Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/110138

Operators of Minimal Norm via Modified Green's Function for the Elastic Two-Dimensional Case Using the Simple Multipole Coefficients

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Received: 7 Jun. 2016, Revised: 21 Sep. 2016, Accepted: 23 Sep. 2016
Published online: 1 Jan. 2017

Abstract: The problem of non-uniqueness arising in the integral formulation of an exterior boundary value problem for the elastic two-dimensional case can be faced using the fundamental solution technique. In this work, a criterion based on the minimization of the norm of the modified integral operator is established using simple multipole coefficients. As applications, the proposed procedure in the case of circle and perturbations of circle are examined.

Keywords: Simple multipole coefficients, fundamental solution, integral equations of Fredholm type, elasticity.

1 Introduction

It is well known that the problem of non-uniqueness arising in the integral formulation of an exterior boundary value has been treated with the addition of series of outgoing waves to the free-space fundamental solution, that is with the modified Green's function technique. This method was introduced by Jones [9] and Ursell [20] to treat the exterior Dirichlet and Neumann problem for the Helmholtz equation. The appropriate choice of the simple multipole coefficients of the added series to the free-space fundamental solution guarantees the uniqueness solvability of the boundary integral equation which describes the problem. Kleinmann and Roach [12], have shown that in addition to uniqueness solvability of the integral equation, the simple multipole coefficients of the modification could be chosen so that the modified Green's function is the best approximation to the actual Green's function for the problem in the least squares sense. In [11], the same authors, motivated by a desire not only to ensure uniqueness solvability, but also to provide a constructive method of solving the integral equation, they have chosen as a criterion, the minimization of the norm of the modified integral operator. In the same way, Kleinmann and Kress [10] presented another criterion choosing the simple multipole coefficients of the modification, that of the minimization of the condition number of the integral equation. All the above mentioned work referred to the acoustical case.

Applying the modified Green's function technique for the elastic case, the problem of the irregular frequencies arising in the integral equation of Fredholm type can be removed as well. Although, the main ideas in both acoustic and elastic cases are the same, however, the corresponding results for the elasticity case require more complicated procedures compared with the acoustical case. This is due to the complexity of the problem in elasticity. The first work which adopts the modified Green's function technique in elasticity is due to Jones [8], who examined the cavity in $\mathbb{R}^3$. In [5,7], Bencheikh have been also consider elastic problems in $\mathbb{R}^2$. In [2,4], the exterior Dirichlet problem in $\mathbb{R}^3$ is investigated by Argyropoulos, Kiriaki and Roach, and the non-uniqueness of the boundary integral equation is overcome with a suitable choice of simple mutipole coefficients in the modification. Not so far, we have presented another criterion choosing the coefficients of the modification, that of the minimization of the norm of the modified fundamental solution for the elastic two-dimensional case [16,17].

In this paper, the criterion of operators of minimal norm via modified Green's function for the elastic two-dimensional case is investigated. If the norm of the modified integral operator can be made small enough then

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the modified integral equation can be solved by iteration. So, if the simple multipole coefficients of the modification are chosen to satisfy this criterion of the minimal norm, then the unique solvability of the integral equation is ensured. In the case of three-dimensional elastic waves, some similar results are given by Argyropoulos and Kiriaki [3].

This paper is organized as follows. In section 2, we present the modified Green's function technique, also the free space fundamental solution and the regular part are expressed via Hankel vector functions. In section 3, a criterion of optimal modification, the criterion of minimization of the norm of the modified integral operator are adopted and based on the optimal simple multipole coefficients for the modification are chosen. The case of circle as an example of the proposed procedure is considered in section 4. In order to give deeper insights, boundaries which can be derived as perturbation of the circle are investigated in section 5.

2 Formulation of the problem using modified Green's function technique

In order to treat an exterior boundary value problem, we can reformulate it as an integral equation, using the direct or indirect method. An exterior Dirichlet boundary value problem for the elastic two-dimensional case can be described through a boundary integral equation of the form [6,15 and 18]:

$$\left(\frac{1}{2}I+\overline{K_0^*}\right)\varphi\left(p\right)=g\left(p\right)\quad p\in\partial D, \qquad (2.1)$$

where g is a Holder continuous density, the integral operator K_0 is defined as: $\left(K_0\varphi\right)(p) = \frac{1}{2\pi}\int_{\partial D}T_pG_0\left(p,q\right).\varphi\left(q\right).ds_q \quad p\in\partial D, \quad (2.2)$

$(*)$ denotes the L_2 adjoint operator and $(-)$ the complex conjugate. G_0 is the fundamental solution and T is the surface stress operator. The superscript (p) on T indicates the action of the operator on the point p.

