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Int-Soft Substructures of Groups and Semirings with Applications

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Abstract: In this paper, we introduce a new type of subgroup and normal subgroup of a group, called intersection-soft subgroup (int-soft subgroup) and intersection-normal subgroup (int-soft normal subgroup), by using Molodtsov's definition of the soft sets. We investigate their related properties with respect to soft set operations and group homomorphisms. Moreover, we introduce intersection-soft subsemiring (int-soft subsemiring) and intersection-soft ideal (int-soft ideal) of a semiring and some related properties are investigated and illustrated by many examples. Finally, we give some applications of these new concepts to group theory and semiring theory.

Keywords: Soft set, int-soft subgroup, int-soft normal subgroup, int-soft subsemiring, int-soft ideal

1 Introduction

Soft set theory was introduced by Molodtsov [17] for modeling vagueness and uncertainty and it has received much attention since Maji et al. [15] and Ali et al. [3] introduced and studied several operations of soft sets. Soft set theory started to progress rapidly in the mean of algebraic structures, since Aktaş and Çağman [2] defined and studied soft groups. Since then, [1,5,6,7,8,11,12,13,14,16,18,19] have studied the soft algebraic structures and soft sets as well. Applying the definition of soft set, Atagün and Sezgin [4] introduced the algebraic soft substructures of rings, fields and modules.
In this paper, applying to soft set theory, we deal with the algebraic intersection-soft substructures of groups and semirings. We define the notions of intersection soft subgroup, abbreviated by int-soft subgroup, and intersection soft normal subgroup of group, abbreviated by int-soft normal subgroup, give several illustrating examples and investigate their related properties with respect to soft set operations. Moreover we introduce the notion of intersection soft subsemiring, abbreviated by int-soft subsemiring and intersection soft ideal of a semiring, abbreviated by int-soft ideal and study their related properties with several examples. Furthermore, we investigate these new concepts with respect to soft set operations and group homomorphisms and give some applications of these new concepts to group theory and semiring theory.

2 Preliminaries

A *semiring* S is a structure consisting of a nonempty set S together with two binary operation on S called *addition* and *multiplication* (denoted in the usual manner) such that

i) S together with addition is a commutative monoid with identity element 0,
ii) S together with multiplication is a monoid with identity element 1,
iii) $(a+b)c = ac+bc$ and $a(b+c) = ab+ac$ for all $a,b,c \in S$.

A zero element of a semiring S is an element 0 such that $0.x = x.0 = 0$ and $0+x = x+0 = x$ for all $x \in S$. A nonempty subset I of a semiring S is called a *left (resp. right) ideal of* S if I is closed under addition and $SI \subseteq I$ (resp. $IS \subseteq I$). We say that I is an ideal of S, denoted by $I \lhd S$, if it is both a left and right ideal of S. Let R and S be semirings. A mapping $f : R \to S$ is called a homomorphism of semirings if it satisfies,

i) $f(a+b) = f(a)+f(b)$
ii) $f(ab) = f(a)f(b)$

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for all $a,b \in R$. A semiring homomorphism $f : R \to S$ is called an *epimorphism* if it is a *surjective* mapping.
Molodtsov [17] defined the soft set in the following manner: Let U be an initial universe set, E be a set of parameters, $P(U)$ be the power set of U and $A \subseteq E$.

Definition 1.([17]) *A pair (F,A) is called a soft set over U, where F is a mapping given by*

$$F : A \to P(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-elements of the soft set (F,A), or as the set of ε-approximate elements of the soft set. To illustrate this idea, Molodtsov considered several examples in [17]. In fact, there exists a mutual correspondence between soft sets and binary relations as shown in [9,10]. That is, let A and B be nonempty sets and assume that α refers to an arbitrary binary relation between an element of A and an element of B. A set-valued function $F : A \to P(B)$ can be defined as $F(x) = \{y \in B \mid (x,y) \in \alpha\}$ for all $x \in A$. Then, the pair (F,A) is a soft set over B, which is derived from the relation α.

Definition 2.([3]) *Let (F,A) and (G,B) be two soft sets over a common universe U such that $A \cap B \neq \emptyset$. The restricted intersection of (F,A) and (G,B) is denoted by $(F,A) \Cap (G,B)$, and is defined as $(F,A) \Cap (G,B) = (H,C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cap G(c)$.*

3 Int-soft substructures of groups

In this section, we first define *intersection-soft subgroup*, abbreviated by *int-soft subgroup* and *intersection-soft normal subgroup*, abbreviated by *int-soft normal subgroup* of a group with illustrative examples. We then study their basic properties with respect to soft set operations. From now on, we denote a group by G and if H is a subgroup (resp. normal subgroup) of G, then it is denoted by $H < G$ (resp. $H \lhd G$).

Definition 3.*Let H be a subgroup of G and (F,H) be a soft set over G. If for all $x,y \in H$, $F(xy^{-1}) \supseteq F(x) \cap F(y)$, then the soft set (F,H) is called an int-soft subgroup of G and denoted by $(F,H)\tilde{<}G$ or simply $F_H\tilde{<}G$.*

*Example 1.*Let $G = \{1,-1,i,-i\}$, $H_1 = \{1,-1\} < G$ and the soft set (F,H_1) over G, where $F : H_1 \to P(G)$ is a set-valued function defined by $F(1) = \{1,-i,i\}$ and $F(-1) = \{1\}$. Then, one can easily show that $F_{H_1}\tilde{<}G$.
Let $H_2 = G < G$ and the soft set (T,H_2) over G, where $T : H_2 \to P(G)$ is a set-valued function defined by

$$T(x) = \{y \in G \mid x\alpha y \Leftrightarrow y \in <x>\}$$

for all $x \in H_2$. Then $T(1) = \{1\}$, $T(-1) = \{-1,1\}$, $T(i) = T(-i) = G$. Since $T((-i).(-i)^{-1}) = T((-i).i) = T(1) \not\supseteq T(-i) \cap T(-i)$, (T,H_2) is not an int-soft subgroup of G.

*Example 2.*Let $G = (\mathbb{Z}_4,+)$, $H_3 = \{0,2\} < G$ and the soft set (M,H_3) over G, where $M : H_3 \to P(G)$ is a set-valued function defined by $M(0) = \mathbb{Z}_4$ and $M(2) = \{0,2\}$. Then, one can easily show that $M_{H_3}\tilde{<}G$.
Let $H_4 = G < G$ and the soft set (J,H_4) over G, where $J : H_4 \to P(G)$ is a set-valued function defined by

$$J(x) = \{y \in G \mid xRy \Leftrightarrow x+y = 0\}$$

for all $x \in H_4$. Then $J(0) = \{0\}$, $J(1) = \{3\}$, $J(2) = \{2\}$ and $J(3) = \{1\}$. Since $J(3-3) = J(0) \not\supseteq J(3) \cap J(3)$, (J,H_4) is not an int-soft subgroup of G.

