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Note on a Picard-like Method for Caputo Fuzzy Fractional Differential Equations

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Abstract: A Picard-like approach which has been used to solve a class of Volterra integro-differential equations, is extended in this manuscript to solve fuzzy fractional differential equations. Such technique uses quadrature rules and Picard's iterations in the fuzzy context. In spite of this, it is conceived to become a non-recursive scheme, in terms of operational matrices, in the linear regime. Some properties of the method are thoroughly discussed, and some numerical examples are provided in order to illustrate the effectiveness of the approach.

Keywords: fractional differential equations, Caputo fuzzy derivative, quadrature rules, Picard-like method, convergence

1 Introduction

The use of fractional calculus has been incorporated in many branches of mathematics, engineering and science to provide more accurate deterministic descriptions of physical phenomena [1,10,18]. However, in reality, the presence of uncertainties needs to be considered to achieve a higher level of reliability [16,23]. In this manuscript, the notion of *uncertainty* will be interpreted as *fuzziness* [28]. Fuzziness is an important characteristic to be considered in realistic decision processes [9], in regression analysis [14], efficient data classifiers [26] and, in general, in modeling complex problems in science and engineering [11].

With those considerations in mind, Agarwal et al. [3] introduced the notion of Fuzzy Fractional Differential Equations (FFDEs). To that end, fractional operators such as the Riemann-Liouville and the Caputo derivatives have been adapted to the fuzzy scenario. Various works have investigated these models using the Riemann-Liouville derivatives [7,27]. The analytical solution, its existence and its uniqueness for a class of FFDEs with Caputo derivatives were discussed in [6]. However, the aim of the present work is to solve numerically FFDEs employing Caputo-type derivatives.

Fuzzy fractional initial-value problems under fuzzy fractional derivatives of the Caputo type were solved by means of a modified fractional Euler method in [20]. Also, the shifted Legendre operational matrix of fuzzy fractional derivatives was used in [5] to solve numerically FFDEs. More recently, FFDEs with Caputo derivatives were used in [4] to model the kinetics behavior of the diluted acid hydrolysis in oil palm frond. In that work, the FFDEs were solved numerically by means of a fuzzy operational matrix of generalized Laguerre polynomials. Also, FFDEs with Caputo derivatives were solved by the differential transform method (DTM) in [22]. Moreover, homotopy techniques have been used for solving fuzzy fractional diffusion equations with Caputo derivatives [24], and related initial-value problems [2].

In the present work, the following problem governed by a FFDE is considered for each $x \in I = [0,T]$ and $0 < \beta < 1$:

$$\begin{cases} {}^{c}D^{\beta}\tilde{y}(x) = F(\tilde{y}) + \tilde{g}(x), \\ \text{subject to } \tilde{y}(0) = \tilde{a}_0. \end{cases} \quad (1)$$

Here, T is a positive number, $\tilde{a}_0$ is a fuzzy number, $\tilde{y}(x)$ is the unknown fuzzy function, $F(\tilde{y})$ a functional form in $\tilde{y}$, and $\tilde{g}(x)$ is in general a given fuzzy-valued function. Note that the FFDE of (1) may be seen as the fuzzification

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through the Zadeh's extension principle of the same equation but without fuzzy variables [28].

The purpose of this work is to solve (1) following an approach analogous to that used in [25], which was employed to solve a class of integro-differential equations and recently a class of partial differential equations [15]. More precisely, a numerical scheme combining quadrature rules and a Picard-like recursion is studied here from the perspective of fuzzy fractional derivatives. Some properties are formally discussed. Numerical results obtained through this technique are compared against analytical and numerical solutions available in the literature, obtaining satisfactory results.

This manuscript is sectioned as follows. Some basic definitions are provided in Section 2. Section 3 is devoted to introducing the proposed Picard-like method to solve (1). Some analytical results are derived in Section 4. Meanwhile, Section 5 is devoted to show simulations and numerical comparisons against some results available in the literature in order to illustrate the performance of our technique. Finally, this work closes with a section of concluding remarks.

