_1(n,l)\lambda^lE_{l,q}^{(k)}.\end{aligned} \tag{2.19}$$

Therefore, by (2.19), we obtain the following theorem.

Theorem 4 *For $n\geq 0$, we have*

$$Bl_{n,q}^{(k)}(\lambda)=\frac{1}{[2]_q^k}\sum_{l=0}^{n}S_1(n,l)\lambda^lE_{l,q}^{(k)}.$$

By replacing t by e^t-1 in (2.17), we get

$$\begin{aligned}[2]_q^k\sum_{n=0}^{\infty}Bl_{n,q}^{(k)}(\lambda)\frac{1}{n!}(e^t-1)^n &= \left(\frac{[2]_q}{qe^{\lambda t}+1}\right)^k\\ &= \sum_{m=0}^{\infty}E_{m,q}^{(k)}\lambda^m\frac{t^m}{m!},\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}[2]_q^k\sum_{n=0}^{\infty}Bl_{n,q}^{(k)}(\lambda)\frac{1}{n!}(e^t-1)^n &= [2]_q^k\sum_{n=0}^{\infty}Bl_{n,q}^{(k)}(\lambda)\sum_{m=n}^{\infty}S_2(m,n)\frac{t^m}{m!}\\ &= [2]_q^k\sum_{m=0}^{\infty}\left\{\sum_{n=0}^{m}Bl_{n,q}^{(k)}(\lambda)S_2(m,n)\right\}\frac{t^m}{m!}.\end{aligned} \tag{2.21}$$

Therefore, by (2.20) and (2.21), we obtain the following theorem.

Theorem 5 *For $m\geq 0$, we have*

$$\sum_{n=0}^{m}Bl_{n,q}^{(k)}(\lambda)S_2(m,n)=\frac{1}{[2]_q^k}E_{m,q}^{(k)}\lambda^m.$$

Let us define the higher-order q-Boole polynomials of the first kind as
follows :

$$\begin{aligned}&[2]_q^kBl_{n,q}^{(k)}(x|\lambda)\\ &=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda x_1+\cdots+\lambda x_k+x)_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k),\end{aligned}$$

where $n\geq 0$ and $k\in\mathbb{N}$. (2.22)

From (2.22), we can derive the generating function of the higher-order q-Boole polynomials of the first kind as follows :

$$\begin{aligned} &[2]_q^k \sum_{n=0}^{\infty} Bl_{n,q}^{(k)}(x|\lambda)\frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda x_1+\cdots+\lambda x_k+x}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k) \\ &= \left(\frac{[2]_q}{1+q(1+t)^{\lambda}}\right)^k (1+t)^x \end{aligned} \tag{2.23}$$

By (2.17), we easily get

$$\begin{aligned} &\left(\frac{[2]_q}{1+q(1+t)^{\lambda}}\right)^k (1+t)^x \\ =&[2]_q^k\left(\sum_{l=0}^{\infty} Bl_{l,q}^{(k)}(\lambda)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} m!\binom{x}{m}\frac{t^m}{m!}\right) \\ =&[2]_q^k\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} m!\binom{x}{m}\frac{n!}{m!(n-m)!}Bl_{n-m,q}^{(k)}(\lambda)\right)\frac{t^n}{n!} \\ =&[2]_q^k\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} m!\binom{x}{m}\binom{n}{m}Bl_{n-m,q}^{(k)}(\lambda)\right)\frac{t^n}{n!}. \end{aligned} \tag{2.24}$$

Therefore, by (2.23) and (2.24), we obtain the following theorem.