It is easily seen that if we take $H = \{e_G\}$, where e_G is the identity element of the group G, then it is obvious that (F,H) is an int-soft subgroup of G no matter how F is defined.

Proposition 1.*If $F_H\tilde{<}G$, then $F(e_G) \supseteq F(x)$ for all $x \in H$.*

*Proof.*Since (F,H) is an int-soft subgroup of G, $F(e_G) = F(xx^{-1}) \supseteq F(x) \cap F(x) = F(x)$ for all $x \in H$.

Theorem 1.*If $F_{H_1}\tilde{<}G$ and $T_{H_2}\tilde{<}G$ then $F_{H_1} \Cap T_{H_2}\tilde{<}G$.*

*Proof.*Since H_1 and H_2 are subgroups of G, then $H_1 \cap H_2$ is a subgroup of G. By Definition 2, let $F_{H_1} \Cap T_{H_2} = (F,H_1) \Cap (T,H_2) = (Q,H_1 \cap H_2)$, where $Q(x) = F(x) \cap T(x)$ for all $x \in H_1 \cap H_2 \neq \emptyset$. Then for all $x,y \in H_1 \cap H_2$,

$$\begin{aligned} Q(xy^{-1}) &= F(xy^{-1}) \cap T(xy^{-1}) \\ &\supseteq (F(x) \cap F(y)) \cap (T(x) \cap T(y)) \\ &= (F(x) \cap T(x)) \cap (F(y) \cap T(y)) \\ &= Q(x) \cap Q(y) \end{aligned}$$

Therefore $F_{H_1} \Cap T_{H_2} = Q_{H_1 \cap H_2}\tilde{<}G$.

Definition 4.*Let G_1 and G_2 be groups and let (F,H_1) and (T,H_2) be two int-soft subgroups of G_1 and G_2, respectively. The product of int-soft subgroups (F,H_1) and (T,H_2) is defined as $(F,H_1) \times (T,H_2) = (Q,H_1 \times H_2)$, where $Q(x,y) = F(x) \times T(y)$ for all $(x,y) \in H_1 \times H_2$.*

Theorem 2.*If $F_{H_1}\tilde{<}G_1$ and $T_{H_2}\tilde{<}G_2$, then $F_{H_1} \times T_{H_2}\tilde{<}G_1 \times G_2$.*

*Proof.*Since H_1 and H_2 are subgroups of G_1 and G_2, respectively, then $H_1 \times H_2$ is a subgroup of $G_1 \times G_2$. By Definition 4, let $F_{H_1} \times T_{H_2} = (F,H_1) \times (T,H_2) = (Q,H_1 \times H_2)$, where $Q(x,y) = F(x) \times T(y)$ for all $(x,y) \in H_1 \times H_2$. Then for all $(x_1,y_1),(x_2,y_2) \in H_1 \times H_2$,

$$\begin{aligned} Q((x_1,y_1)(x_2,y_2)^{-1}) &= Q(x_1x_2^{-1}, y_1y_2^{-1}) \\ &= F(x_1x_2^{-1}) \times T(y_1y_2^{-1}) \\ &\supseteq (F(x_1) \cap F(x_2)) \times (T(y_1) \cap T(y_2)) \\ &= (F(x_1) \times T(y_1)) \cap (F(x_2) \times T(y_2)) \\ &= Q(x_1,y_1) \cap Q(x_2,y_2) \end{aligned}$$

Hence $F_{H_1} \times T_{H_2} = Q_{H_1 \times H_2}\tilde{<}G_1 \times G_2$.

To illustrate Theorem 1 and Theorem 2, we have the following example:

Example 3. Consider $(M,H_3)\widetilde{<}G=\mathbb{Z}_4$ in Example 2. Let $G=(\mathbb{Z}_4,+)$, $H_5=\mathbb{Z}_4<G$ and the soft set (K,H_5) over G, where $K:H_5\to P(G)$ is a set-valued function defined by $K(0)=\mathbb{Z}_4$, $K(1)=K(3)=\{0,1,3\}$ and $K(2)=\{1,2\}$. Then, it is easy to show that $K_{H_5}\widetilde{<}G$.

By Definition 2, $M_{H_3}\Cap K_{H_5}=(M,H_3)\Cap(K,H_5)=(W,H_3\cap H_5)$, where $W(x)=M(x)\cap K(x)$ for all $x\in H_3\cap H_5=\{0,2\}$. Then $W(0)=\mathbb{Z}_4$ and $W(2)=\{2\}$. Since $W(0-0)=W(0)\supseteq W(0)\cap W(0)=W(0)$, $W(2-2)=W(0)=\mathbb{Z}_4\supseteq W(2)\cap W(2)=W(2)=\{2\}$ and $W(2-0)=W(0-2)=W(2)=\{2\}\supseteq W(0)\cap W(2)=\{2\}$, it follows that $W_{H_3\cap H_5}\widetilde{<}G$.

Now we consider the int-soft subgroup (F,H_1) of $G=\{1,-1,i,-i\}$ in Example 1 and the int-soft subgroup (M,H_3) of $\mathbb{Z}_4$ in Example 2. By Definition 4, $F_{H_1}\times M_{H_3}=(F,H_1)\times(M,H_3)=(P,H_1\times H_3)$, where $P(x,y)=F(x)\times M(y)$ for all $(x,y)=\{(1,0),(1,2),(-1,0),(-1,2)\}\in H_1\times H_3$. Then it can be easily seen that $P_{H_1\times H_3}\widetilde{<}G\times\mathbb{Z}_4$. We show the the operation for some elements of $H_1\times H_3$:

$$\begin{aligned}P((1,0)(-1,2)^{-1}) &= P(1.(-1)^{-1},0-2)\\ &= P(-1,-2)=F(-1)\times M(2)\\ &= \{(1,0),(1,2)\}\\ &\supseteq P(1,0)\cap P(-1,2)\\ &= \{\{1,-i,i\}\times\mathbb{Z}_4\}\cap\{(1,0),(1,2)\}=\{(1,0),(1,2)\}.\end{aligned}$$

It is worth noting that if A and B are two subgroups of a group $(G,+)$, then the sum of these two subgroups is defined as the following: $A+B=\{a+b\mid a\in A\wedge b\in B\}$.