In order to remove the lack of uniqueness appears when the boundary value problem is formulated as a boundary integral equation we follow the modified Green's function technique. Introducing a regular solution $H(P,Q)$ [6], the modified Green's function is written as the superposition of the fundamental solution and the regular part as:

$$G_1\left(P,Q\right)=G_0\left(P,Q\right)+H\left(P,Q\right). \qquad (2.3)$$

We modify the kernel of Eq. (2.2) with another defined by using G_1, and so the operator K_0 is modified to K_1. The boundary integral equation obtains following a layer theoretic approach is given by :

$$\left(\frac{1}{2}I+\overline{K_1^*}\right)\varphi\left(p\right)=g\left(p\right)\quad p\in\partial D. \qquad (2.4)$$

We note that the operators K_0 and K_1 arn't compact. Indeed, their kernels are singular, but the singular integral Eq. (2.1) and the corresponding modified Eq. (2.4) admit a regularization procedure as it is described in [13].

Next, we will take the eigenvector expansion of the Green's function introduced in [6]. So, for the free-space fundamental solution we have the following expansion :

$$G_0\left(P,Q\right) = \frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}\left(F_m^{\sigma l}\left(P_>\right)\otimes F_m^{\sigma l}\left(P_<\right)\right). \qquad (2.5)$$

$F_m^{\sigma l}$ are the vector Hankel functions [1], where $P_> = \left\{\begin{array}{ll} P, & R_P>R_Q \\ Q, & R_P<R_Q\end{array}\right.$ and $P_< = \left\{\begin{array}{ll} P, & R_P<R_Q \\ Q, & R_P>R_Q\end{array}\right.$. (2.6)

The $\widehat{F}_m^{\sigma l}$ are obtained by changing the function of Hankel H_m^1 of the vector Hankel functions into the function of Bessel J_m^1 [1]. For the regular part of the modification, apart of the usage of dyads similar to those appeared in Eq. (2.5), we introduce simple terms as in [6]:

$$H(P,Q)=\frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}[a_m^{\sigma l}F_m^{\sigma l}(P)\otimes F_m^{\sigma l}(Q), \qquad (2.7)$$

where

$$\begin{array}{l} F_m^{\sigma 1}(P)=grad(H_m^1(k\,R_P).E_m^{\sigma}(\theta_P)), \\ F_m^{\sigma 2}(P)=rot\left(H_m^1(K\,R_P).E_m^{\sigma}(\theta_P).\widehat{e}_3\right), \end{array} \qquad (2.8)$$

and $a_m^{\sigma l}$ is the simple multipole coefficients. (R_P,θ_P) are the polar coordinates of the point P,

and $E_m^{\sigma}(\theta_P) = \sqrt{\varepsilon_m}.\left\{\begin{array}{ll}\cos(m\theta_p) & \sigma=1 \\ \sin(m\theta_p) & \sigma=2\end{array}\right.$, with $\varepsilon_m = \left\{\begin{array}{ll} 1, & m=0 \\ 2, & m>0\end{array}\right.$.

In what follows, we use the usually assumption taken under consideration that the series in (2.7) converges uniformly. As it has been proved in [6], the set $\{F_m^{\sigma l}\}_{m=0:\infty}^{\sigma l=1:2}$ is a complete set in $L_2(\partial D)$ and linearly independent. Since the elements of that set are not orthogonal, then in order to proceed, we define the following set $\left\{F_m^{\sigma l\,\perp}\right\}_{m=0:\infty}^{\sigma l=1:2}$, where its elements have the following property:

$$< F_m^{\sigma l}, F_m^{\nu k\,\perp} >= \delta_{mn}\,\delta_{\sigma\nu}. \qquad (2.9)$$

In fact, we can represent every element of the new set as a linear combination of Hankel vectors:

$$F_m^{\nu k\,\perp}(P)=\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}C_{mn}^{\sigma\nu\ lk}F_m^{\sigma l}(P). \qquad (2.10)$$

By taking the inner products of Eq. (2.10) with $F_m^{\sigma l}(P)$ in the L_2 sense, we obtain a linear system with the unknowns $C_{mn}^{\sigma\nu\ lk}$ having non-vanishing determinant. This is established by the linear independence of $\{F_m^{\sigma l}\}_{m=0:\infty}^{\sigma l=1:2}$. Solving this system to calculate the coefficients $C_{mn}^{\sigma \nu\ lk}$ of Eq. (2.10). So $F_m^{\nu k\ \perp}$ can be computed through Eq. (2.10) explicitly. It is obvious from their definition that $\{F_m^{\sigma l\ \perp}\}_{m=0:\infty}^{\sigma l=1:2}$ are linearly independent.