Theorem 6 *For $n \geq 0$, we have*

$$Bl_{n,q}^{(k)}(x|\lambda) = \sum_{m=0}^{n}\binom{n}{m}Bl_{n-m,q}^{(k)}(\lambda)(x)_m.$$

Replacing t by e^t-1 in (2.23), we have

$$\begin{aligned} [2]_q^k\sum_{n=0}^{\infty}Bl_{n,q}^{(k)}(x|\lambda)\frac{(e^t-1)^n}{n!} &= \left(\frac{[2]_q}{1+qe^{\lambda t}}\right)^k e^{xt} \\ &= \sum_{m=0}^{\infty}E_{m,q}^{(k)}\left(\frac{x}{\lambda}\right)\lambda^m\frac{t^m}{m!}, \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} &[2]_q^k\sum_{n=0}^{\infty}Bl_{n,q}^{(k)}(x|\lambda)\frac{(e^t-1)^n}{n!} \\ =&[2]_q^k\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}Bl_{n,q}^{(k)}(x|\lambda)S_2(m,n)\right)\frac{t^m}{m!}. \end{aligned} \tag{2.26}$$

Thus, from (2.25) and (2.26), we have the following theorem.

Theorem 7 *For $m \geq 0$ and $k \in \mathbb{N}$, we have*

$$\sum_{n=0}^{m}Bl_{n,q}^{(k)}(x|\lambda)S_2(m,n) = \frac{1}{[2]_q^k}\lambda^m E_{m,q}^{(k)}\left(\frac{x}{\lambda}\right).$$

From (2.22), we have

$$\begin{aligned} &[2]_q^k Bl_{n,q}^{(k)}(x|\lambda) \\ =&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda x_1+\cdots+\lambda x_k+x)_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k) \\ =&\sum_{l=0}^{n}S_1(n,l) \\ &\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda x_1+\cdots+\lambda x_k+x)^l d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k) \\ =&\sum_{l=0}^{n}S_1(n,l)\lambda^l E_{l,q}^{(k)}\left(\frac{x}{\lambda}\right). \end{aligned} \tag{2.27}$$

Therefore, by (2.27), we obtain the following theorem.

Theorem 8 *For $n \geq 0$, $k \in \mathbb{N}$, we have*

$$Bl_{n,q}^{(k)}(x|\lambda) = \frac{1}{[2]_q^k}\sum_{l=0}^{n}S_1(n,l)\lambda^l E_{l,q}^{(k)}\left(\frac{x}{\lambda}\right).$$

Now, we consider the q-analogue of Boole polynomials of the second kind as follows :

$$\widehat{Bl}_{n,q}(x|\lambda) = \frac{1}{[2]_q}\int_{\mathbb{Z}_p}(-\lambda y+x)_n d\mu_{-q}(y), (n \geq 0). \tag{2.28}$$

Thus, by (2.28), we get

$$\begin{aligned} \widehat{Bl}_{n,q}(x|\lambda) &= \frac{1}{[2]_q}\sum_{l=0}^{n}S_1(n,l)(-1)^l\lambda^l\int_{\mathbb{Z}_p}\left(-\frac{x}{\lambda}+y\right)^l d\mu_{-q}(y) \\ &= \frac{1}{[2]_q}\sum_{l=0}^{n}S_1(n,l)(-1)^l\lambda^l E_{l,q}\left(-\frac{x}{\lambda}\right). \end{aligned} \tag{2.29}$$

When $x=0$, $\widehat{Bl}_{n,q}(\lambda) = \widehat{Bl}_{n,q}(0|\lambda)$ are called the q-Boole numbers of the second kind. From (2.28), we can derive the generating function of $\widehat{Bl}_{n,q}(x|\lambda)$ as follows:

$$\begin{aligned} \sum_{n=0}^{\infty}\widehat{Bl}_{n,q}(x|\lambda)\frac{t^n}{n!} &= \frac{1}{[2]_q}\int_{\mathbb{Z}_p}(1+t)^{-\lambda y+x}d\mu_{-q}(y) \\ &= \frac{(1+t)^{\lambda}}{q+(1+t)^{\lambda}}(1+t)^x. \end{aligned} \tag{2.30}$$

By replacing t by e^t-1 in (2.30), we get

$$\begin{aligned} \sum_{n=0}^{\infty}\widehat{Bl}_{n,q}(x|\lambda)\frac{(e^t-1)^n}{n!} &= \frac{e^{\lambda t}}{q+e^{\lambda t}}e^{xt} \\ &= \frac{1}{qe^{-\lambda t}+1}e^{xt} \\ &= \frac{1}{[2]_q}\sum_{m=0}^{\infty}(-1)^m\lambda^m E_{m,q}(-\frac{x}{\lambda})\frac{t^m}{m!}, \end{aligned} \tag{2.31}$$

and

$$\sum_{n=0}^{\infty}\widehat{Bl}_{n,q}(x|\lambda)\frac{(e^t-1)^n}{n!} = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\widehat{Bl}_{n,q}(x|\lambda)S_2(m,n)\right)\frac{t^m}{m!}. \tag{2.32}$$

Therefore, by (2.31) and (2.32), we obtain the following theorem.