Definition 5. *Let (F,H_1) and (T,H_2) be two int-soft subgroups of $(G,+)$. If $H_1\cap H_2=\{e_G\}$, then the sum of int-soft subgroups (F,H_1) and (T,H_2) is defined as $(F,H_1)+(T,H_2)=(Q,H_1+H_2)$, where $Q(x+y)=F(x)+T(y)$ for all $x+y\in H_1+H_2$.*

Theorem 3. *If $F_{H_1}\widetilde{<}G$ and $T_{H_2}\widetilde{<}G$, where $(G,+)$ is an abelian group and $H_1\cap H_2=\{e_G\}$, then $F_{H_1}+T_{H_2}\widetilde{<}G$.*

Proof. Since H_1 and H_2 are subgroups of an abelian group G, then H_1+H_2 is a subgroup of G. By Definition 5, let $F_{H_1}+T_{H_2}=(F,H_1)+(T,H_2)=(Q,H_1+H_2)$, where $Q(x+y)=F(x)+T(y)$ for all $x+y\in H_1+H_2$. Since $H_1\cap H_2=\{e_G\}$, then the sum operation is well defined. Thus, for all $x_1+y_1,x_2+y_2\in H_1+H_2$,

$$\begin{aligned}Q((x_1+y_1)-(x_2+y_2)) &= Q((x_1-x_2)+(y_1-y_2))\\ &= F(x_1-x_2)+T(y_1-y_2)\\ &\supseteq (F(x_1)\cap F(x_2))+(T(y_1)\cap T(y_2))\\ &= (F(x_1)+T(y_1))\cap(F(x_2)+T(y_2))\\ &= Q(x_1+y_1)\cap Q(x_2+y_2)\end{aligned}$$

Therefore $F_{H_1}+T_{H_2}=Q_{H_1+H_2}\widetilde{<}G$.

To illustrate Theorem 3, we have the following example:

Example 4. Let $G=(\mathbb{Z}_6,+)$, $H_1=\{0,2,4\}<G$ and the soft set (F,H_1) over G, where $F:H_1\to P(G)$ is a set-valued function defined by $F(0)=\mathbb{Z}_6$, $F(2)=\{1,5\}$ and $F(4)=\{3\}$. It can be easily seen that (F,H_1) is an int-soft subgroup of G.

Let $H_2=\{0,3\}<G$ and the soft set (T,H_2) over G, where $T:H_2\to P(G)$ is a set-valued function defined by $T(0)=\mathbb{Z}_6$ and $T(3)=\{4\}$. It is obvious that (T,H_2) is an int-soft subgroup of G.

By Definition 5, $F_{H_1}+T_{H_2}=(F,H_1)+(T,H_2)=(Q,H_1+H_2)$, where $Q(x)=F(x)+T(x)$ for all $x\in H_1+H_2=\mathbb{Z}_6$. It can be easily seen that $Q_{H_1+H_2}\widetilde{<}G$. We show the operations for some elements of H_1+H_2: We consider $2+3\in H_1+H_2$ and $4+3\in H_1+H_2$. Then $Q(2+3)=F(2)+T(3)=\{3,5\}$ and $Q(4+3)=F(4)+T(3)=\{1\}$.

$$\begin{aligned}Q((2+3)-(4+3)) &= Q((2-4)+(3-3))\\ &= Q(4+0)=F(4)+T(0)\\ &= \mathbb{Z}_6\\ &\supseteq Q(2+3)\cap Q(4+3)=\emptyset.\end{aligned}$$

Definition 6. *Let N be a normal subgroup of G and let (F,N) be a soft set over G. Then, (F,N) is called an int-soft normal subgroup of G, denoted by $(F,N)\widetilde{\lhd}G$ or simply $F_N\widetilde{\lhd}G$, if the following conditions are satisfied:*

i_1) $F(xy^{-1})\supseteq F(x)\cap F(y)$ and
i_2) $F(gxg^{-1})\supseteq F(x)$,

for all $x,y\in N$ and $g\in G$.

Example 5. Let $G=(S_3,\circ)$, $N=A_3=\{e,(123),(132)\}\lhd G$ and the soft set (F,N) over G, where $F:N\to P(G)$ is a set-valued function defined by $F(e)=\{e,(12),(123),(132)\}$ and $F(123)=F(132)=\{e,(123),(132)\}$. It can be easily shown that $F_N\widetilde{\lhd}G$.
Let $G=(S_3,\circ)$, $N=A_3=\{e,(123),(132)\}\lhd G$ and the soft set (T,N) over G, where $T:N\to P(G)$ is a set-valued function defined by $T(e)=\{e,(12),(23),(123),(132)\}$, $T(123)=\{(12),(123)\}$ and $T(132)=\{(23),(132)\}$. Since $T((12)(123)(12)^{-1})=T((12)(123)(12))=T(132)=\{(23),(132)\}\not\supseteq T(123)=\{(12),(123)\}$, it follows that T_N is not an int-soft normal subgroup of G.

Theorem 4. *If $F_{N_1}\widetilde{\lhd}G$ and $T_{N_2}\widetilde{\lhd}G$, then $F_{N_1}\Cap T_{N_2}\widetilde{\lhd}G$.*

Proof. Since $N_1,N_2\lhd G$, then $N_1\cap N_2\lhd G$. By Definition 2, $F_{N_1}\Cap T_{N_2}=(F,N_1)\Cap(T,N_2)=(W,N_1\cap N_2)$, where $W(x)=F(x)\cap T(x)$ for all $x\in N_1\cap N_2\neq\emptyset$. Then for all $x,y\in N_1\cap N_2$ and for all $g\in G$,

$$\begin{aligned}W(xy^{-1}) &= F(xy^{-1})\cap T(xy^{-1})\\ &\supseteq (F(x)\cap F(y))\cap(T(x)\cap T(y))\\ &= (F(x)\cap T(x))\cap(F(y)\cap T(y))\\ &= W(x)\cap W(y),\\ W(gxg^{-1}) &= F(gxg^{-1})\cap T(gxg^{-1})\\ &\supseteq F(x)\cap T(x)\\ &= W(x).\end{aligned}$$

Therefore $F_{N_1} \Cap T_{N_2} = W_{N_1 \cap N_2} \widetilde{\lhd} G$.

Definition 7. *Let G_1 and G_2 be groups and (F,N_1) and (T,N_2) be two SI-normal subgroups of G_1 and G_2, respectively. Theproduct of int-soft normal subgroup (F,N_1) and (T,N_2) is defined as $(F,N_1) \times (T,N_2) = (Q,N_1 \times N_2)$, where $Q(x,y) = F(x) \times T(y)$ for all $(x,y) \in N_1 \times N_2$.*

Theorem 5. *If $F_{N_1} \widetilde{\lhd} G_1$ and $T_{N_2} \widetilde{\lhd} G_2$, then $F_{N_1} \times T_{N_2} \widetilde{\lhd} G_1 \times G_2$.*

Proof. Since N_1 and N_2 are normal subgroups of G_1 and G_2, respectively, then $N_1 \times N_2$ is a normal subgroup of $G_1 \times G_2$. By Definition 7, $F_{N_1} \times T_{N_2} = (F,N_1) \times (T,N_2) = (Q,N_1 \times N_2)$, where $Q(x,y) = F(x) \times T(y)$ for all $(x,y) \in N_1 \times N_2$. Then for all $(x_1,y_1),(x_2,y_2) \in N_1 \times N_2$ and $(g_1,g_2) \in G_1 \times G_2$,

$$\begin{aligned}
Q((x_1,y_1)(x_2,y_2)^{-1}) &= Q(x_1x_2^{-1},y_1y_2^{-1}) \\
&= F(x_1x_2^{-1}) \times T(y_1y_2^{-1}) \\
&\supseteq (F(x_1) \cap F(x_2)) \times (T(y_1) \cap T(y_2)) \\
&= (F(x_1) \times T(y_1)) \cap (F(x_2) \times T(y_2)) \\
&= Q(x_1,y_1) \cap Q(x_2,y_2), \\
Q((g_1,g_2)(x_1,y_1)(g_1,g_2)^{-1}) &= Q(g_1x_1g_1^{-1},g_2y_1g_2^{-1}) \\
&= F(g_1x_1g_1^{-1}) \times T(g_2y_1g_2^{-1}) \\
&\supseteq F(x_1) \times T(y_1) \\
&= Q(x_1,y_1).
\end{aligned}$$

Therefore $F_{N_1} \times T_{N_2} = Q_{N_1 \times N_2} \widetilde{\lhd} G_1 \times G_2$.