2 Preliminaries

Throughout, the set U will represent a nonempty and fixed (though arbitrary) closed interval of $\mathbb{R}$. The term *crisp* will mean 'not fuzzy'.

Definition 1.*A* fuzzy number $\tilde{u}$ *is defined by a membership function* $\mu_u(x): U \to [0,1]$ *which satisfies the following properties:*

- $\tilde{u}$ *is* **normal**, *meaning that* $\sup_{x\in U} \mu_u(x) = 1$,
- $\tilde{u}$ *is* **convex** *on* U, *that is,*

$$\mu_u(\alpha x + (1-\alpha)y) \geq \min(\mu_u(x), \mu_u(y)) \quad (2)$$

for each $x, y \in U$ *and each* $\alpha \in [0,1]$,
- $\tilde{u}$ *is upper semi-continuous, and*
- $[\tilde{u}]_0 = \mathrm{cl}(\{x \in U : \mu_u(x) > 0\})$ *is compact. Here* cl *denotes closure in the standard topology of* U.

Definition 2.*Let* $\alpha \geq 0$. *The* α-cut *of the fuzzy number* $\tilde{u}$ *is the crisp set*

$$[\tilde{u}]_\alpha = \{x \in U : \mu_u(x) \geq \alpha\} \quad (3)$$

when $\alpha > 0$, *otherwise* $[\tilde{u}]_0$ *is given as in Definition 1.*

Definition 3.*The* parametric form *of the fuzzy number* $\tilde{u}$ *is a pair of functions* $\underline{u}(\alpha), \overline{u}(\alpha): U \to \mathbb{R}$ *for each* $\alpha \in [0,1]$, *which satisfy the following properties:*

1. $\underline{u}(\alpha)$ *is a bounded, left-continuous, monotonic increasing function,*
2. $\overline{u}(\alpha)$ *is a bounded, left-continuous, monotonic decreasing function, and*
3. $\underline{u}(\alpha) \leq \overline{u}(\alpha)$.

The notation $[\tilde{u}]_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)]$ *is employed if such form is available.*

Definition 4.*A fuzzy number* $\tilde{u}$ *is called* triangular *if there exist real numbers* d_C, d_L *and* d_R, *such that*

$$[\tilde{u}]_\alpha = [d_C + (\alpha-1)d_L, d_C + (1-\alpha)d_R], \quad (4)$$

for each $\alpha \in [0,1]$. *If that is the case then* $\tilde{u}$ *is identified by the ordered triplet* (d_C, d_L, d_R), *and the numbers* d_C, d_L *and* d_R *are called the* center, *the* left *and the* right spreads, *respectively.*

In the sequel, $\tilde{f}(x)$ will denote a continuous and Lebesgue-integrable fuzzy-valued function on $[a,b] \subset \mathbb{R}$.

Definition 5.*The* fuzzy Riemann-Liouville integral *of order* β *of* $\tilde{f}(x)$ *is given as*

$$J^\beta \tilde{f}(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \tilde{f}(s) ds, \quad (5)$$

where Γ *represents the Gamma function. According to [20], the* α*-cut representation of* $J^\beta \tilde{f}(x)$ *is provided by*

$$[J^\beta \tilde{f}(x)]_\alpha = [J^\beta \underline{f}(x,\alpha), J^\beta \overline{f}(x,\alpha)]. \quad (6)$$

Definition 6.*Let* $f(x)$ *be a crisp continuous function, and* $0 < \beta < 1$. *The* Caputo fractional derivative *of order* β *of* f *is defined in [21] as*

$$D^\beta f(x) = \frac{1}{\Gamma(1-\beta)} \int_0^x (x-s)^{-\beta} \frac{df}{dx}(s) ds. \quad (7)$$

The concept of strong generalized H-differentiability studied in [8] was extended in [20] to the context of fractional derivatives. In the latter work, the following definition of differentiability was considered.

