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On the Calculus of Spatial Relations

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Abstract: A logical calculus based a method is compared with topological relation of 4 Intersection Model (4-IM) of spatial regions. The first result of this paper is answer of the following question: what is the smallest formula represents the four intersection model?. The second main result, as an algebraic view, under certain conditions, the family of all relations from 9-IM using the 4-IM of two closed regions A and B are obtained.

Keywords: Spatial topological relations, 4-Intersection model, 9-Intersection model, logic of spatial relations.

1 Introduction

Representation of spatial information is important in many applications such as Geographic Information System (GIS). In 1990, [4] and [5] M. Egenhofer and others introduced a mathematical method for classifying topological relationships between spatial regions in the plane. Many researchers follows Egenhofer and established the relations between two regions, two lines, two points, region to line, region to point, and line to point. (see [2,3,6,8]). One object of this paper, in section3, is to give the connection between the spatial geographic relations and the logical calculus of 4-IM, and obtain the smallest set of formulas which represents the spatial topological relations. The next section 4, we give an algebraic view of all possible Boolean matrices of the topological relations of two spatial regions with a non-empty exterior.

2 Preliminaries

We recall some general notation and results taking mostly from [1,2,4,6] these will be required in the paper .

Let (X,τ) be a topological space. For a subset $A \subseteq X$, we write write A° be the interior of A in X.i.e. the largest open set contained in A, and $\overline{A}$ be the closure of A in X i.e. the smallest closed set containing A. The boundary of A, written by ∂A is defined by the set of difference between $\overline{A}$ and A°.i.e. $\partial A = \overline{A} - A^\circ$. Moreover, the exterior of A , denoted by A^{ext}, which is defined by

$$A^{ext} = (X - \overline{A}) = (X - A)^\circ .$$

It is known that, the set $\{A^\circ, \partial A, A^{ext}\}$ forms a partition of X.

A closed subset $A \subseteq X$ is called regular if $A = \overline{A^\circ}$, that is the smallest regular closed set containing A°.

A region is a nonempty proper regular closed subset of the real plane R^2. As a topological space, the collection of all regions in the real plane together with $\varnothing$ and R^2 forms a complete Boolean algebra.

In this paper, we use the simple region, which is a homeomorphic to the unit closed disk, and hence a simple region (for short, a region) has a connected boundary and connected exterior and has no holes.

For more details and properties of topological spaces, we refer to [1]. In the sense of Egenhofer [4] and [5], the topological relation between two connected closed regions A and B can be characterized by considering the intersections of interiors and boundaries of the two regions. The results can be written as the following matrix

$$m(A,B) = \begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B \\ \partial A \cap B^\circ & \partial A \cap \partial B \end{pmatrix}$$

If the intersection is empty, we write 0 for the entry in the corresponding matrix , and write 1 for a non empty intersection. By this way, the topological relations between two regions A and B can be represents as a Boolean matrices (i.e. each element in the matrix takes only 0 or 1) of degree 2×2. Hence, there are $2^4 = 16$ possible Boolean matrices.

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It is known that, the Boolean matrices representation of geographic relations between two regions are:

$$m_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, m_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, m_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, m_4 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$m_5 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, m_6 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, m_7 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, m_8 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Where m_1, m_2, m_3, m_4, m_5, m_6, m_7 and m_8 represents the disjoint, meet, equal, inside, contains, cover, covered by and overlap relations between two regions, respectively. For more details and properties of Egenhofer relations, we refer to [2,4,5] and [6].

3 Logical calculus of 4-IM

In the present section, we give the answer of the following question: what is the smallest formula represents the four intersection model?

A logical sentence (or proposition) is the sentence deals only with the value 1 (true) or 0 (false).

A logical sentence combined in different ways by using the connectives *"not"*, *"and"*, *"or"*, *"if...then"* and *"if and only if"*. We use the symbols, *p, q, r, s,...* for logical sentences (logical sentences are called variables). We use the symbols $\neg$, $\wedge$, $\vee$, $\longrightarrow$, $\longleftrightarrow$ for the connectives *"not"*, *"and"*, *"or"*, *"if...then"* and *"if and only if"*, respectively. The symbols $\neg$, $\wedge$, $\vee$, $\longrightarrow$, $\longleftrightarrow$ are called a negation, a conjunction, a disjunction, a conditional and biconditional connective, respectively.

Formulas are expressions build by means logical connectives and variables are denoted by F_1, F_2, $F_3, \ldots$.

For any formula F, let $Var(F)$ be the set of all propositional variables appearing in F. The function

$$v : Var(F) \to \{0,1\}$$

is called a truth assignment restricted to F

Theorem 3.1.[9] For any formula F there are $2^{|Var(F)|}$ possible truth assignments restricted to F.