It is well-known that every subgroup of an abelian group G is also a normal subgroup of G. A similar relationship exists for int-soft subgroups and int-soft normal subgroups as following:

Proposition 2. *Let H be a subgroup of G, where G is an abelian group and let (F,H) be a soft set over G. If (F,H) is an int-soft subgroup of G, then it is also an int-soft normal subgroup of G.*

Proof. Since H is a subgroup of an abelian group of G, then $H \lhd G$. Moreover $F(gxg^{-1}) = F(gg^{-1}x) = F(x) \supseteq F(x)$ for all $x \in H$ and $g \in G$. Thus, $F_H \widetilde{\lhd} G$.

Corollary 1. *Every int-soft normal subgroup of a group G is an int-soft subgroup of G; however the converse is true when G is an abelian group.*

Definition 8. *Let (F,N_1) and (T,N_2) be two SI-normal subgroups of $(G,+)$. If $N_1 \cap N_2 = \{e_G\}$, then the the sum of int-soft normal subgroup (F,N_1) and (T,N_2) is defined as $(F,N_1) + (T,N_2) = (Q,N_1 + N_2)$, where $Q(x+y) = F(x) + T(y)$ for all $x+y \in N_1 + N_2$.*

Theorem 6. *If $F_{N_1} \widetilde{\lhd} G$ and $T_{N_2} \widetilde{\lhd} G$, where G is an abelian group and $N_1 \cap N_2 = \{e_G\}$, then $F_{N_1} + T_{N_2} \widetilde{\lhd} G$.*

Proof. The proof follows from Theorem 3 and Corollary 1.

Definition 9. *Let (F,N) be an int-soft subgroup (int-soft normal subgroup) of G. Then,*

i) (F,N) is said to be trivial if $F(x) = \{e_G\}$ for all $x \in N$.
ii) (F,N) is said to be whole if $F(x) = G$ for all $x \in N$.

Proposition 3. *Let (F,N_1) and (T,N_2) be int-soft subgroups (resp. int-soft normal subgroups) of G. Then,*

i) If (F,N_1) and (T,N_2) are trivial int-soft subgroups (resp. int-soft normal subgroups) of G, then $(F,N_1) \Cap (T,N_2)$ is a trivial int-soft subgroup (resp. int-soft normal subgroup) of G.
ii) If (F,N_1) and (T,N_2) are whole int-soft subgroups (resp. int-soft normal subgroups) of G, then $(F,N_1) \Cap (T,N_2)$ is a whole int-soft subgroup (resp. int-soft normal subgroup) of G.
iii) If (F,N_1) is a trivial int-soft subgroup (resp. int-soft normal subgroup) of G and (T,N_2) is a whole int-soft subgroup (resp. int-soft normal subgroup) of G, then $(F,N_1) \Cap (T,N_2)$ is a trivial int-soft subgroup (resp. int-soft normal subgroup) of G.
iv) If (F,N_1) and (T,N_2) are trivial int-soft subgroups (resp. int-soft normal subgroups) of G, where G is abelian and $N_1 \cap N_2 = \{e_G\}$, then $(F,N_1) + (T,N_2)$ is a trivial int-soft subgroup (resp. int-soft normal subgroup) of G.
v) If (F,N_1) and (T,N_2) are whole int-soft subgroups (resp. int-soft normal subgroups) of G, where G is abelian and $N_1 \cap N_2 = \{e_G\}$, then $(F,N_1) + (T,N_2)$ is a whole int-soft subgroup (resp. int-soft normal subgroup) of G.
vi) If (F,N_1) is a trivial int-soft subgroup (resp. int-soft normal subgroup) of G and (T,N_2) is a whole int-soft subgroup (resp. int-soft normal subgroup) of G, where G is abelian and $N_1 \cap N_2 = \{e_G\}$, then $(F,N_1) + (T,N_2)$ is a whole int-soft subgroup (resp. int-soft normal subgroup) of G.

Proof. Straightforward, hence omitted.

Proposition 4. *Let (F,N_1) and (T,N_2) be two int-soft subgroups (resp. int-soft normal subgroups) of G_1 and G_2, respectively. Then,*

i) If (F,N_1) and (T,N_2) are trivial int-soft subgroups (resp. int-soft normal subgroups) of G_1 and G_2, respectively, then $(F,N_1) \times (T,N_2)$ is a trivial int-soft subgroup (resp. int-soft normal subgroup) of $G_1 \times G_2$.
ii) If (F,N_1) and (T,N_2) are whole int-soft subgroups (resp. int-soft normal subgroups) of G_1 and G_2, respectively, then $(F,N_1) \times (T,N_2)$ is a whole int-soft subgroup (resp. int-soft normal subgroup) of $G_1 \times G_2$.

Proof. The proof is obvious, hence omitted.

4 Some applications of int-soft substructures of groups

In this section, we give some applications of int-soft subgroups and int-soft normal subgroups of a group to group theory.

Proposition 5. *If $F_H \tilde{<} G$, then $H_F = \{x \in H \mid F(x) = F(e_G)\}$ is a subgroup of H.*

Proof. It is clear that $e_G \in H_F$ and $\emptyset \neq H_F \subseteq H$. We need to show that $xy^{-1} \in H_F$ for all $x,y \in H_F$, which means that $F(xy^{-1}) = F(e_G)$ has to be satisfied. Since $x,y \in H_F$, then $F(x) = F(y) = F(e_G)$. By Proposition 1, $F(e_G) \supseteq F(xy^{-1})$ for all $x,y \in H_F$. Since (F,H) is an int-soft subgroup of G, then $F(xy^{-1}) \supseteq F(x) \cap F(y) = F(e_G)$ for all $x,y \in H_F$. Therefore H_F is a subgroup of H.