3 Optimal choice of the simple multipole coefficients

In the sequel, we will consider a different criterion for choosing the simple multipole coefficients in the modification (2.3), from the criterion presented in [15]. A similar criterion is considered for the acoustical case by Kleinmann and Roach [11]. As in their work it is mentioned this criterion does not only assure the unique solvability of the boundary integral equation but also leads to a constructive method of solving the equation. We will prove that the same holds for the elastic two-dimensional case. This argument is established by the following theorem.

Theorem 3.1. The norm $\|K_1\|$ of the modified integral operator K_1 is minimized if we choose the simple multipole coefficients of the modification (2.3) through the relation:

$$a_m^{\sigma l} \cdot \frac{i}{4\mu K^2} = -\frac{f_m^{\sigma l}}{\alpha_m^{\sigma l}}, \qquad (3.1)$$

where

$$\alpha_m^{\sigma l} = \left\| TF_m^{\sigma l} \right\|^2 \qquad (3.2)$$

and

$$f_m^{\sigma l} = < \overline{K_0^*} T\overline{F_m^{\sigma l}}, F_m^{\sigma l\ \perp} > . \qquad (3.5)$$

Proof. The operator norm will be minimized if the simple multipole coefficients in the modification minimize $\|K_1 w\|^2$ for each function $w \in L_2(\partial D)$. So we will calculate the norm of $\|K_1 w\|^2$, using the expansion for the kernel given by (2.5) and (2.7), we have :

$$\|K_1 w\|^2 = \|K_0 w\|^2$$

$$+\frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}[\overline{a_m^{\sigma 1}} < K_0 w, TF_m^{\sigma 1} >< \overline{F_m^{\sigma 1}}, w >$$

$$+a_m^{\sigma l} < TF_m^{\sigma 1}, K_0 w >< F_m^{\sigma 1}, \overline{w} >]$$

$$+\frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}[\overline{a_m^{\sigma 2}} < K_0 w, TF_m^{\sigma 2} >< \overline{F_m^{\sigma 2}}, w >$$

$$+a_m^{\sigma 2} < TF_m^{\sigma 2}, K_0 w >< F_m^{\sigma 2}, \overline{w} >]$$

$$+\left(\frac{i}{4\mu K^2}\right)^2\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{n=0}^{\infty}\sum_{\nu=1}^{2}[a_m^{\sigma 1}\,\overline{a_n^{\nu 1}} < TF_m^{\sigma 1}, TF_n^{\nu 1} >$$

$$< F_m^{\sigma 1}, \overline{w} >< \overline{F_n^{\nu 1}}, w >]$$

$$+\left(\frac{i}{4\mu K^2}\right)^2\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{n=0}^{\infty}\sum_{\nu=1}^{2}[a_m^{\sigma 1}\,\overline{a_n^{\nu 2}} < TF_m^{\sigma 1}, TF_n^{\nu 2} >$$

$$< F_m^{\sigma 1}, \overline{w} >< \overline{F_n^{\nu 2}}, w >]$$

$$+\left(\frac{i}{4\mu K^2}\right)^2\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{n=0}^{\infty}\sum_{\nu=1}^{2}[a_m^{\sigma 2}\,\overline{a_n^{\nu 1}} < TF_m^{\sigma 2}, TF_n^{\nu 1} >$$

$$< F_m^{\sigma 2}, \overline{w} >< \overline{F_n^{\nu 1}}, w >]$$

$$+\left(\frac{i}{4\mu K^2}\right)^2\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{n=0}^{\infty}\sum_{\nu=1}^{2}[a_m^{\sigma 2}\,\overline{a_n^{\nu 2}} < TF_m^{\sigma 2}, TF_n^{\nu 2} >$$

$$< F_m^{\sigma 2}, \overline{w} >< \overline{F_n^{\nu 2}}, w >]. \qquad (3.4)$$

In (3.4) the inner products and norms are in L_2 sense. Necessary conditions for the minimum of (3.4) are the vanishing of the gradient, with respect to the coefficients. So, first, differentiating with respect to $a_n^{\nu 1}$ and $a_n^{\nu 2}$ we obtain the relations:

$$\begin{pmatrix} \frac{\partial \|K_1 w\|^2}{\partial a_n^{\nu 1}} \\ \frac{\partial \|K_1 w\|^2}{\partial a_n^{\nu 2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \forall w \in L_2(\partial D). \qquad (3.5)$$

From Eq. (3.5) we conclude that: $K_0^* TF_n^{\nu 1} - \frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}\overline{a_m^{\sigma l}} < T\overline{F_m^{\sigma 1}}, T\overline{F_n^{\nu 1}} >$ $F_n^{\nu 1} = 0,$ (3.6)
and
$K_0^* TF_n^{\nu 1} - \frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}\overline{a_m^{\sigma l}} < T\overline{F_m^{\sigma 2}}, T\overline{F_n^{\nu 2}} >$ $F_n^{\nu 2} = 0.$ (3.7)