Definition 7.*Let* $0 < \beta < 1$. *The fuzzy-valued function* $\tilde{f}(x)$ *is a* Caputo fuzzy fractional differentiable function *of order* β *at* $x_0 \in I$ *if either*

$$\begin{aligned} \tilde{f}'(x_0) &= \lim_{h\to 0^+} \frac{\tilde{f}(x_0+h) \ominus \tilde{f}(x_0)}{h} \\ &= \lim_{h\to 0^+} \frac{\tilde{f}(x_0) \ominus \tilde{f}(x_0-h)}{h} \end{aligned} \quad (8)$$

or

$$\begin{aligned} \tilde{f}'(x_0) &= \lim_{h\to 0^+} \frac{\tilde{f}(x_0) \ominus \tilde{f}(x_0+h)}{-h} \\ &= \lim_{h\to 0^+} \frac{\tilde{f}(x_0-h) \ominus \tilde{f}(x_0)}{-h}, \end{aligned} \quad (9)$$

and

$${}^cD^\beta \tilde{f}(x_0) = \frac{1}{\Gamma(1-\beta)} \int_0^{x_0} \tilde{f}'(s)(x_0-s)^{-\beta} ds, \quad (10)$$

where ${}^cD^\beta \tilde{f}(x_0)$ denotes the Caputo fuzzy fractional derivative of $\tilde{f}$ at x_0.

Let $0 \leq \alpha \leq 1$. Following [20], the α-cut form of the Caputo fuzzy fractional derivative is

$$[{}^cD^\beta \tilde{f}(x_0)]_\alpha = \begin{cases} [{}^cD^\beta \underline{f}(x_0,\alpha), {}^cD^\beta \overline{f}(x_0,\alpha)], \text{ for (8)}, \\ [{}^cD^\beta \overline{f}(x_0,\alpha), {}^cD^\beta \underline{f}(x_0,\alpha)], \text{ for (9)}. \end{cases} \tag{11}$$

For briefness, in the following we will refer to the first equation of (11) to illustrate our numerical approach.

Let $\tilde{y}(x)$ be a fuzzy function. Using [12], one may readily check that $[\tilde{y}(x)]_\alpha = [\underline{y}(x,\alpha), \overline{y}(x,\alpha)]$. As shown in [20], problem (1) is equivalent to

$$\begin{cases} {}^cD^\beta \mathbf{Y}(x,\alpha) = \mathbf{F}(\mathbf{Y}(x,\alpha),\alpha) + \mathbf{G}(x,\alpha), \\ \mathbf{Y}(0,\alpha) = \mathbf{A}(\alpha), \end{cases} \tag{12}$$

where

$$\mathbf{F}(\mathbf{Y}(x,\alpha),\alpha)^T = (\underline{F}(\underline{y}(x,\alpha), \overline{y}(x,\alpha),\alpha), \overline{F}(\underline{y}(x,\alpha), \overline{y}(x,\alpha),\alpha)), \tag{13}$$

$$\mathbf{Y}(x,\alpha)^T = (\underline{y}(x,\alpha), \overline{y}(x,\alpha)), \tag{14}$$

$$\mathbf{G}(x,\alpha)^T = (\underline{g}(x,\alpha), \overline{g}(x,\alpha)) \tag{15}$$

3 Methodology

In this section, the approach employed in [25] is extended in order to solve the FFDE of (12). Letting $\gamma = \frac{1}{\Gamma(\beta)}$, applying the operator J^β to both sides of that equation and arguing as in [20], one obtains

$$\mathbf{Y}(x,\alpha) = \mathbf{A}(\alpha) + \gamma \int_0^x [\mathbf{F}(\mathbf{Y}(x,\alpha),\alpha) + \mathbf{G}(x,\alpha)](x-s)^{\beta-1} ds. \tag{16}$$