Definition 3.2.[7] Let V_n be the set of all n-tuples $(a_1,\ a_2,\ldots,\ a_n)$ over the Boolean algebra $B = \{0,\ 1\}$. An element $v \in V_n$ is called a Boolean vector of dimension n.

Now, let $U = (P_1, P_2, P_3, P_4)$ be a Boolean vector of four components, where $P_1 = A^\circ \cap B^\circ$, $P_2 = A^\circ \cap \partial B$, $P_3 = \partial A \cap B^\circ$ and $P_4 = \partial A \cap \partial B$. Let $V = \{v_i : 1 \le i \le 16\}$ be the set of all functions v_i from the set U into the set $\{0,1\}$. It is clear that $|V| = 2^4$. We can describe the set V by the table (1).

For any $v_i \in V$ can be represented as a Boolean vector as v_i=($v_i(P_1)$,$v_i(P_2)$,$v_i(P_3$,$v_4)$.

From table (1), we have

Table 1: The set $V = \{v_i : 1 \le i \le 16\}$

	P_1	P_2	P_3	P_4			P_1	P_2	P_3	P_4
v_1	0	0	0	0		v_9	1	0	0	0
v_2	0	0	0	1		v_{10}	0	1	0	0
v_3	1	0	0	1		v_{11}	0	0	1	0
v_4	1	0	1	0		v_{12}	0	1	1	0
v_5	1	1	0	0		v_{13}	0	1	0	1
v_6	1	1	0	1		v_{14}	0	0	1	1
v_7	1	0	1	1		v_{15}	1	1	1	0
v_8	1	1	1	1		v_{16}	0	1	1	1

$$v_1 = (v_1(P_1), v_1(P_2), v_1(P_3), v_1(P_4)) = (0,0,0,0),$$
$$v_2 = (v_2(P_1), v_2(P_2), v_2(P_3), v_2(P_4)) = (0,0,0,1),$$
$$\ldots \ldots \ldots$$

$$v_{16} = (v_{16}(P_1), v_{16}(P_2), v_{16}(P_3), v_{16}(P_4)) = (0,1,1,1).$$

Define the order relation "$\le$" on the set V as:

$$v_i \le v_j \quad \text{iff} \quad v_i(P_k) \le v_j(P_k), \quad 1 \le i,j \le 16, \quad k = 1,2,3,4.$$

We observe that, the algebraic structure $(V, \wedge, \vee, ', v_1, v_8)$ forms a Boolean algebra, where

$$\begin{aligned} v_i \wedge v_j &= (\min(v_i(P_1), v_j(P_1)), \min(v_i(P_2), v_j(P_2)), \\ &\quad \min(v_i(P_3), v_j(P_3)), \min(v_i(P_4), v_j(P_4))) \\ v_i \vee v_j &= \max(v_i(P_1), v_j(P_1)), \max(v_i(P_2), v_j(P_2)), \\ &\quad \max(v_i(P_3), v_j(P_3)), \max(v_i(P_4), v_j(P_4))), \end{aligned}$$

and $v_i' = (1 - v_i(P_1), 1 - v_i(P_2), 1 - (v_i(P_3), 1 - v_i(P_4))$, $1 \le i \le 16$.

Also, we see that v_1 and v_8 are the smallest and largest element of the set V, respectively. Now, we formulate.

Theorem 3.3. The algebraic structure $(V; \wedge, \vee, ', v_1, v_8)$ forms a Boolean algebra.

The Hass diagram of the Boolean algebra V shown as Figure (1). The corresponding possible values make sense in GIS called the Egenhofer relations, see[6], given by the set

$$G = \{v_i : v_i = (v_i(P_1), v_i(P_2), v_i(P_3),\ v_i(P_4)), 1 \le i \le 8\}.$$

We indicate v_1 as the disjoint relation, v_2 as the meet relation, v_3 as the equal relation, v_4 as the inside relation, v_5 as the cover relation, v_6 as the covered by relation, v_7 as the contains relation and v_8 as the overlap relation.

Clearly, the poset $(G; \le)$ is a partially ordered set which is neither meet nor join semi-lattice with the same operations defined on the Boolean algebra V because $(1,0,1,0) \vee (1,1,0,0) = (1,1,1,0)$ and $(1,0,1,0) \wedge (1,1,0,0) = (1,0,0,0)$ do not belong to G. Figure(2) represents the Hass diagram of the pair $(G; \le)$.