Taking the inner products of (3.6) and (3.7) with $F_n^{\nu 1\ \perp}$ and $F_n^{\nu 2\ \perp}$. The unique solution of this system gives us $a_m^{\sigma l}$ and $a_m^{\sigma 2}$ as they are expressed via (3.1). It remains to prove that this choice of simple multipole coefficients provides a minimum, that is if we denote by K_1^0 the modified operator with the optimal simple multipole coefficients as specified by (3.1) and by K_1 the modified operator with any other choice of simple multipole coefficients, we have to verify that :

$$\left\|K_1^0 w\right\| \le \|K_1 w\| \quad \forall w \in L_2(\partial D). \qquad (3.8)$$

Let the simple multipole coefficients in the arbitrary modification be denoted by :

$$a_m^{\sigma l} = a_m^{\sigma l}(0) + \varepsilon_m^{\sigma l}, \qquad (3.9)$$

where $a_m^{\sigma l}(0)$ is defined by (3.1). Then we can calculate $\|K_1 w\|$ taking into account (3.9) :

$$\|K_1 w\|^2 = \left\|K_1^0 w\right\|^2 \qquad (3.10)$$

$$+ < K_1^0 w, \sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2} h_m^{\sigma l}\overline{a_m^{\sigma l}} < w, \overline{F_m^{\sigma l}} >>$$

$$+ < \sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2} h_m^{\sigma l}\overline{a_m^{\sigma l}} < w, \overline{F_m^{\sigma l}} >, K_1^0 w >$$

$$+\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}\sum_{n=0}^{\infty}\sum_{\nu=1}^{2}\sum_{k=1}^{2} < h_m^{\sigma l}, h_n^{\nu k} >< w, \overline{F_m^{\sigma l}} >< \overline{w}, F_n^{\nu k} >,$$

where

$$h_m^{\sigma l}(p) = \varepsilon_m^{\sigma l}\, T F_m^{\sigma l}(P). \qquad (3.11)$$

From (3.6) and (3.7) we have the vanishing of all terms in the first two inner products of the sum in (3.10). Then (3.10) becomes :

$$\|K_1 w\|^2 = \left\|K_1^0 w\right\|^2$$

$$+\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}\sum_{n=0}^{\infty}\sum_{\nu=1}^{2}\sum_{k=1}^{2} Z_m^{\sigma l}\,\overline{Z_n^{\nu k}} < h_m^{\sigma l}, h_n^{\nu k} >, \qquad (3.12)$$

where

$$Z_m^{\sigma l} = < w, \overline{F_m^{\sigma l}} > . \qquad (3.13)$$

So to establish our argument we need to prove that the quantity :

$$\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}\sum_{n=0}^{\infty}\sum_{\nu=1}^{2}\sum_{k=1}^{2} Z_m^{\sigma l}\,\overline{Z_n^{\nu k}} < h_m^{\sigma l}, h_n^{\nu k} >, \qquad (3.14)$$

is positive semi-defined.

But, by constructing an orthonormal set $\left\{U_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma l=1:2}$ from $\left\{h_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma l=1:2}$, e.g. by using a Gram-Schmidt procedure [14] and the linear independence of $\left\{h_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma l=1:2}$, there exists a set of coefficients $\left\{d_{mp}^{\sigma\mu ls}\right\}$ such that :

$$h_m^{\sigma l} = \sum_{p=0}^{\infty}\sum_{\mu=1}^{2}\sum_{s=1}^{2} d_{mp}^{\sigma\mu ls} U_p^{\mu s}. \qquad (3.15)$$

Then

$$< h_m^{\sigma l}, h_n^{\nu k} >= \sum_{p=0}^{\infty}\sum_{\mu,s=1}^{2}\sum_{q=0}^{\infty}\sum_{\lambda,r=1}^{2} d_{mp}^{\sigma\mu ls}\,\overline{d_{nq}^{\nu\lambda kr}} < U_p^{\mu s}, U_q^{\lambda r} >$$

$$= \sum_{p=0}^{\infty}\sum_{\mu,s=1}^{2} d_{mp}^{\sigma\mu ls}\,\overline{d_{np}^{\nu\mu ks}} = D\,D^*, \qquad (3.16)$$

where D is the matrix with elements $d_{mp}^{\sigma\mu ls}$ and D^* is the Hermitian conjugate. However, DD^* is positive semidefined [14], which completes the proof. □