Proposition 6. *If $F_N \tilde{\lhd} G$, then $N_F = \{x \in N \mid F(x) = F(e_G)\}$ is a normal subgroup of G.*

Proof. It is obvious that $e_G \in N_F$ and $\emptyset \neq N_F \subseteq G$. We need to show that (i) $xy^{-1} \in N_F$ and (ii) $gxg^{-1} \in N_F$ for all $x,y \in N_F$ and $g \in G$. If $x,y \in N_F$, then $F(x) = F(y) = F(e_G)$. By Proposition 1, $F(e_G) \supseteq F(xy^{-1})$ and $F(e_G) \supseteq F(gxg^{-1})$ for all $g \in G$ and $x,y \in N_F$. Since (F,N) is an int-soft normal subgroup of G, then for all $x,y \in N_F$ and $g \in G$, (i) $F(xy^{-1}) \supseteq F(x) \cap F(y) = F(e_G)$ and (ii) $F(gxg^{-1}) \supseteq F(x) = F(e_G)$. Hence $F(xy^{-1}) = F(e_G)$ and $F(gxg^{-1}) = F(e_G)$ for all $g \in G$ and $x,y \in N_F$. Thus, N_F is a normal subgroup of N.

Theorem 7. *Let G_1 and G_2 be two groups and $(F_1,H_1) \tilde{<} G_1$, $(F_2,H_2) \tilde{<} G_2$. If $f : H_1 \to H_2$ is a group homomorphism, then*

a) If f is an epimorphism, then $(F_1, f^{-1}(H_2)) \tilde{<} G_1$,
b) $(F_2, f(H_1)) \tilde{<} G_2$,
c) $(F_1, Ker f) \tilde{<} G_1$.

Proof. **a)** Since $H_1 < G_1$, $H_2 < G_2$ and $f : H_1 \to H_2$ is a group epimorphism, then it is clear that $f^{-1}(H_2) < G_1$. Since $(F_1,H_1) \tilde{<} G_1$ and $f^{-1}(H_2) \subseteq H_1$, $F_1(xy^{-1}) \supseteq F_1(x) \cap F_1(y)$ for all $x,y \in f^{-1}(H_2)$. Hence $(F_1, f^{-1}(H_2)) \tilde{<} G_1$.

b) Since $H_1 < G_1$, $H_2 < G_2$ and $f : H_1 \to H_2$ is a group homomorphism, then $f(H_1) < G_2$. Since $f(H_1) \subseteq H_2$, the result is obvious by Definition 3.

c) Since $Ker f < G_1$ and $Ker f \subseteq H_1$, the rest of the proof is clear by Definition 3.

Corollary 2. *Let $(F,H_1) \tilde{<} G_1$, $(F,H_2) \tilde{<} G_2$ and $f : H_1 \to H_2$ is a group homomorphism, then $(F_2, \{e_{G_2})\} \tilde{<} G_2$.*

Proof. By Theorem 7 (c), $(F_1, Ker f) \tilde{<} G_1$. Then $(F_2, f(Ker f)) =$
$(F_2, \{e_{G_2}\}) \tilde{<} G_2$ by Theorem 7 (b).

5 Int-soft substructures of semirings

In this section, we first define *intersection-soft subsemiring*, abbreviated by *int-soft subsemiring* and *intersection-soft ideal*, abbreviated by *int-soft ideal* of a semiring with illustrative examples. We then study their basic properties with respect to soft set operations. From now on, S denotes a semiring with zero element 0 and if M is a subsemiring (resp. ideal) of S, then it is denoted by $M < S$ (resp. $M \lhd S$).

Definition 10. *Let M be a subsemiring of S anD (F,M) be a soft set over S. If for all $x,y \in M$,*

s1) $F(x+y) \supseteq F(x) \cap F(y)$ and
s2) $F(xy) \supseteq F(x) \cap F(y)$

then, the soft set (F,M) is called an int-soft subsemiring of S and denoted by $(F,M) \tilde{<} S$ or simply $F_M \tilde{<} S$.

Example 6. Let $S = \{1,2,3\}$. If we define $x+y = max\{x,y\}$ and $x.y = min\{x,y\}$ for all $x,y \in S$, then $(S,+,.)$ is a semiring and its zero element is 1. Moreover, S is not a ring. Let $M = \{1,2\}$, then M is a subsemiring of S. Let the soft set (F,M) over S, where $F : M \to P(S)$ is a set valued function defined by $F(1) = M$ and $F(2) = \{2\}$. Then it can be easily seen that $(F,M) \tilde{<} S$. Let $H = \{1,3\}$, then H is a semiring of S. Let the soft set (G,H) over S, where $G : H \to P(S)$ is a set valued function defined by $G(1) = S$ and $G(3) = H$, then $(G,H) \tilde{<} S$, too.

Proposition 7. *Let $(S,+,.)$ be a semiring and let M be a subsemiring of S such that $(M,+)$ is a group. If $F_M \tilde{<} S$, then $F(0) \supseteq F(x)$ for all $x \in M$.*

Proof. Since (F,M) is an int-soft subsemiring of S, then for all $x,y \in M$, $F(x+y) \supseteq F(x) \cap F(y)$. Since $(M,+)$ is a group, if we take $y = -x$ then $F(x-x) = F(0) \supseteq F(x) \cap F(x) = F(x)$ for all $x \in M$.

Theorem 8. *If $F_{M_1} \tilde{<} S$ and $G_{M_2} \tilde{<} S$ such that $M_1 \cap M_2 \neq \emptyset$, then $F_{M_1} \Cap G_{M_2} \tilde{<} S$.*

Proof. Since M_1 and M_2 are subsemirings of S, where $M_1 \cap M_2 \neq \emptyset$, then $M_1 \cap M_2$ is a subsemiring of S. By Definition 2, let $F_{M_1} \Cap G_{M_2} = (F,M_1) \Cap (G,M_2) = (T, M_1 \cap M_2)$, where $T(x) = F(x) \cap G(x)$ for all $x \in M_1 \cap M_2 \neq \emptyset$. Then for all $x,y \in M_1 \cap M_2$,

s1) $T(x+y) = F(x+y) \cap G(x+y) \supseteq (F(x) \cap F(y)) \cap (G(x) \cap G(y)) = (F(x) \cap G(x)) \cap (F(y) \cap G(y)) = T(x) \cap T(y)$,
s2) $T(xy) = F(xy) \cap G(xy) \supseteq (F(x) \cap F(y)) \cap (G(x) \cap G(y)) = (F(x) \cap G(x)) \cap (F(y) \cap G(y)) = T(x) \cap T(y)$.

Therefore $F_{M_1} \Cap G_{M_2} = T_{M_1 \cap M_2} \tilde{<} S$.