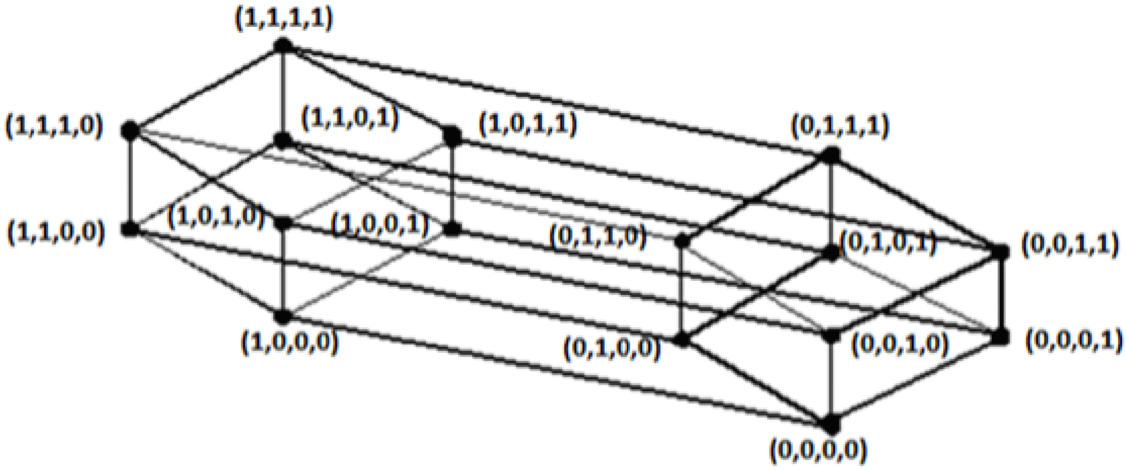

Fig. 1: The Boolean algebra $V = (V; \wedge, \vee, ', v_1, v_8)$

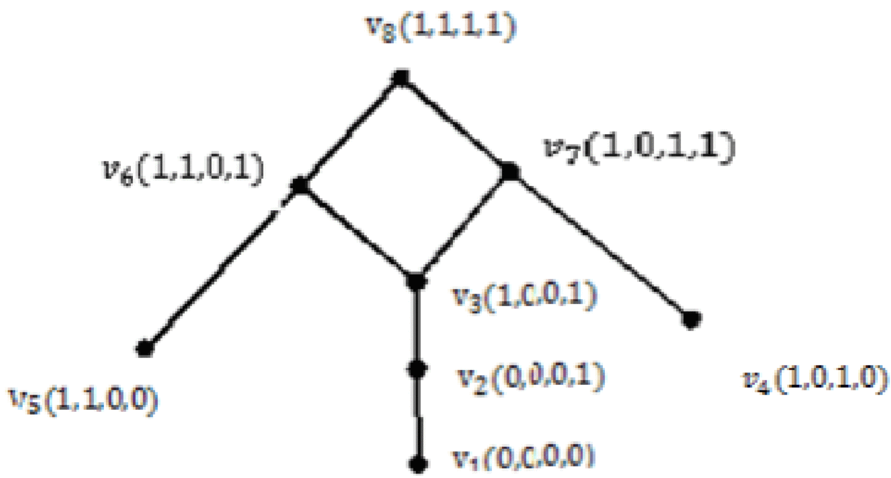

Fig. 2: The partially order set $(G, \leq)$

Now, consider the map $F_i : B^2 \to B,\ B = \{0,1\},\ 1 \leq i \leq 8$.

Let $F = \{F_i(p,q) :\ p,\ q \in \{1,0\},\ \ 1 \leq i \leq 8\}$, where

$F_1(p,q) = ((p' \vee q) \vee (p \vee q'))'$,
$F_2(p,q) = (p \vee q)'$,
$F_3(p,q) = (p' \vee q) \wedge (p \vee q')$,
$F_4(p,q) = \begin{cases} 1 \ \ if \ \ p = q = 1 \ \ or \ \ p = 0, \ \ q = 1 \\ 0 \ \ if \ \ p = q = 0 \ \ or \ \ p = 1, \ \ q = 0 \end{cases}$

$F_5(p,q) = \begin{cases} 1 \ \ if \ \ p = q = 1 \ \ or \ \ p = 1, \ \ q = 0 \\ 0 \ \ if \ \ p = 1, q = 0 \ \ or \ \ p = 0, \ \ q = 0 \end{cases}$
$F_6(p,q) = (p \vee q')$,
$F_7(p,q) = (p' \vee q)$ and
$F_8(p,q) = ((p' \vee q) \vee (p \vee q'))$.