4 Optimal choice of the simple multipole coefficients for the case of the circle

Lemma 4.1. If ∂D is a circle of radius a, then the expansion for $\left\{F_m^{\sigma l\,\perp}\right\}_{m=0:\infty}^{\sigma l=1:2}$ are given by the following equations:

$$F_m^{11\,\perp} = \frac{\left(a_m^2 F_m^{11} - c_m F_m^{22}\right)}{\triangle_m}, \quad F_m^{22\,\perp} = \frac{\left(a_m^1 F_m^{22} - \overline{c_m} F_m^{11}\right)}{\triangle_m} \quad (4.1)$$

and

$$F_m^{12\,\perp} = \frac{\left(a_m^1 F_m^{12} + \overline{c_m} F_m^{21}\right)}{\triangle_m}, \quad F_m^{21\,\perp} = \frac{\left(a_m^2 F_m^{21} + c_m F_m^{12}\right)}{\triangle_m}, \quad (4.2)$$

where

$$a_m^1 = 2\pi a k^2\left[\left|H_m'(ka)\right|^2 + \frac{m^2}{(ka)^2}\left|H_m(ka)\right|^2\right], \qquad (4.3)$$

$$a_m^2 = 2\pi a K^2\left[\left|H_m'(Ka)\right|^2 + \frac{m^2}{(Ka)^2}\left|H_m(Ka)\right|^2\right], \qquad (4.4)$$

$$c_m = 2\pi akK\left[\frac{m}{Ka}H_m'(ka)\,\overline{H_m(Ka)} + \frac{m}{ka}H_m(ka)\,\overline{H_m'(Ka)}\right], \quad (4.5)$$

and

$$\triangle_m = a_m^1 a_m^2 - |c_m|^2. \qquad (4.6)$$

Proof. Taking the inner product of Eq. (2.10) with $F_m^{\sigma l}$ and using eq. (2.9), we obtain that

$$\sum_{p=0}^{\infty}\sum_{\mu=1}^{2}\sum_{s=1}^{2} C_{mn}^{\sigma\nu lk}\, I_{mn}^{\sigma\nu lk} = \delta_{mn}\delta_{\sigma\nu}\delta_{lk}, \qquad (4.7)$$

such that

$$I_{mn}^{\sigma\nu lk} = < F_m^{\sigma l}, F_n^{\nu k} > \qquad (4.8)$$

and the inner product is defined on the circle.

The values of Eq. (4.8) for each $\sigma, \nu, l, k = 1:2$ are given in [18] as follows:

$$I_{mn}^{\sigma\nu 11} = a_m^1\,\delta_{mn}\delta_{\sigma\nu}, \qquad (4.9)$$

$$I_{mn}^{\sigma\nu 22} = a_m^2\,\delta_{mn}\delta_{\sigma\nu}, \qquad (4.10)$$

$$I_{mn}^{\sigma\nu 12} = -(-1)^{\sigma} c_m\,\delta_{mn}\left(1-\delta_{\sigma\nu}\right), \qquad (4.11)$$

and

$$I_{mn}^{\sigma\nu 21} = -(-1)^{\nu}\,\overline{c_m}\,\delta_{mn}\left(1-\delta_{\sigma\nu}\right). \qquad (4.12)$$

By using Eqs. (4.9)-(4.12), we obtain the following linear systems :

$$\left\{\begin{array}{l} a_m^1 C_{mn}^{\sigma 1l1} + \overline{c_m} C_{mn}^{\sigma 2l2} = \delta_{mn}\delta_{\sigma 1}\delta_{l1} \\ c_m C_{mn}^{\sigma 1l1} + a_m^2 C_{mn}^{\sigma 2l2} = \delta_{mn}\delta_{\sigma 2}\delta_{l2} \end{array}\right. \qquad (4.13)$$

$$\left\{\begin{array}{l} a_m^1 C_{mn}^{\sigma 2l1} + \overline{c_m} C_{mn}^{\sigma 1l2} = \delta_{mn}\delta_{\sigma 2}\delta_{l1} \\ -c_m C_{mn}^{\sigma 2l1} + a_m^2 C_{mn}^{\sigma 1l2} = \delta_{mn}\delta_{\sigma 1}\delta_{l2} \end{array}\right., \qquad (4.14)$$

with the same non-vanishing determinant given by Eq. (4.6). This is established by the linear independence of $\left\{F_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma l=1:2}$ in $L_2(\partial D)$ and the Schwartz inequality. The unique solution of the system (4.13) gives us $F_m^{11\,\perp}$ and $F_m^{22\,\perp}$ as they are expressed by (4.1), and the unique solution of the system (4.14) gives us $F_m^{12\,\perp}$ and $F_m^{21\,\perp}$ as they are expressed by (4.2). □