Let N be a positive integer and consider a (not necessarily uniform) partition of $[0,T]$ consisting of N points and partition norm equal to h, say,

$$0 = x_1 < x_2 < \ldots < x_N = T. \tag{17}$$

Then (16) may be rewritten as

$$\mathbf{Y}(x,\alpha) = \mathbf{A}(\alpha) + \gamma \mathbf{C}(x) [\mathbf{F}(\mathbf{Y}(\alpha),\alpha) + \mathbf{Q}(\alpha)], \tag{18}$$

in which

$$\mathbf{F}(\mathbf{Y}(\alpha),\alpha)^T = (\underline{F}(\mathbf{Y}_1,\alpha), \ldots, \underline{F}(\mathbf{Y}_N,\alpha), \overline{F}(\mathbf{Y}_1,\alpha), \ldots, \overline{F}(\mathbf{Y}_N,\alpha)), \tag{19}$$

$$\mathbf{Y}_i^T = (\underline{y}(x_i,\alpha), \overline{y}(x_i,\alpha)), \tag{20}$$

$$\mathbf{A}(\alpha)^T = (\underline{a}_0(\alpha), \overline{a}_0(\alpha)), \tag{21}$$

$$\mathbf{Q}^T(\alpha) = (\underline{g}(x_1,\alpha), \ldots, \underline{g}(x_N,\alpha), \overline{g}(x_1,\alpha), \ldots, \overline{g}(x_N,\alpha)), \tag{22}$$

and

$$\mathbf{C}(x) = \begin{pmatrix} C_1^\beta(x) \ldots C_N^\beta(x) & 0 & \ldots & 0 \\ 0 & \ldots & 0 & C_1^\beta(x) \ldots C_N^\beta(x) \end{pmatrix}. \tag{23}$$

Here, for each $i = 1, 2, \ldots, N$,

$$C_i^\beta(x) = \int_0^x l_i(s)(x-s)^{\beta-1} ds, \tag{24}$$

with $l_i(s)$ being the ith Lagrange polynomial. By successive approximations, the solution $\mathbf{Y}(x,\alpha)$ is

$$\mathbf{Y}(x,\alpha) = \sum_{k=0}^\infty \mathbf{Y}_k(x,\alpha), \tag{25}$$

where each $\mathbf{Y}_k^T(x,\alpha)$ has to be determined recursively using the formulas

$$\mathbf{Y}_0(x,\alpha) = \mathbf{A}(\alpha) + \gamma \mathbf{C}(x) \mathbf{Q}(\alpha), \tag{26}$$

$$\mathbf{Y}_{k+1}(x,\alpha) = \gamma \mathbf{C}(x) \mathbf{F}(\mathbf{Y}_k(\alpha),\alpha). \tag{27}$$

On the other hand, truncating (25) at the pth term yields

$$\mathbf{Y}^{[p]}(x,\alpha) = \mathbf{Y}_0(x,\alpha) + \gamma \mathbf{C}(x) \sum_{k=1}^{p-1} \mathbf{F}(\mathbf{Y}_k(\alpha),\alpha). \tag{28}$$

Note that (27) reduces to $\mathbf{Y}_{k+1}(x,\alpha) = \gamma_0 \mathbf{C}(x) \mathbf{Y}_k(\alpha)$ when $F(\tilde{y})$ is a linear function, where

$$\mathbf{Y}_k(\alpha)^T = (\underline{y}_k(x_1,\alpha), \ldots, \underline{y}_k(x_N,\alpha), \overline{y}_k(x_1,\alpha), \ldots, \overline{y}_k(x_N,\alpha)) \tag{29}$$

and γ_0 is a constant. Let $\mathbf{D}$ be the matrix whose entries are $D_{ij} = C_j(x_i)$. Since $\mathbf{Y}_k(\alpha) = \gamma_0 \mathbf{D} \mathbf{Y}_{k-1}(\alpha) = \gamma_0^k \mathbf{D}^k \mathbf{Y}_0(\alpha)$, the truncation of (25) after p terms becomes

$$\mathbf{Y}^{[p]}(x,\alpha) = \mathbf{Y}_0(x,\alpha) + \gamma_0 \mathbf{C}(x) \sum_{k=0}^{p-1} \gamma_0^k \mathbf{D}^k \mathbf{Y}_0(\alpha). \tag{30}$$