Consider the set

$$\overline{F} = \{\overline{F_i} : \overline{F_i} = (F_i(1,1), F_i(1,0), F_i(0,1), F_i(0,0)),\ 1 \leq i \leq 8\}.$$

Define the map $\alpha : G \to \overline{F_i}$ by $\alpha(v_i) = \overline{F_i}$ for all $v_i \in G,\ 1 \leq i \leq 8$, where

$$\alpha(v_i(P_1)) = F_i(1,1),\ (v_i(P_2)) = F_i(1,0),$$
$$\alpha(v_i(P_3)) = F_i(0,1) \text{ and } \alpha(v_i(P_4)) = F_i(0,0).$$

If $v_3 = (v_3(P_1), v_3(P_2), v_3(P_3), v_3(P_4)) = (1,0,0,1)$, then

$$\begin{aligned}\alpha(v_3(P_1)) &= F_3(1,1) = (1' \vee 1) \wedge (1 \vee 1') \\ &= (0 \vee 1) \wedge (1 \vee 0) = 1, \\ \alpha(v_3(P_2)) &= F_3(1,0) = (1' \vee 0) \wedge (1 \vee 0') \\ &= (0 \vee 0) \wedge (1 \vee 1) = 0, \\ \alpha(v_3(P_3)) &= F_3(0,1) = (0' \vee 1) \wedge (0 \vee 1') \\ &= (1 \vee 1) \wedge (0 \vee 0) = 0, \\ \alpha(v_3(P_4)) &= F_3(0,0) = (0' \vee 0) \wedge (0 \vee 0') \\ &= (1 \vee 0) \wedge (0 \vee 1) = 1.\end{aligned}$$

Hence,

$$\alpha(v_3(P_1)) = F_3(1,1)\ ,\ \alpha(v_3(P_2)) = F_3(1,0),$$
$$\alpha(v_3(P_3)) = F_3(0,1)\ ,\ \alpha(v_3(P_4)) = F_3(0,0).$$

That is

$$\begin{aligned}\alpha(v_3) &= \alpha((v_3(P_1)), \alpha(v_3(P_2)), \alpha(v_3(P_3)), \alpha(v_3(P_4))) \\ &= (F_3(1,1), F(1,0), F_3(0,1), F_3(0,0)) \\ &= F_3 = (1,0,0,1).\end{aligned}$$

Hence, v_3 corresponding to $\bar{F_3}$. By similar way, for all $v_i,\quad 1 \leq i \leq 8$. Set $v_4 \leq v_7$. Then $v_4(P_k) \leq v_7(P_k)$, $k = 1,2,3,4$. Now,

$$\begin{aligned}\alpha(v_4(P_1)) &= F_4(1,1) = 1, \\ \alpha(v_7(P_1)) &= F_7(1,1) = (1' \vee 1) = 1, \\ \alpha(v_4(P_2)) &= F_4(1,0) = 0, \\ \alpha(v_7(P_2)) &= F_7(1,0) = (1' \vee 0) = 0, \\ \alpha(v_4(P_3)) &= F_4(0,1) = 1, \\ \alpha(v_7(P_3)) &= F_7(0,1) = (0' \vee 1) = 1, \\ \alpha(v_4(P_4)) &= F_4(0,0) = 0, \text{ and} \\ \alpha(v_7(P_4)) &= F_7(0,0) = (0' \vee 0) = 1.\end{aligned}$$

Hence, in all cases, we get

$$(\alpha(v_4(P_1)), \alpha(v_4(P_2)), \alpha(v_4(P_3)), \alpha(v_4(P_4))) \leq (\alpha(v_7(P_1)), \alpha(v_7(P_2)), \alpha(v_7(P_3)), \alpha(v_7(P_4)))$$

That is $\alpha(v_4) \leq \alpha(v_7)$.
Clearly, v_4 is incomparable with v_5 . By similar way , we can prove that for any $1 \leq i,j \leq 8$, we have $v_i \leq v_j$ implies that $\alpha(v_i) \leq \alpha(v_j),\ 1 \leq i,j \leq 8$.

From the above discussion , we can investigate that

Theorem 3.4. There is one-one correspondence between the set of Egenhofer relations G and the set of Boolean vectors $\overline{F}$.

The following Theorem gives the answer of the following question: what is the smallest number of formulas can be represents the set $\overline{F}$?.

Theorem 3.5. For each $\overline{F}_i \in \overline{F}, 1 \leq i \leq 8$ can be written as a new formula by using $\overline{F}_2$, $\overline{F}_6$ or $\overline{F}_7$.

Proof

Since for any $p,q \in \{1,0\}$. Let $\overline{F}_i = (F_i(1,1),\ F_i(1,0),\ F_i(0,1),\ F_i(0,0)),\ 1 \leq i \leq 8$, where $\overline{F}_2(p,q) = (p \vee q)'$, $\overline{F}_6(p,q) = (p \vee q')$ and $\overline{F}_7(p,q) = (p' \vee q)$. Then

$$\begin{aligned}\overline{F}_1(p,q) &= \neg(\overline{F}_6(p,q) \wedge \neg\overline{F}_7(p,q)), \\ \overline{F}_3(p,q) &= \neg\overline{F}_6(p,q) \wedge \overline{F}_7(p,q), \\ \overline{F}_4)(p,q) &= \neg\overline{F}_2(p,q) \leftrightarrow \overline{F}_7(p,q), \\ \overline{F}_5(p,q) &= \neg\overline{F}_2(p,q) \leftrightarrow \overline{F}_6(p,q), \text{ and} \\ \overline{F}_8(p,q) &= \overline{F}_6(p,q) \vee \overline{F}_7(p,q).\end{aligned}$$

Theorem 3.5, tell us the basic relations of Egenhofer relations are the meet, cover and contains relations.