Theorem 4.2. If ∂D is a circle of radius a, then the optimal multipole coefficients of the modification (2.3), which minimize the norm of the modified integral operator, given by Eq. (3.1), take the following forms :

$$a_m^{11} = -\frac{1}{2}\left[\frac{\widehat{\alpha}_m^1}{\alpha_m^1} + \frac{\left(\widehat{a}_m^1 a_m^2 - \overline{c_m}\widehat{c}_m\right)}{\triangle_m} + \frac{\overline{\beta_m}}{\alpha_m^1}\cdot\frac{\left(\widehat{d}_m a_m^2 - \overline{c_m}\widehat{a}_m^2\right)}{\triangle_m}\right], \quad (4.15)$$

$$a_m^{12} = -\frac{1}{2}\left[\frac{\widehat{\alpha}_m^2}{\alpha_m^2} + \frac{\left(a_m^1\widehat{a}_m^2 - c_m\widehat{d}_m\right)}{\triangle_m} + \frac{\beta_m}{\alpha_m^2}\cdot\frac{\left(a_m^1\widehat{c}_m - c_m\widehat{a}_m^1\right)}{\triangle_m}\right], \quad (4.16)$$

$$a_m^{21} = -\frac{1}{2}\left[\frac{\widehat{\alpha}_m^1}{\alpha_m^1} + \frac{\left(\widehat{a}_m^1 a_m^2 - \overline{c_m}\widehat{c}_m\right)}{\triangle_m} + \frac{\overline{\beta_m}}{\alpha_m^1}\cdot\frac{\left(\widehat{d}_m a_m^2 - \overline{c_m}\widehat{a}_m^2\right)}{\triangle_m}\right], \quad (4.17)$$

and

$$a_m^{22} = -\frac{1}{2}\left[\frac{\widehat{\alpha}_m^2}{\alpha_m^2} + \frac{\left(a_m^1\widehat{a}_m^2 - c_m\widehat{d}_m\right)}{\triangle_m} + \frac{\beta_m}{\alpha_m^2}\cdot\frac{\left(a_m^1\widehat{c}_m - c_m\widehat{a}_m^1\right)}{\triangle_m}\right]. \quad (4.18)$$

Proof. To calculate the simple multipole coefficients $a_m^{\sigma l}$, we must calculate the values of $f_m^{\sigma l}$ given by (3.3). We know by [18] that:

$$\overline{K_0^*}T\overline{F_m^{\sigma l}}(p) = \int_{\partial D} T\overline{F_m^{\sigma l}}(q)\, T_q G_0(q,p)\, ds_q. \quad (4.19)$$

Using the following expansion for the Green's function

$$G_0(q,p) = \frac{i}{4\mu K^2}\frac{1}{2}\sum_{n=0}^{\infty}\sum_{\nu,k=1}^{2}\left[\begin{array}{c} F_n^{\nu k}(q)\otimes\widehat{F}_n^{\nu k}(P) \\ \widehat{F}_n^{\nu k}(q)\otimes F_m^{\sigma l}(p)\end{array}\right], \quad (4.20)$$

we obtain that

$$f_m^{\sigma l} = \frac{i}{8\mu K^2}\left[\begin{array}{c} < T\widehat{F}_m^{\sigma l}, TF_m^{\sigma l} > \sum_{n=0}^{\infty}\sum_{\nu=1}^{2}\sum_{k=1}^{2} \\ < TF_n^{\nu k}, TF_m^{\sigma l} >< \widehat{F}_n^{\nu k}, F_m^{\sigma l\,\perp} > \end{array}\right]. \quad (4.21)$$

Using Eqs. (4.3)-(4.6), (4.19),(4.20) and the facts that [18]

$$\widehat{a}_m^1 = 2\pi a k^2\left[J_m'(ka)\overline{H_m'(ka)} + \frac{m^2}{(ka)^2}J_m(ka)\overline{H_m(ka)}\right],$$

$$\widehat{a}_m^2 = 2\pi a K^2\left[J_m'(Ka)\overline{H_m'(Ka)} + \frac{m^2}{(Ka)^2}J_m(Ka)\overline{H_m(Ka)}\right],$$

$$\widehat{c}_m = 2\pi akK\left[\frac{m}{Ka}J_m'(ka)\overline{H_m(Ka)} + \frac{m}{ka}J_m(ka)\overline{H_m'(Ka)}\right],$$

$$\widehat{d}_m = 2\pi akK\left[\frac{m}{ka}J_m'(Ka)\overline{H_m(ka)} + \frac{m}{Ka}J_m'(Ka)\overline{H_m'(Ka)}\right.$$