Definition 11. *Let S_1 and S_2 be semirings and let (F,M_1) and (G,M_2) be two int-soft subsemirings of S_1 and S_2, respectively. The product of int-soft subsemiring (F,M_1) and (G,M_2) is defined as $(F,M_1) \times (G,M_2) = (Q, M_1 \times M_2)$, where $Q(x,y) = F(x) \times G(y)$ for all $(x,y) \in M_1 \times M_2$.*

Theorem 9. *If $F_{M_1} \tilde{<} S_1$ and $G_{M_2} \tilde{<} S_2$, then $F_{M_1} \times G_{M_2} \tilde{<} S_1 \times S_2$.*

Proof. Since M_1 and M_2 are subsemirings of S_1 and S_2, respectively, then $M_1 \times M_2$ is a subsemiring of $S_1 \times S_2$. By Definition 11, let $F_{M_1} \times G_{M_2} = (F, M_1) \times (G, M_2) = (Q, M_1 \times M_2)$, where $Q(x,y) = F(x) \times G(y)$ for all $(x,y) \in M_1 \times M_2$. Then for all $(x_1,y_1),(x_2,y_2) \in M_1 \times M_2$,

s1) $Q((x_1,y_1)+(x_2,y_2)) = Q(x_1+x_2, y_1+y_2) = F(x_1+x_2) \times G(y_1+y_2) \supseteq (F(x_1) \cap F(x_2)) \times (G(y_1) \cap G(y_2)) = (F(x_1) \times G(y_1)) \cap (F(x_2) \times G(y_2)) = Q(x_1,y_1) \cap Q(x_2,y_2)$,
s2) $Q((x_1,y_1)(x_2,y_2)) = Q(x_1x_2, y_1y_2) = F(x_1x_2) \times G(y_1y_2) \supseteq (F(x_1) \cap F(x_2)) \times (G(y_1) \cap G(y_2)) = (F(x_1) \times G(y_1)) \cap (F(x_2) \times G(y_2)) = Q(x_1,y_1) \cap Q(x_2,y_2)$.

Hence $F_{M_1} \times G_{M_2} = Q_{M_1 \times M_2} \widetilde{<} S_1 \times S_2$.

To illustrate Theorem 8 and Theorem 9, we have the following example:

Example 7. Let $(F,M) \widetilde{<} S$ and $(G,H) \widetilde{<} S$ in Example 6. By Definition 2, $(F,M) \Cap (G,H) = (T, M \cap H)$, where $T(x) = F(x) \cap G(x)$ for all $x \in M \cap H = \{1\} \neq \emptyset$. Then $T(1) = F(1) \cap G(1) = M \cap S = M$. Since $T(1+1) = T(1) = M \supseteq T(1) \cap T(1) = M$ and $T(1.1) = T(1) = M \supseteq T(1) \cap T(1) = M$, it follows that $(T, M \cap H) \widetilde{<} S$.

By Definition 11, $(F,M) \times (G,H) = (Q, M \times H)$, where $Q(x,y) = F(x) \times G(y)$ for all $(x,y) \in M \times H = \{(1,1),(1,3),(2,1),(2,3)\}$. Then it can be easily seen that $(Q, M \times H) \widetilde{<} S \times S$. We show the operations for some elements of $M \times H$:

$$\begin{aligned} Q((1,1)+(2,3)) = Q(1+2,1+3) &= Q(2,3) \\ &= F(2) \times G(3) = \{2\} \times \{1,3\} \\ &= \{(2,1),(2,3)\} \\ Q(1,1) \cap Q(2,3) &= (F(1) \times G(1)) \cap (F(2) \times G(3)) \\ &= (\{1,2\} \times \{1,2,3\}) \cap (\{2\} \times \{1,3\}) \\ &= \{(2,1),(2,3)\}. \\ Q((1,1)(2,3)) = Q(1.2,1.3) &= Q(1,1) \\ &= F(1) \times G(1) = M \times S. \end{aligned}$$

Therefore $Q((1,1)+(2,3)) \supseteq Q(1,1) \cap Q(2,3)$ and $Q((1,1).(2,3)) \supseteq Q(1,1) \cap Q(2,3)$.

Definition 12. *Let I be an ideal of S and let (F,I) be a soft set over S. If for all $x,y \in I$ and $s \in S$,*

i_1) $F(x+y) \supseteq F(x) \cap F(y)$,
i_2) $F(sx) \supseteq F(x)$ and
i_3) $F(xs) \supseteq F(x)$,

then (F,I) is called an int-soft ideal of S and denoted by $(F,I) \widetilde{\lhd} S$ or simply $F_I \widetilde{\lhd} S$. If $I \lhd_l S$, (F,I) is a soft set over S and if the conditions i_1 and i_2 are satisfied, then (F,I) is called a int-soft left ideal of S and denoted by $(F,I) \widetilde{\lhd_l} S$ or simply $F_I \widetilde{\lhd_l} S$. If $I \lhd_r S$, (F,I) is a soft set over S and if the conditions i_1 and i_3 are satisfied, then (F,I) is called a int-soft right ideal of S and denoted by $(F,I) \widetilde{\lhd_r} S$ or simply $F_I \widetilde{\lhd_r} S$.

Example 8. Consider the semiring S and the soft set (F,M) over S in Example 6. Then it is seen that $F_M \widetilde{\lhd} S$. Let $K = S \lhd S$ in Example 6 and the soft set (T,K) over S is defined as $T: K \to P(S)$, where $T(1) = \{1,3\}$, $T(2) = \{1,2\}$, $T(3) = \{2,3\}$. Since $T(2.3) = T(2) \not\supseteq T(3)$, (T,K) is not an int-soft ideal of S.

Theorem 10. *If $(F,I) \widetilde{\lhd} S$ (resp. $(F,I) \widetilde{\lhd_l} S$, $(F,I) \widetilde{\lhd_r} S$), $(G,J) \widetilde{\lhd} S$ (resp. $(G,J) \widetilde{\lhd_l} S$, $(G,J) \widetilde{\lhd_r} S$) and $I \cap J \neq \emptyset$, then $F_I \Cap G_J \widetilde{\lhd} S$ (resp. $F_I \Cap G_J \widetilde{\lhd_l} S$, $F_I \Cap G_J \widetilde{\lhd_r} S$).*

Proof. We give the proof for int-soft ideals, the same proof can be seen for int- soft left ideals and int- soft right ideals, too. Since $I, J \lhd S$ and $I \cap J \neq \emptyset$, then $I \cap J \lhd S$. By Definition 2, $F_I \Cap G_J = (F,I) \Cap (G,J) = (H, I \cap J)$, where $H(x) = F(x) \cap G(x)$ for all $x \in I \cap J \neq \emptyset$. Then for all $x,y \in I \cap J$ and for all $s \in S$,

i_1) $H(x+y) = F(x+y) \cap G(x+y) \supseteq (F(x) \cap F(y)) \cap (G(x) \cap G(y)) = (F(x) \cap G(x)) \cap (F(y) \cap G(y)) = H(x) \cap H(y)$,
i_2) $H(sx) = F(sx) \cap G(sx) \supseteq (F(x) \cap G(x)) = H(x)$,
i_3) $H(xs) = F(xs) \cap G(xs) \supseteq (F(x) \cap G(x)) = H(x)$.

Therefore, $F_I \Cap G_J \widetilde{\lhd} S$.