4 Construction of Boolean matrices for two regions with the same exterior

The 9-IM intersection model based on considering the exterior beside the interior and boundary of two connected closed regions A and B. Consider the matrix of nine intersections given by

$$M(A,B) = \begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B & A^\circ \cap B^{ext} \\ \partial A \cap B^\circ & \partial A \cap \partial B & \partial A \cap B_{ext} \\ A^{ext} \cap B^\circ & A^{ext} \cap \partial B & A^{ext} \cap B_{ext} \end{pmatrix}.$$

By considering the empty and non-empty intersections of such nine sets, we have the $2^9 = 512$ possible combination. Excluding the impossible cases , we get the same 8-relationships as the 4-IM, see [5]. In this section, we discus in a algebraic view, under certain conditions, how many relations can be obtained from 9-IM using the 4-IM of two closed regions A and B ?. The matrices of 4-IM in the sense of Egenhofer relations are given by

$$m_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ m_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ m_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ m_4 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$m_5 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},\ m_6 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ m_7 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ m_8 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The matrices representation of 9-IM are given by

$$D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ M_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},\ E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$I_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix},\ N_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ C_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1\,0\,0 \\ 1\,1\,0 \\ 1\,1\,1 \end{pmatrix}, \; V_1 = \begin{pmatrix} 1\,1\,1 \\ 1\,1\,1 \\ 1\,1\,1 \end{pmatrix}.$$

Clearly, for each Boolean matrix of degree 33 of 9-IM, there is no row or column can be $(0,0,0)$ as:

Lemma 4.1.[6] For each matrix M of the 9-IM, there is no row or column be $(0,0,0)$.

Lemma 4.2. For each matrix M of the 9-IM, then the element lies in the position $\alpha_{33} = 1$.

Proof. From the definition of the 9-IM, we have for any two regions A and B of a topological space (X,τ), the position $\alpha_{33} = A^{ext} \cap B^{ext}$. Thus

$$A^{ext} \cap B^{ext} = (X - \overline{A}) \cap (X - \overline{B}) = X - (\overline{A} \cap \overline{B}) \neq \emptyset.$$

And hence in the 9-IM matrix the element lies in the position $\alpha_{33} = 1$. Now, we can embedding of 4-IM into the 9-IM as

$$\begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B \\ \partial A \cap B^\circ & \partial A \cap \partial B \end{pmatrix} \rightarrow \begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B & A^\circ \cap B^{ext} \\ \partial A \cap B^\circ & \partial A \cap \partial B & \partial A \cap B_{ext} \\ A^{ext} \cap B^\circ & A^{ext} \cap \partial B & A^{ext} \cap B_{ext} \end{pmatrix}.$$

So,

$$\begin{pmatrix} 0\,0 \\ 0\,0 \end{pmatrix} \rightarrow \begin{pmatrix} 0\,0\,1 \\ 0\,0\,1 \\ 0\,0\,1 \end{pmatrix}, \begin{pmatrix} 0\,0 \\ 0\,1 \end{pmatrix} \rightarrow \begin{pmatrix} 0\,0\,1 \\ 0\,1\,1 \\ 1\,1\,1 \end{pmatrix},$$

$$\begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix} \rightarrow \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix}, \begin{pmatrix} 1\,0 \\ 1\,0 \end{pmatrix} \rightarrow \begin{pmatrix} 1\,0\,0 \\ 1\,0\,0 \\ 1\,1\,1 \end{pmatrix},$$

$$\begin{pmatrix} 1\,1 \\ 0\,0 \end{pmatrix} \rightarrow \begin{pmatrix} 1\,1\,1 \\ 0\,0\,1 \\ 0\,0\,1 \end{pmatrix}, \begin{pmatrix} 1\,1 \\ 0\,1 \end{pmatrix} \rightarrow \begin{pmatrix} 1\,1\,1 \\ 0\,1\,1 \\ 0\,0\,1 \end{pmatrix},$$

$$\begin{pmatrix} 1\,0 \\ 1\,1 \end{pmatrix} \rightarrow \begin{pmatrix} 1\,0\,0 \\ 1\,1\,0 \\ 1\,1\,1 \end{pmatrix}, \begin{pmatrix} 1\,1 \\ 1\,1 \end{pmatrix} \rightarrow \begin{pmatrix} 1\,1\,1 \\ 1\,1\,1 \\ 1\,1\,1 \end{pmatrix}.$$

For The Eignhofer relations, the question is : How many Boolean matrices can be obtained under the constraints of Lemma 4.1 and Lemma 4.2.