$$\widehat{\alpha}_m^1 = 2\pi a(k^4(2\mu J_m''(ka) - \lambda J_m''(ka))\left(2\mu\overline{H_m''(ka)} - \lambda\overline{H_m''(ka)}\right) + \left(\frac{2\mu m}{a}\right)^2 \left(k\overline{H_m'(ka)} - \frac{\overline{H_m''(ka)}}{a}\right)\left(kJ_m'(ka) - \frac{J_m''(ka)}{a}\right)),$$

$$\widehat{\alpha}_m^2 = 2\pi a(\left(\mu K^2\right)^2\left(2J_m''(Ka) + J_m(Ka)\right) \left(2\overline{H_m''(Ka)} + \overline{H_m''(Ka)}\right) + \left(\frac{2\mu m}{a}\right)^2 \left(KJ_m'(Ka) - \frac{J_m''(Ka)}{a}\right)\left(K\overline{H_m'(Ka)} - \frac{\overline{H_m''(Ka)}}{a}\right)),$$

$$\widehat{\beta}_m = 4\pi\mu m(k^2(2\mu J_m''(ka) - \lambda J_m''(ka))\left(K\overline{H_m'(Ka)} - \frac{\overline{H_m''(Ka)}}{a}\right) + \mu K^2\left(kJ_m'(ka) - \frac{J_m''(ka)}{a}\right)\left(2\overline{H_m''(Ka)}\right) + \overline{H_m''(Ka)}),$$

and

$$\triangle_m = (2\pi\mu)^2\left(\mu k^2K^2\left(2\mu H_m''(ka) - \lambda H_m''(ka)\right) \left(2H_m''(Ka) + H_m''(Ka)\right) - \left(\frac{2\mu m}{a}\right)^2\left(kH_m'(ka) - \frac{H_m''(ka)}{a}\right)\left(KH_m'(Ka) - \frac{H_m(Ka)}{a}\right)\right),$$

we obtain the following relations :

$$f_m^{11} = f_m^{21} = \frac{i}{8\mu K^2}(\widehat{\alpha}_m^1 + \widehat{\alpha}_m^1\frac{\left(\widehat{a}_m^1 a_m^2 - \overline{c_m}\widehat{c}_m\right)}{\triangle_m} + \overline{\beta_m}\frac{\left(\widehat{d}_m a_m^2 - \overline{c_m}\widehat{a}_m^2\right)}{\triangle_m}) \quad (4.22)$$

and

$$f_m^{12} = f_m^{22} = \frac{i}{8\mu K^2}(\widehat{\alpha}_m^2 + \widehat{\alpha}_m^2\frac{\left(a_m^1\widehat{a}_m^2 - c_m\widehat{d}_m\right)}{\triangle_m} + \beta_m\frac{\left(\widehat{c}_m a_m^1 - c_m\widehat{a}_m^1\right)}{\triangle_m}). \quad (4.23)$$

Using (4.22) and (4.23) we obtain the expressions of the multipole coefficients $a_m^{\sigma l}$ as they are expressed via (4.15)-(4.18). □

Theorem 4.3.
If ∂D is a circle of radius a and the optimal multipole coefficients of the modification in Eq. (2.3) are given by Eqs. (4.15)-(4.18), then it holds that

$$\|K_1\| = 0$$

Proof. In view of Lemma 4.1 and Theorem 4.2, the modified Green's function admits the following development:

$$G_1(p,q) = \frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}F_m^{\sigma l}(P_<) \otimes\left[\widehat{F}_m^{\sigma l}(P_>) + a_m^{\sigma l}F_m^{\sigma l}(P_>)\right] = \frac{1}{2}\left[G^D(p,q) + G^N(p,q)\right] \quad (4.33)$$

where G^D is the Green's function for the exterior Dirichlet problem while G^N is the Green's function for the exterior Neumann problem for the circle. So :

$$G^D(p,q) = 0 \quad and \quad T_qG^N(p,q) = 0, \quad (4.34)$$

for $R_p > a$ and $R_q = a$. After calculus we obtain that

$$T_qG^D(p,q) = -T_qG^N(p,q), \quad (4.35)$$

for $R_p = R_q = a$. Then in the circle it holds that

$$T_q G_1 (p,q) = 0. \quad (4.36)$$

This last result implies that

$$K_1 w = 0 , \quad \forall w \in L_2 (\partial D) . \quad (4.37)$$

Hence $\|K_1 w\| = 0$ and the integral equation is uniquely solvable. □

5 Optimal choice of the simple multipole coefficients for a perturbation of a circle

As in [11] we will consider a family of non-circular boundaries given parametrically by the relation:

$$R_\varepsilon = a + \varepsilon \varphi(\theta_p),\ 0 \le \theta_p \le 2\pi, \quad (5.1)$$