4 Properties

The present section summarizes the main properties of the method of Section 3. For the remainder, the spectral radius of the matrix $\mathbf{D}$ will be represented by $\rho(\mathbf{D})$, and the symbol $\mathbf{I}$ will denote the identity matrix of size $2N$.

Lemma 1. *If $F(\tilde{y})$ is linear and $\rho(\mathbf{D}) < \frac{1}{|\gamma_0|}$ then the solution of (25) is approximated by*

$$\mathbf{Y}(x,\alpha) = \mathbf{Y}_0(x,\alpha) + \gamma_0 \mathbf{C}(x)(\mathbf{I} - \gamma_0 \mathbf{D})^{-1} \mathbf{Y}_0(\alpha). \tag{31}$$

Proof. The proof follows as that of Theorem 1 in [25], considering a geometric series approximation of (30) with matrices and taking then the limit when $p \to \infty$. □

Table 1: Values of $\underline{y}(1,\alpha)$ and $\overline{y}(1,\alpha)$, for various values of α and several methods. The FFDE and the parameters used are described in Example 1.

VALUES OF $\underline{y}(1,\alpha)$						
α	**Present 7 GCL**	**Present 15 GCL**	**Method of [5]**	**Method of [20]**	**Method of [22]**	**Exact**
0.0	0.196543	0.196554	0.1962	0.183	0.1967	0.1966
0.1	0.216197	0.216209	0.2159	0.201	0.2164	0.2162
0.2	0.235851	0.235864	0.2355	0.219	0.2360	0.2359
0.3	0.255505	0.255520	0.2551	0.238	0.2557	0.2555
0.4	0.275160	0.275175	0.2747	0.256	0.2754	0.2752
0.5	0.294814	0.294831	0.2944	0.274	.2950	0.2948
0.6	0.314468	0.314486	0.3140	0.293	0.3147	0.3145
0.7	0.334122	0.334141	0.3336	0.311	0.3344	0.3341
0.8	0.353777	0.353797	0.3532	0.329	.3540	0.3538
0.9	0.373431	0.373452	0.3729	0.348	0.3737	0.3735
1.0	0.393085	0.393107	0.3925	0.366	0.3934	0.3931

VALUES OF $\overline{y}(1,\alpha)$						
α	**Present 7 GCL**	**Present 15 GCL**	**Method of [5]**	**Method of [20]**	**Method of [22]**	**Exact**
0.0	0.589627	0.589661	0.5887	0.549	0.5901	0.5897
0.1	0.569973	0.570005	0.5691	0.534	0.5704	0.5700
0.2	0.550319	0.550350	0.5495	0.513	0.5507	0.5504
0.3	0.530664	0.530694	0.5298	0.494	0.5311	0.5307
0.4	0.511010	0.511039	0.5102	0.476	0.5114	0.5110
0.5	0.491356	0.491384	0.4906	0.458	0.4917	0.4914
0.6	0.471702	0.471728	0.4710	0.439	0.4721	0.4717
0.7	0.452047	0.452073	0.4514	0.421	0.4524	0.4521
0.8	0.432393	0.432418	0.4317	0.403	0.4327	0.4324
0.9	0.412739	0.412762	0.4121	0.384	0.4130	0.4128
1.0	0.393085	0.393107	0.3925	0.366	0.3934	0.3931