Definition 13. *Let S_1 and S_2 be semirings and let (F,I) and (G,J) be two int-soft ideals of S_1 and S_2, respectively. The product of int-soft ideal (F,I) and (G,J) is defined as $(F,I) \times (G,J) = (Q, I \times J)$, where $Q(x,y) = F(x) \times G(y)$ for all $(x,y) \in I \times J$.*

Theorem 11. *If $F_I \widetilde{\lhd} S_1$ (resp. $F_I \widetilde{\lhd_l} S_1$, $F_I \widetilde{\lhd_r} S_2$) and $G_J \widetilde{\lhd} S_2$ (resp. $G_J \widetilde{\lhd_l} S_2$, $G_J \widetilde{\lhd_r} S_2$), then $F_I \times G_J \widetilde{\lhd} S_1 \times S_2$ (resp. $F_I \times G_J \widetilde{\lhd_l} S_1 \times S_2$, $F_I \times G_J \widetilde{\lhd_r} S_1 \times S_2$).*

Proof. We give the proof for int-soft ideals, the same proof can be seen for int- soft left ideals and int- soft left ideals, too. Since I and J are ideals of S_1 and S_2, respectively, then $I \times J$ is an ideal of $S_1 \times S_2$. By Definition 13, $F_I \times G_J = (F,I) \times (G,J) = (Q, I \times J)$, where $Q(x,y) = F(x) \times G(y)$ for all $(x,y) \in I \times J$. Then for all $(x_1,y_1),(x_2,y_2) \in I \times J$ and $(s_1,s_2) \in S_1 \times S_2$,

i_1) $Q((x_1,y_1)+(x_2,y_2)) = Q(x_1+x_2, y_1+y_2) = F(x_1+x_2) \times G(y_1+y_2) \supseteq (F(x_1) \cap F(x_2)) \times (G(y_1) \cap G(y_2)) = (F(x_1) \times G(y_1)) \cap (F(x_2) \times G(y_2)) = Q(x_1,y_1) \cap Q(x_2,y_2)$,
i_2) $Q((s_1,s_2)(x_1,y_1)) = Q(s_1x_1, s_2y_1) = F(s_1x_1) \times G(s_2y_1) \supseteq F(x_1) \times G(y_1) = Q(x_1,y_1)$,
i_3) $Q((x_1,y_1)(s_1,s_2)) = Q(x_1s_1, y_1s_2) = F(x_1s_1) \times G(y_1s_2) \supseteq F(x_1) \times G(y_1) = Q(x_1,y_1)$.

Therefore, $F_I \times G_J = Q_{I \times J} \widetilde{\lhd} S_1 \times S_2$.

It is worth noting that if I_1 and I_2 are two ideals of a semiring $(S,+,.)$, then the sum of these two ideals is defined as the following: $I_1 + I_2 = \{i_1 + i_2 \mid i_1 \in I_1 \wedge i_2 \in I_2\}$.

Definition 14. *Let (F,I_1) and (G,I_2) be two int-soft ideals of S. If $I_1 \cap I_2 = \{0\}$, then sum of int-soft ideals (F,I_1) and (G,I_2) is defined as $(F,I_1)+(G,I_2) = (Q,I_1+I_2)$, where $Q(x+y) = F(x)+G(y)$ for all $x+y \in I_1+I_2$.*

Theorem 12. *If $F_{I_1} \widetilde{\lhd} S$ (resp. $F_{I_1} \widetilde{\lhd_l} S$, $F_{I_1} \widetilde{\lhd_r} S$) and $G_{I_2} \widetilde{\lhd} S$ (resp. $G_{I_2} \widetilde{\lhd_l} S$, $G_{I_2} \widetilde{\lhd_r} S$), where $(I_1,+)$ and $(I_2,+)$ are abelian groups and $I_1 \cap I_2 = \{0\}$, then $F_{I_1}+G_{I_2} \widetilde{\lhd} S$ (resp. $F_{I_1}+G_{I_2} \widetilde{\lhd_l} S$, $F_{I_1}+G_{I_2} \widetilde{\lhd_r} S$).*

Proof. We give the proof for int-soft ideals, the same proof can be seen for int-soft-left ideal and int-soft-right ideals, too. Since I_1 and I_2 are ideals of S, where $(I_1,+)$ and $(I_2,+)$ are groups, then I_1+I_2 is an ideal of S. By Definition 14, let $F_{I_1}+G_{I_2} = (F,I_1)+(G,I_2) = (Q,I_1+I_2)$, where $Q(x+y) = F(x)+G(y)$ for all $x+y \in I_1+I_2$. Then, for all $x_1+y_1, x_2+y_2 \in I_1+I_2$ and $s \in S$,

$$\begin{aligned}
Q((x_1+y_1)+(x_2+y_2)) &= Q((x_1+x_2)+(y_1+y_2)) \\
&= F(x_1+x_2)+G(y_1+y_2) \\
&\supseteq (F(x_1)\cap F(x_2))+(G(y_1)\cap G(y_2)) \\
&= (F(x_1)+G(y_1))\cap(F(x_2)+G(y_2)) \\
&= Q(x_1+y_1)+Q(x_2+y_2), \\
Q(s(x_1+y_1)) &= Q(sx_1+sy_1) \\
&= F(sx_1)+G(sy_1) \\
&\supseteq F(x_1)+G(y_1) \\
&= Q(x_1+y_1), \\
Q((x_1+y_1)s) &= Q(x_1s+y_1s) \\
&= F(x_1s)+G(y_1s) \\
&\supseteq F(x_1)+G(y_1) \\
&= Q(x_1+y_1).
\end{aligned}$$

Therefore $F_{I_1}+G_{I_2} = Q_{I_1+I_2} \widetilde{\lhd} S$.

Definition 15. *Let (F,I) be a soft int-soft subsemiring (int-soft ideal) of S. Then,*

i) (F,I) is said to be whole if $F(x) = 0$ for all $x \in S$.
ii) (F,I) is said to be whole if $F(x) = S$ for all $x \in S$.

Proposition 8. *Let (F,I_1) and (G,I_2) be int-soft subsemirings (resp. int-soft ideals) of S such that $I_1 \cap I_2 \neq \emptyset$. Then,*

i) If (F,I_1) and (G,I_2) are trivial int-soft subsemirings (resp. int-soft ideals) of S, then $(F,I_1) \Cap (G,I_2)$ is a trivial int-soft subsemiring (resp. int-soft ideal) of S.
ii) If (F,I_1) and (G,I_2) are whole int-soft subsemirings (resp. int-soft ideals) of S, then $(F,I_1) \Cap (G,I_2)$ is a whole int-soft subsemiring (resp. int-soft ideal) of S.
iii) If (F,I_1) is a trivial int-soft subsemiring (resp. int-soft ideal) of S and (G,I_2) is a whole int-soft subsemiring (resp. int-soft ideal) of S, then $(F,I_1) \Cap (G,I_2)$ is a trivial int-soft subsemiring (resp. int-soft ideal) of S.
iv) If (F,I_1) and (G,I_2) are trivial int-soft ideals of S, where $(I_1,+)$ and $(I_2,+)$ are abelian groups and $I_1 \cap I_2 = \{0\}$, then $(F,I_1)+(G,I_2)$ is a trivial int-soft ideal of S.
v) If (F,I_1) and (G,I_2) are whole int-soft ideals of S, where $(I_1,+)$ and $(I_2,+)$ are abelian groups and $I_1 \cap I_2 = \{0\}$, then $(F,I_1)+(G,I_2)$ is a whole int-soft ideal of S.
vi) If (F,I_1) is a trivial int-soft ideal of S and (G,I_2) is a whole int-soft ideal of S, where $(I_1,+)$ and $(I_2,+)$ are abelian groups and $I_1 \cap I_2 = \{0\}$, then $(F,I_1)+(G,I_2)$ is a whole int-soft ideal of S.