For the disjoint relation, i.e. the two region A and B are disjoint, under the constraint of Lemmas 4.1 and 4.2, there is one and only one matrix obtained form the disjoint relation as.

$$D = \begin{pmatrix} 0\,0\,1 \\ 0\,0\,1 \\ 0\,0\,1 \end{pmatrix}.$$

For the meet relation, the two region A and B are meet in point or line, by using the constraint of lemmas 4.1 and 4.2, then the remaining value of the intersection value of $\alpha_{23} = \partial A \cap B^{ext}$ and $\alpha_{32} = A^{ext} \cap \partial B$,

$$M = \begin{pmatrix} 0\,0\,1 \\ 0\,1\,? \\ 1\,?\,1 \end{pmatrix}.$$

Then there are 2^2 possible Boolean matrices for meet relations as

$$M_1 = \begin{pmatrix} 0\,0\,1 \\ 0\,1\,1 \\ 1\,1\,1 \end{pmatrix}, M_2 = \begin{pmatrix} 0\,0\,1 \\ 0\,1\,0 \\ 1\,0\,1 \end{pmatrix},$$
$$M_3 = \begin{pmatrix} 0\,0\,1 \\ 0\,1\,1 \\ 1\,0\,1 \end{pmatrix}, M_4 = \begin{pmatrix} 0\,0\,1 \\ 0\,1\,0 \\ 1\,1\,1 \end{pmatrix}.$$

The four matrices are realizable, see [6].

For equal relation, the two region A and B are coincide, we embed the equal relation of 4-IM into the 9-IM and apply Lemmas 4.1 and 4.2,

$$E = \begin{pmatrix} 0\,0\,? \\ 0\,1\,? \\ ?\,?\,1 \end{pmatrix},$$

It remain the value of the intersections of $A^o \cap B^{ext}$, and $\partial A \cap B^{ext}$, $A^{ext} \cap B^o$ and $A^{ext} \cap \partial B$. Then there are 2^4 possible Boolean matrices as:

$$E_1 = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix}, E_2 = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 1\,0\,1 \end{pmatrix}, E_3 = \begin{pmatrix} 1\,0\,1 \\ 0\,1\,1 \\ 0\,0\,1 \end{pmatrix},$$
$$E_4 = \begin{pmatrix} 1\,0\,1 \\ 0\,1\,0 \\ 1\,0\,1 \end{pmatrix}, \; E_5 = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 1\,1\,1 \end{pmatrix}, \; E_6 = \begin{pmatrix} 1\,0\,1 \\ 0\,1\,1 \\ 1\,0\,1 \end{pmatrix},$$
$$E_7 = \begin{pmatrix} 1\,0\,1 \\ 0\,1\,0 \\ 1\,1\,1 \end{pmatrix}, \; E_8 = \begin{pmatrix} 1\,0\,1 \\ 0\,1\,1 \\ 1\,1\,1 \end{pmatrix}, \; E_9 = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,1 \\ 0\,1\,1 \end{pmatrix},$$
$$E_{10} = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,1 \\ 1\,0\,1 \end{pmatrix}, \; E_{11} = \begin{pmatrix} 1\,0\,1 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix}, \; E_{12} = \begin{pmatrix} 1\,0\,1 \\ 0\,1\,1 \\ 0\,1\,1 \end{pmatrix},$$
$$E_{13} = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,1 \\ 0\,0\,1 \end{pmatrix}, \; E_{14} = \begin{pmatrix} 1\,0\,1 \\ 0\,1\,0 \\ 0\,1\,1 \end{pmatrix}, \; E_{15} = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,1 \\ 1\,1\,1 \end{pmatrix},$$
$$E_{16} = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,1\,1 \end{pmatrix}.$$

It is possible that the matrices E_1 to E_8 are represented in [6].

Let X^T be the transpose matrix of a matrix X. Then we have
$E_1 = E_1^T,\; E_4 = E_4^T,\; E_7 = E_6^T,\; E_9 = E_9^T,\; E_{10} = E_{14}^T,$

$E_{11}=E_2^T, E_{12}=E_{15}^T, E_{13}=E_{16}^T$.