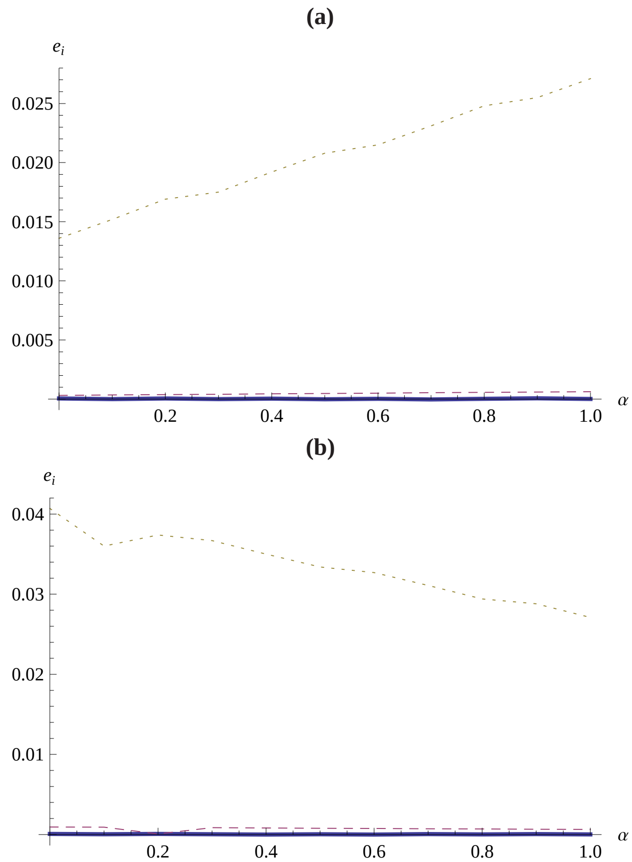

Fig. 1: Absolute errors with respect to the exact solution for the calculated values of **(a)** $\underline{y}(1,\alpha)$ and **(b)** $\overline{y}(1,\alpha)$ for $\alpha \in [0,1]$, using the present method (solid), the method used in [5] (dashed), and that employed in [20] (dotted). The FFDE and the parameters used are described in Example 1.

The generalized Taylor's formula under the Caputo fractional derivative was established in [20]. That formula will be a useful tool in the next results. In the sequel, the exact solution of the FFDE in (12) will be represented by $\mathbf{Y}(x,\alpha) = [\underline{y}(x,t,\alpha), \overline{y}(x,\alpha)]$. The norm of L_∞ will be employed next in view of the continuity of the functions involved (see [13]). So, in what follows,

$$e_\infty(x_j,\alpha) = \left\| \mathbf{Y}^{[p]}(x_j,\alpha) - \mathbf{Y}(x_j,\alpha) \right\|_\infty . \tag{32}$$

Theorem 1. *Let $0 < \beta < 1$ and $0 \le \alpha \le 1$. Let $\underline{F}(\mathbf{Y}(x,\alpha),\alpha)$, $\overline{F}(\mathbf{Y}(x,\alpha),\alpha)$, $\underline{g}(\alpha)$ and $\overline{g}(\alpha)$ be continuous 2β-times differentiable crisp functions with respect to x. The following bound holds for any $0 < h < 1$:*

$$\begin{aligned} e_\infty(x_j,\alpha) \le \Bigg\| & \gamma \mathbf{C}((j-1)h) \sum_{k=1}^{p-1} \mathbf{F}(\mathbf{Y}_k(\alpha),\alpha) \\ & - \frac{(j-1)^\beta h^\beta}{\Gamma(\beta+1)} [\mathbf{F}(\mathbf{Y}(0,\alpha),\alpha) \\ & + \mathbf{G}(0,\alpha)] \Bigg\|_\infty + \mathcal{O}(h^{2\beta}), \end{aligned} \tag{33}$$