where φ and $\frac{\partial \varphi}{\partial \theta}$ are all bounded. Using the estimates for the multipole vectors which are established in [18]:

$$F_m^{\sigma l} (P_\varepsilon) = F_m^{\sigma l} (P_a) + O(\varepsilon) , \quad (5.2)$$

$$T F_m^{\sigma l} (P_\varepsilon) = T F_m^{\sigma l} (P_a) + O(\varepsilon) , \quad (5.3)$$

$$< F_m^{\sigma l}, F_n^{vk} >_\varepsilon = < F_m^{\sigma l}, F_n^{vk} >_a + O(\varepsilon) , \quad (5.4)$$

$$< TF_m^{\sigma l}, TF_n^{vk} >_\varepsilon = < TF_m^{\sigma l}, TF_n^{vk} >_a + O(\varepsilon) , \quad (5.5)$$

and

$$F_m^{\sigma l\,\perp} (P_\varepsilon) = F_m^{\sigma l\,\perp} (P_a) + O(\varepsilon) , \quad (5.6)$$

where P_ε is a point in the perturbed circle while P_a describes points on the circle of radius a, and $< \ , \ >_\varepsilon$ is the inner product on the perturbed circle, and $< \ , \ >_a$ is the inner product on the circle.

Theorem 5.1. If ∂D is defined by (5.1), then the optimal multipole coefficients of the modification in Eq. (2.3) minimize the norm of the modified integral operator given by Eq. (3.1) take the forms:

$$a_m^{11} = a_m^{11} (a) + O(\varepsilon) , \quad (5.7)$$

$$a_m^{12} = a_m^{12} (a) + O(\varepsilon) , \quad (5.8)$$

$$a_m^{21} = a_m^{21} (a) + O(\varepsilon) , \quad (5.9)$$

and

$$a_m^{22} = a_m^{22} (a) + O(\varepsilon) , \quad (5.10)$$

where $a_m^{\sigma l} (a)$ are the optimal multipole coefficients for the circle of radius a.

Proof.

Suppose that ∂D is defined by (5.1). Then from Eq. (5.5), it follows that

$$\alpha_m^{\sigma l} (P_\varepsilon) = \alpha_m^{\sigma l} (P_a) + O(\varepsilon) , \quad (5.11)$$

and

$$\triangle_m^{\sigma l\,\prime} (P_\varepsilon) = \triangle_m^{\sigma l\,\prime} (P_a) + O(\varepsilon) . \quad (5.12)$$

Using Eqs. (5.2), (5.6), (5.11) and (5.12), we obtain that

$$f_m^{\sigma l} (P_\varepsilon) = f_m^{\sigma l} (P_a) + O(\varepsilon) , \quad (5.13)$$

This leads to Eqs. (5.7)-(5.10). □.

Theorem 5.2. If ∂D is defined by Eq. (5.1) and the optimal multipole coefficients of the modification in Eq. (2.3) are given by Eqs. (5.7)-(5.11), then it holds that

$$\|K_1\| = O(\varepsilon). \quad (5.14)$$

Proof. Suppose that ∂D is defined by Eq. (5.1) and the optimal multipole coefficients of the modification in Eq. (2.3) are given by Eqs. (5.7)-(5.11). In view of Theorem 5.1, it holds that

$$T_{p_\varepsilon} G_0 (p_\varepsilon, q_\varepsilon) = T_{p_a} G_0 (p_a, q_a) + O(\varepsilon) , \quad (5.15)$$

$$T_{q_\varepsilon} G_0 (p_\varepsilon, q_\varepsilon) = T_{q_a} G_0 (p_a, q_a) + O(\varepsilon) , \quad (5.16)$$

$$T_{p_\varepsilon} G_1 (p_\varepsilon, q_\varepsilon) = T_{p_a} G_1 (p_a, q_a) + O(\varepsilon) , \quad (5.17)$$

$$T_{p_\varepsilon} G_1 (p_\varepsilon, q_\varepsilon) = T_{p_a} G_1 (p_a, q_a) + O(\varepsilon) \quad (5.18)$$

and

$$(K_1^\varepsilon w) (p_\varepsilon) = (K_1^a w) (p_a) + O(\varepsilon) . \quad (5.19)$$

By using (4.37) and (5.19) we obtain that

$$(K_1^\varepsilon w)(p_\varepsilon) = O(\varepsilon) , \quad (5.20)$$

which leads to $\|K_1\| = O(\varepsilon)$. □.

6 Conclusion

In this paper, we have proposed a new criterion of optimization of the simple multipole coefficients used in the modified Green's function for the elastic two-dimensional case. To that end, we have based on the minimization of the norm of the modified integral operator. Some interesting results for the special circular case have been shown.

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