Theorem 9 *For $m \geq 0$, we have*

$$\frac{(-1)^m \lambda^m}{[2]_q} E_{m,q}(-\frac{x}{\lambda}) = \sum_{n=0}^{m} \widehat{Bl}_{n,q}(x|\lambda) S_2(m,n),$$

and

$$\widehat{Bl}_{m,q}(x|\lambda) = \frac{1}{[2]_q} \sum_{l=0}^{m} S_1(m,l)(-1)^l \lambda^l E_{l,q}\left(-\frac{x}{\lambda}\right).$$

For $k \in \mathbb{N}$, let us define the q-Boole polynomials of the second kind with order k as follows :

$$\begin{aligned} &\widehat{Bl}_{n,q}^{(k)}(x|\lambda) \\ &= \frac{1}{[2]_q^k} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-(\lambda x_1 + \cdots + \lambda x_k) + x)_n \\ &\quad \times d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k). \end{aligned} \tag{2.33}$$

Then we have

$$[2]_q^k \widehat{Bl}_{n,q}^{(k)}(x|\lambda) = \sum_{l=0}^{n} S_1(n,l) \lambda^l (-1)^l E_{l,q}\left(-\frac{x}{\lambda}\right).$$

From (2.33), we can derive the generating function of $\widehat{Bl}_{n,q}^{(k)}(x|\lambda)$ as follows :

$$\begin{aligned} &\sum_{n=0}^{\infty} \widehat{Bl}_{n,q}^{(k)}(x|\lambda) \frac{t^n}{n!} \\ &= \frac{1}{[2]_q^k} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{-(\lambda x_1 + \cdots + \lambda x_k) + x} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\ &= \left(\frac{(1+t)^\lambda}{q + (1+t)^\lambda}\right)^k (1+t)^x \\ &= \left(\frac{1}{q(1+t)^{-\lambda} + 1}\right)^k (1+t)^x \\ &= \sum_{n=0}^{\infty} Bl_{n,q}^{(k)}(x|-\lambda) \frac{t^n}{n!}. \end{aligned} \tag{2.34}$$

Thus, by (2.34), we get

$$\widehat{Bl}_{n,q}^{(k)}(x|\lambda) = Bl_{n,q}^{(k)}(x|-\lambda), (n \geq 0). \tag{2.35}$$

Indeed,

$$\begin{aligned} (-1)^n [2]_q \frac{Bl_{n,q}(x|\lambda)}{n!} &= (-1)^n \int_{\mathbb{Z}_p} \binom{x+\lambda y}{n} d\mu_{-q}(y) \\ &= \int_{\mathbb{Z}_p} \binom{-y\lambda - x + n - 1}{n} d\mu_{-q}(y) \\ &= \sum_{m=0}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \binom{-y\lambda - x}{m} d\mu_{-q}(y) \\ &= \sum_{m=1}^{n} \frac{\binom{n-1}{m-1}}{m!} m! \int_{\mathbb{Z}_p} \binom{-y\lambda - x}{m} d\mu_{-q}(y) \\ &= [2]_q \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{Bl}_{m,q}(-x|\lambda)}{m!}, \end{aligned}$$

and

$$\begin{aligned} (-1)^n [2]_q \frac{\widehat{Bl}_{n,q}(x|\lambda)}{n!} &= \sum_{m=0}^{n} \binom{n-1}{m-1} \int_{\mathbb{Z}_p} \binom{-x+y\lambda}{m} d\mu_{-q}(y) \\ &= [2]_q \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{Bl}_{m,q}(-x|\lambda)}{m!}. \end{aligned}$$

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