Proof. The truncation of the generalized Taylor expansion of $\mathbf{Y}(x,\alpha)$ around $x=0$ at the first two terms is

$$\begin{aligned} \mathbf{Y}(x,\alpha) = \mathbf{A}(\alpha) + & \frac{x^\beta}{\Gamma(\beta+1)} D^\beta \mathbf{Y}(0,\alpha) \\ & + D^{2\beta} \mathbf{Y}(\chi,\alpha) \frac{x^{2\beta}}{\Gamma(2\beta+1)}, \end{aligned} \tag{34}$$

for some $\chi \in [0,T]$. Using (12), the following inequality results for each node x_j and each $\alpha \in [0,1]$:

$$\begin{aligned} e_\infty(x_j,\alpha) \le \Bigg\| & \gamma \mathbf{C}(x_j) \sum_{k=1}^{p-1} \mathbf{F}(\mathbf{Y}_k(\alpha),\alpha) \\ & - \frac{x_j^\beta}{\Gamma(\beta+1)} [\mathbf{F}(\mathbf{Y}(0,\alpha),\alpha) \\ & + \mathbf{G}(0,\alpha)] \Bigg\|_\infty + \mathcal{O}(x_j^{2\beta}), \end{aligned} \tag{35}$$

For each $j = 1,\ldots,N$, note that $x_j = x_1 + (j-1)h$ or $x_j \le x_1 + (j-1)h$ in the case of uniform and nonuniform partitions, respectively. The conclusion of the theorem readily follows now. □

Theorem 1 implies that the error tends to zero when $h \to 0$, and that the approximation error has order h^β.

Table 2: Values and errors of $\underline{y}(1,\alpha)$ and $\overline{y}(1,\alpha)$, for various values of α and several methods. The FFDE and the parameters used are described in Example 3.

VALUES OF $\underline{y}(1,\alpha)$				
α	Present 11 GCL	Exact	Error	Error with [5]
0.0	0.720607	0.7206	7.60×10^{-6}	1.44×10^{-4}
0.1	0.722132	0.7221	3.25×10^{-5}	1.44×10^{-4}
0.2	0.723657	0.7237	4.26×10^{-5}	1.45×10^{-4}
0.3	0.725182	0.7252	1.76×10^{-5}	1.45×10^{-4}
0.4	0.726707	0.7267	7.28×10^{-6}	1.46×10^{-4}
0.5	0.728232	0.7282	3.22×10^{-5}	1.47×10^{-4}
0.6	0.729757	0.7298	4.29×10^{-5}	1.47×10^{-4}
0.7	0.731282	0.7313	1.18×10^{-5}	1.48×10^{-4}
0.8	0.732807	0.7328	6.96×10^{-6}	1.49×10^{-4}
0.9	0.734332	0.7343	3.19×10^{-5}	1.49×10^{-4}
1.0	0.735856	0.7359	4.32×10^{-5}	1.50×10^{-4}
VALUES OF $\overline{y}(1,\alpha)$				
α	Present 11 GCL	Exact	Error	Error with [5]
0.0	0.739669	0.7397	3.09×10^{-5}	1.51×10^{-4}
0.1	0.739288	0.7393	1.21×10^{-5}	1.51×10^{-4}
0.2	0.738907	0.7389	6.65×10^{-6}	1.51×10^{-4}
0.3	0.738525	0.7385	2.54×10^{-5}	1.51×10^{-4}
0.4	0.738144	0.7381	4.42×10^{-5}	1.46×10^{-4}
0.5	0.737763	0.7378	3.70×10^{-5}	1.47×10^{-4}
0.6	0.737382	0.7374	1.83×10^{-5}	1.47×10^{-4}
0.7	0.737000	0.7370	4.96×10^{-7}	1.48×10^{-4}
0.8	0.736619	0.7366	1.92×10^{-5}	1.49×10^{-4}
0.9	0.736238	0.7362	3.80×10^{-5}	1.49×10^{-4}
1.0	0.735856	0.7359	4.32×10^{-5}	1.50×10^{-4}