For inside relation, the region A inside the region B, under the constraints of Lemmas 4.1 and 4.2, we have

$$I=\begin{pmatrix}1&0&?\\0&1&?\\?&1&1\end{pmatrix}.$$

Then there are 2^3 possible for meet relations as

$$I_1=\begin{pmatrix}1&0&0\\1&0&0\\1&1&1\end{pmatrix}, I_2=\begin{pmatrix}1&0&1\\1&0&1\\1&1&1\end{pmatrix}, I_3=\begin{pmatrix}1&0&0\\1&0&0\\0&1&1\end{pmatrix},$$
$$I_4=\begin{pmatrix}1&0&1\\1&0&0\\0&1&1\end{pmatrix}, I_5=\begin{pmatrix}1&0&0\\1&0&1\\0&1&1\end{pmatrix}, I_6=\begin{pmatrix}1&0&1\\1&0&1\\0&1&1\end{pmatrix},$$
$$I_7=\begin{pmatrix}1&0&1\\1&0&0\\1&1&1\end{pmatrix}, I_8=\begin{pmatrix}1&0&0\\1&0&1\\1&1&1\end{pmatrix}.$$

The matrices $I_1$and I_2 are represented in [6].

For the contains relation, the two region B inside the region A, Consider the constraints of Lemmas 4.1 and 4.2, we have

$$N=\begin{pmatrix}1&1&?\\0&0&1\\?&?&1\end{pmatrix}$$

Then there are 2^3 possible for meet relation as

$$N_1=\begin{pmatrix}1&1&1\\0&0&1\\0&0&1\end{pmatrix}, N_2=\begin{pmatrix}1&1&1\\0&0&1\\1&1&1\end{pmatrix}, N_3=\begin{pmatrix}1&1&0\\0&0&1\\0&0&1\end{pmatrix},$$
$$N_4=\begin{pmatrix}1&1&0\\0&0&1\\1&0&1\end{pmatrix}, N_5=\begin{pmatrix}1&1&0\\0&0&1\\0&1&1\end{pmatrix}, N_6=\begin{pmatrix}1&1&0\\0&0&1\\1&1&1\end{pmatrix},$$
$$N_7=\begin{pmatrix}1&1&1\\0&0&1\\1&0&1\end{pmatrix}, N_8=\begin{pmatrix}1&1&1\\0&0&1\\0&1&1\end{pmatrix}.$$

The matrices $N_1$and N_2 are represented in [6]. For the transpose matrices, we have
$I_1=N_1^T$, $I_2=N_2^T$, $I_3=N_3^T$, $I_4=N_4^T$, $I_5=N_5^T$, $I_6=N_6^T$, $I_7=N_7^T$, $I_8=N_8^T$.

For the cover relation, the region A cover the region B, we get

$$C=\begin{pmatrix}1&1&?\\0&1&?\\?&?&1\end{pmatrix}.$$

By lemmas 4.1 and 4.2, we have 2^4 possible matrices

$$C_1=\begin{pmatrix}1&1&1\\0&1&1\\1&1&1\end{pmatrix}, C_2=\begin{pmatrix}1&1&1\\0&1&1\\1&0&1\end{pmatrix}, C_3=\begin{pmatrix}1&1&1\\0&1&0\\1&1&1\end{pmatrix},$$
$$C_4=\begin{pmatrix}1&1&1\\0&1&0\\1&0&1\end{pmatrix}, C_5=\begin{pmatrix}1&1&0\\0&1&0\\1&1&1\end{pmatrix}, C_6=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},$$
$$C_7=\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}, C_8=\begin{pmatrix}1&1&1\\0&1&1\\0&1&1\end{pmatrix}, C_9=\begin{pmatrix}1&1&0\\0&1&1\\0&1&1\end{pmatrix},$$
$$C_{10}=\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}, C_{11}=\begin{pmatrix}1&1&0\\0&1&1\\1&1&1\end{pmatrix}, C_{12}=\begin{pmatrix}1&1&0\\0&1&0\\1&0&1\end{pmatrix},$$
$$C_{13}=\begin{pmatrix}1&1&0\\0&1&0\\0&1&1\end{pmatrix}, C_{14}=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}, C_{15}=\begin{pmatrix}1&1&1\\0&1&0\\0&0&1\end{pmatrix},$$
$$C_{16}=\begin{pmatrix}1&1&1\\0&1&0\\0&1&1\end{pmatrix}.$$

The matrices from C_1 to C_4 are represented in [6].