Proof. The proof is obvious, hence omitted.

Proposition 9. *Let (F,I_1) and (G,I_2) be two int-soft subsemirings (resp. int-soft ideals) of S_1 and S_2, respectively. Then,*

i) If (F,I_1) and (G,I_2) are trivial int-soft subsemirings (resp. int-soft ideals) of S_1 and S_2, respectively, then $(F,I_1)\times(G,I_2)$ is a trivial int-soft subsemiring (resp. int-soft ideal) of $S_1 \times S_2$.
ii) If (F,I_1) and (G,I_2) are whole int-soft subsemirings (resp. int-soft ideals) of S_1 and S_2, respectively, then $(F,I_1)\times(G,I_2)$ is a whole int-soft subsemiring (resp. int-soft ideal) of $S_1 \times S_2$.

Proof. Straightforward, hence omitted.

6 Some applications of int-soft substructures of semirings

In this section, we give some applications of int-soft subsemirings and int-soft ideals of a semiring to semiring theory.

Proposition 10. *Let $(S,+,.)$ be a semiring and let M be a subsemiring of S such that $(M,+)$ is a group. If $F_M \widetilde{<} S$, then $M_F = \{x \in M \mid F(x) = F(0)\}$ is a subsemiring of M.*

Proof. It is obvious that $0 \in M_F$ and $\emptyset \neq M_F \subseteq M$. We need to show that $x+y \in M_F$ and $xy \in M_F$ for all $x,y \in M_F$, which means that $F(x+y) = F(0)$ and $F(xy) = F(0)$ have to be satisfied. Since $x,y \in M_F$, then $F(x) = F(y) = F(0)$. Since (F,M) is an int-soft subsemiring of S, then $F(x+y) \supseteq F(x)\cap F(y) = F(0)$ and $F(xy) \supseteq F(x)\cap F(y) = F(0)$ for all $x,y \in M_F$. Moreover, by Proposition 7, $F(0) \supseteq F(x+y)$ and $F(0) \supseteq F(xy)$. Therefore, M_F is a subsemiring of M.

Theorem 13. *Let $(S,+,.)$ be a semiring and I be an ideal (resp. left ideal, right ideal) of S such that $(I,+)$ is a group. If $F_I \widetilde{\lhd} S$ (resp. $F_I \widetilde{\lhd_l} S$, $F_I \widetilde{\lhd_r} S$), then $I_F = \{x \in I \mid F(x) = F(0)\}$ is an ideal (resp. left ideal, right ideal) of S.*

Proof. It is obvious that $0 \in I_F$ and $\emptyset \neq I_F \subseteq S$. In view of Proposition 7, $F(0) \supseteq F(x)$ for all $x \in I$. We need to show that $F(x+y) = F(0)$, $F(sx) = F(0)$ and $F(xs) = F(0)$ for all $x,y \in I_F$ and $s \in S$. Since $x+y \in I$, $sx \in I$ and $xs \in I$, $F(0) \supseteq F(x+y)$, $F(0) \supseteq Fsx)$ and $F(0) \supseteq F(xs)$. Moreover for all $x,y \in I_F$ and $s \in S$, $F(x+y) \supseteq F(x)\cap F(y) = F(0)$, $F(sx) \supseteq F(x) = F(0)$ and $F(xs) \supseteq F(x) =$

$F(0)$ since $F_I \widetilde{\lhd} S$. Therefore $x+y \in I_F$, $sx \in I_F$ and $xs \in I_F$ for all $x,y \in I_F$ and $s \in S$. Hence, I_F is an ideal of S. The proof is seen for int-soft-left and int-soft-right ideals, too.

Theorem 14. *Let S_1 be a semiring with zero 0_{S_1}, S_2 be a semiring with zero 0_{S_2} and $(F_1,M_1)\widetilde{<}S_1$, $(F_2,M_2)\widetilde{<}S_2$. If $f : M_1 \to M_2$ is a semiring homomorphism, then*

a) If f is an epimorphism, then $(F_1, f^{-1}(M_2))\widetilde{<}S_1$,
b) $(F_2, f(M_1))\widetilde{<}S_2$,
c) $(F_1, Ker f)\widetilde{<}S_1$.

Proof. **a)** Since $M_1 < S_1$, $M_2 < S_2$ and $f : M_1 \to M_2$ is a semiring epimorphism, then it is clear that $f^{-1}(M_2) < S_1$. Since $(F_1,M_1)\widetilde{<}S_1$ and $f^{-1}(M_2) \subseteq M_1$, $F_1(x+y) \supseteq F_1(x) \cap F_1(y)$ and $F_1(xy) \supseteq F_1(x) \cap F_1(y)$ for all $x,y \in f^{-1}(M_2)$. Hence $(F_1, f^{-1}(M_2))\widetilde{<}S_1$.

b) Since $M_1 < S_1$, $M_2 < S_2$ and $f : M_1 \to M_2$ is a semiring homomorphism, then $f(M_1) < S_2$. Since $f(M_1) \subseteq M_2$, the result is obvious by Definition 10.

c) Since $Ker f < S_1$ and $Ker f \subseteq M_1$, the rest of the proof is clear by Definition 10.

Corollary 3. *Let $(F_1,M_1)\widetilde{<}S_1$, $(F_2,M_2)\widetilde{<}S_2$ and $f : M_1 \to M_2$ is a semiring homomorphism, then $(F_2, \{0_{S_2}\})\widetilde{<}S_2$.*

Proof. By Theorem 14 (c), $(F_1, Ker f)\widetilde{<}S_1$. Then $(F_2, f(Ker f)) =$ $(F_2, \{0_{S_2}\})\widetilde{<}S_2$ by Theorem 14 (b).

7 Conclusion

Throughout this paper, we deal with the algebraic intersection-soft substructures of a group. We first introduce int-soft subgroups and int-soft normal subgroups of a group. Then, we have investigated the relations between int-soft subgroups and int-soft normal subgroups under certain conditions of the group and obtain their related properties. Moreover, we introduce the algebraic intersection-soft substructures of a semiring, that is, int-soft subsemiring and int-soft ideal of a semiring and some related properties are investigated and illustrated by many examples. Finally, we give some applications of these new concepts to group theory and semiring theory.

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