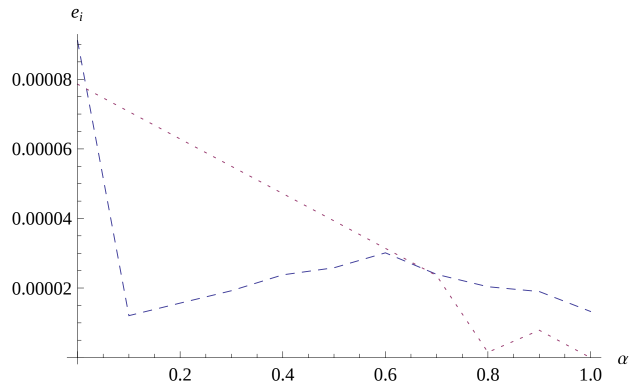

Fig. 2: Absolute errors with respect to the exact solution for the calculated values of $\overline{y}(1,\alpha)$ for $\alpha\in[0,1]$, using the present method (dashed) and the method in [5] (dotted). The FFDE and the parameters used are described in Example 2.

5 Simulations

The present section shows comparisons of known exact solutions of (1) against approximations obtained via (28) when (31) holds. Throughout, Gauss-Tchebyshev-Lobatto (GCL) grid points [17] are used, and all fuzzy numbers are triangular with $T=1$. The exact solution of the first example was taken from [20], and the solutions of the last two are borrowed from [5].

*Example 1.*Let $\beta=0.75$, $d_c=1$, $d_L=0.5$, $d_R=0.5$, $F(\tilde{y})=-\tilde{y}(x)$, and fix $g\equiv 0$. Table 1 shows the solutions $\underline{y}$ and $\overline{y}$ at $x=1$ using the method proposed in the present paper (with different number of GCL points), and other methods available in the literature. In turn, Figure 1 shows the absolute errors of the present method and the methods of the literature with respect to the exact solutions. Clearly, the present methodology yields more accurate results. □

*Example 2.*Let $\beta=0.75$, $d_c=0$, $d_L=1$, $d_R=1$, and consider

$$F(\tilde{y}(x))=-\tilde{y}(x), \tag{36}$$

$$g(x)=\frac{2x^{2-\beta}}{\Gamma(3-\beta)}-\frac{x^{1-\beta}}{\Gamma(2-\beta)}+x^2-x. \tag{37}$$

The method introduced in this manuscript is used to calculate $\overline{y}(1,\alpha)$ for various values of α using 20 GCL points. Figure 2 shows the absolute error using the present method (dashed line) and the technique reported in [5] (dotted line). The graph shows that the approximation reported in this manuscript produces more accurate results for values of $\alpha\in[0.05,0.6]$. In any case, the order of the error is 10^{-5}. The simulations corresponding to $\underline{y}$ have been omitted in view that they are qualitatively similar to those of $\overline{y}$. □

*Example 3.*Fix $\beta=0.85$, $d_c=1$, $d_L=0.04$, $d_R=0.01$. Let $F(\tilde{y}(x))=-\tilde{y}(x)$ and $g(x)=\sin x$. Table 2 provides the approximate solution $\underline{y}$ and $\overline{y}$ at $x=1$ for several values of α, calculated using the present method. The results show that the absolute error using the present technique with 11 GCL points is substantially smaller than the error obtained using the technique reported in [5]. □

6 Conclusions

A Picard-like numerical scheme, which was previously employed to solve a class of Volterra integro-differential equations, has been extended in this manuscript to solve problems involving Caputo fuzzy fractional differential equations.

The proposed approach presents two main advantages:

– it is able to reproduce exactly the initial condition;
– in the linear regime, it becomes a non-recursive scheme in terms of known operational matrices.

For the general nonlinear scenario, some formal considerations on the error have been discussed. Some numerical examples illustrate the effectiveness of the proposed approach. However, ongoing research is focusing on extending the method towards piecewise solutions for certain classes of problems, similarly to the approach proposed in [19] for fuzzy partial differential equations.

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