For the covered relation, the region B cover the region A, we have

$$B=\begin{pmatrix}1&0&?\\1&1&?\\?&?&1\end{pmatrix}.$$

By Lemma 4.1 and 4.2, then there are 2^4 possible

$$B_1=\begin{pmatrix}1&0&1\\1&1&1\\1&1&1\end{pmatrix}, B_2=\begin{pmatrix}1&0&1\\1&1&0\\1&1&1\end{pmatrix}, B_3=\begin{pmatrix}1&0&1\\1&1&1\\1&0&1\end{pmatrix},$$
$$B_4=\begin{pmatrix}1&0&1\\1&1&0\\1&0&1\end{pmatrix}, B_5=\begin{pmatrix}1&0&1\\1&1&1\\0&0&1\end{pmatrix}, B_6=\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix},$$
$$B_7=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}, B_8=\begin{pmatrix}1&0&0\\1&1&1\\1&1&1\end{pmatrix}, B_9=\begin{pmatrix}1&0&0\\1&1&1\\0&1&1\end{pmatrix},$$
$$B_{10}=\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}, B_{11}=\begin{pmatrix}1&0&1\\1&1&1\\0&1&1\end{pmatrix}, B_{12}=\begin{pmatrix}1&0&1\\1&1&0\\0&0&1\end{pmatrix},$$
$$B_{13}=\begin{pmatrix}1&0&0\\1&1&1\\0&0&1\end{pmatrix}, B_{14}=\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix}, B_{15}=\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix},$$
$$B_{16}=\begin{pmatrix}1&0&0\\1&1&1\\1&0&1\end{pmatrix}.$$

The matrices from B_1 to B_6 are represented in [6]. For the transpose matrices, we have
$C_1=B_1^T, C_2=B_2^T, C_3=B_3^T, C_4=B_4^T, C_5=B_5^T, C_6=B_6^T, C_7=B_7^T, C_8=B_8^T, C_9=B_9^T, C_{10}=B_{10}^T, C_{11}=B_{11}^T, C_{12}=B_{12}^T, C_{13}=B_{13}^T, C_{14}=B_{14}^T, C_{15}=B_{15}^T, C_16=B_{16}^T$

For the overlap relation, the matrices are

$$V=\begin{pmatrix}1&1&?\\1&1&?\\?&?&1\end{pmatrix}$$

Then there are 2^4 possible

$$V_1=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix},\ V_2=\begin{pmatrix}1&1&1\\1&1&0\\1&0&1\end{pmatrix},\ V_3=\begin{pmatrix}1&1&1\\1&1&1\\1&0&1\end{pmatrix},$$

$$V_4=\begin{pmatrix}1&1&1\\1&1&0\\1&1&1\end{pmatrix},\ V_5=\begin{pmatrix}1&1&0\\1&1&1\\1&1&1\end{pmatrix},\ V_6=\begin{pmatrix}1&1&1\\1&1&1\\0&1&1\end{pmatrix},$$

$$V_7=\begin{pmatrix}1&1&1\\1&1&0\\0&1&1\end{pmatrix},\ V_8=\begin{pmatrix}1&1&0\\1&1&1\\1&0&1\end{pmatrix},V_9=\begin{pmatrix}1&1&0\\1&1&1\\0&1&1\end{pmatrix},$$

$$V_{10}=\begin{pmatrix}1&1&1\\1&1&1\\0&0&1\end{pmatrix},\ V_{11}=\begin{pmatrix}1&1&0\\1&1&0\\1&1&1\end{pmatrix},\ V_{12}=\begin{pmatrix}1&1&0\\1&1&0\\1&0&1\end{pmatrix},$$

$$V_{13}=\begin{pmatrix}1&1&1\\1&1&0\\0&0&1\end{pmatrix},\ V_{14}=\begin{pmatrix}1&1&0\\1&1&0\\0&1&1\end{pmatrix},\ V_{15}=\begin{pmatrix}1&1&0\\1&1&1\\0&0&1\end{pmatrix},$$

$$V_{16}=\begin{pmatrix}1&1&0\\1&1&0\\0&0&1\end{pmatrix}.$$

The matrices from V_1 to V_4 are represented in [6]. For the transpose matrices, we have
$V_1=V_1^T,\ V_2=V_2^T, V_3=V_4^T, V_5=V_6^T, V_7=V_8^T, V_9=V_9^T,$
$V_{10}=V_{11}^T, V_{12}=V_{13}^T, V_{14}=V_{15}^T, V_{16}=V_{16}^T.$

5 Conclusion

Representation of spatial information is important in many applications such as Geographic Information System (GIS). In 1990 M. Egenhofer and others introduced a mathematical method for classifying topological relationships between spatial regions in the plane. Many researchers follows Egenhofer and established the relations between two regions, two lines, two points, region to line, region to point, and line to point. In this presentation, we observed the connection between the spatial geographic relations and the logical calculus of 4-IM. The smallest set of formulas which represents the spatial topological relations is obtained. Also, we gave an algebraic view of all possible Boolean matrices of the spatial relations of two spatial regions with a non-empty exterior. As a future work on this topic, we hope to study more applications of the spatial geographic relations by using our representation.

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