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Blending Type Approximation by Bivariate Bernstein-Kantorovich operators

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Abstract: Pop and Fărcas [12] introduced the bivariate operators of the Bernstein-Kantorovich type and the associated GBS(Generalized Boolean sum) operators of the Kantorovich type. The concern of this paper is to obtain the rate of convergence in terms of the partial and complete modulus of continuity and the degree of approximation by means of Lipschitz class for the above bivariate operators. We also study the simultaneous approximation for the first order partial derivative of the operator. In the last section, we estimate the degree of approximation by means of the Lipschitz class for Bögel continuous functions and the rate of convergence with the help of Peetre's K- functional for the GBS operator of Bernstein-Kantorovich type.

Keywords: Blending type approximation, partial moduli of continuity, Bögel-continuity, GBS operators, degree of approximation.

1 Introduction

For a bounded function $f(x)$ defined on the interval $J=[0,1]$, the Bernstein polynomial of nth degree is defined as

$$(R_n f)(x)=\sum_{k=0}^{n} p_{n.k}(x) f(\frac{k}{n}),$$

where

$$p_{n,k}(t)=\binom{n}{k} t^k (1-t)^{n-k}$$

are the fundamental polynomials.
Lorentz gave a brief presentation of the principle results concerning these polynomials in his book [8]. If $f(x)$ is continuous on J, the fundamental property of these polynomials is that $(R_n f)(x)\to f(x)$, as $n\to\infty$, uniformly on J.In 1930, to approximate Lebesgue integrable functions on $[0,1]$, Kantorovich [10] introduced and studied the operators $\mathscr{L}_m : L_1([0,1])\to C([0,1])$ defined for any function f in $L_1([0,1])$ and for any $m\in\mathbb{N}$ as follows:

$$(\mathscr{L}_m f)(x)=(m+1)\sum_{k=0}^{m} p_{m,k}(x)\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt, \qquad (1)$$

for $x\in[0,1]$, $k\in\{0,1,2,\cdots,m\}$.
In [12], for $m\in\mathbb{N}$, $f\in L_1(\Delta_2)$, where

$$\Delta_2=\{(x,y)\in\mathbb{R}\times\mathbb{R}|\, x,y\geq 0, x+y\leq 1\}.$$

Pop and Fărcas constructed bivariate operators of Kantorovich type $\mathscr{K}_m : L_1(\Delta_2)\to C([0,1]\times[0,1])$ defined as:

$$(\mathscr{K}_m f;x,y) = (m+1)^2 \sum_{k,j=0,k+j\leq m} p_{m,k,j}(x,y) \times \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}}\int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}} f(t,s)dt\, ds, \qquad (2)$$

where
$p_{m,k,j}(x,y)=$

$$\binom{m}{k}\binom{m-k}{j} x^k y^j (1-x-y)^{m-k-j},$$

for any $k,j\geq 0, k+j\leq m$ and $(x,y)\in\Delta_2$.
The method of construction of bivariate operators of Kantorovich type was inspired by the construction of Bernstein bivariate operators [8], $B_m : \mathscr{F}(\Delta_2)\to\mathscr{F}(\Delta_2)$ defined as:

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$$(B_m f)(x,y) = \sum_{k,j=0,k+j\le m} p_{m,k,j}(x,y) f\left(\frac{k}{m},\frac{j}{m}\right),$$

for any $(x,y) \in \Delta_2$. In [12], Pop and Fărcas also constructed the GBS operators associated with the bivariate operators $\mathscr{K}_m$ of Kantorovich type.

The purpose of this paper is to discuss the rate of convergence of the bivariate Bernstein-Kantorovich type operators $\mathscr{K}_m$ defined on a triangle and investigate the rate of convergence by the associated GBS operators by using Lipschitz class and mixed K- functional.

2 PRELIMINARIES

Let $C(\Delta_2) = \{f : \Delta_2 \to \mathbb{R} |\ f$ be continuous on $\Delta_2\}$. The norm in $C(\Delta_2)$ is given by

$$||f|| = sup_{(x,y)\in\Delta_2} |f(x,y)|.$$

In the bivariate case, for $f \in C(\Delta_2)$, the complete modulus of continuity is given as:

$$\bar{\omega}(f;\delta_1,\delta_2) = \sup\left\{|f(t,s)-f(x,y)| : (t,s),(x,y)\in\Delta_2 \right.$$
$$\left. \text{and } |t-x|\le\delta_1,\ |s-y|\le\delta_2\right\},$$

where $\bar{\omega}(f,\delta_1,\delta_2)$ satisfies the following properties:

1.$\bar{\omega}(f,\delta_1,\delta_2)\to 0$, if $\delta_1\to 0$ and $\delta_2\to 0$;
2.$|f(t,s)-f(x,y)|\le\bar{\omega}(f,\delta_1,\delta_2)$
$\times\left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|s-y|}{\delta_2}\right).$

The details of the complete modulus of continuity for the bivariate case can be found in [1].

Further, for $f\in C(\Delta_2)$ the partial moduli of continuity with respect to x and y is given by

$$\omega_1(f;\delta) = \sup\left\{|f(x_1,y)-f(x_2,y)| : y\in[0,1],\right.$$
$$\left.|x_1-x_2|\le\delta\right\}, \tag{3}$$

and

$$\omega_2(f;\delta) = \sup\left\{|f(x,y_1)-f(x,y_2)| : x\in[0,1],\right.$$
$$\left.|y_1-y_2|\le\delta\right\}. \tag{4}$$

Apparently, they satisfy the properties of the usual modulus of continuity.

Let $C^2(\Delta_2) := \left\{f\in C(\Delta_2) : f_{xx}, f_{xy}, f_{yx}, f_{yy}\in C(\Delta_2)\right\}$
equipped with the norm

$$||f||_{C^{(2)}(\Delta_2)} = ||f|| + \sum_{i=1}^{2}\left(\left\|\frac{\partial^i f}{\partial x^i}\right\| + \left\|\frac{\partial^i f}{\partial y^i}\right\|\right).$$

For $f\in C(\Delta_2)$, let us consider the following K-functional:

$$K_2(f,\delta) = inf\{||f-g|| + \delta||g||_{C^2(\Delta_2)} : g\in C^2(\Delta_2)\}, \tag{5}$$

where $\delta > 0$.
By [7], there exists an absolute constant $C>0$ such that

$$K_2(f,\delta)\le C\bar{\omega}_2(f,\sqrt{\delta}), \tag{6}$$

where $\bar{\omega}_2(f,\sqrt{\delta})$ denotes the second order modulus of continuity for the bivariate case.

Next, we give some definitions which will be required for the GBS operator. A function $f:\Delta_2\longrightarrow$ R is called a B-continuous (Bögel continuous) function at $(x_0,y_0)\in\Delta_2$ if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta f[(x_0,y_0);(x,y)] = 0,$$

where $\Delta f[(x_0,y_0);(x,y)]$ denotes the mixed difference defined by

$$\Delta f[(x_0,y_0);(x,y)] = f(x,y) - f(x,y_0) - f(x_0,y) + f(x_0,y_0).$$

A function $f:\Delta_2\longrightarrow\mathbb{R}$ is called a B-differentiable (Bögel differentiable) function at $(x_0,y_0)\in\Delta_2$ if the limit

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta f[(x_0,y_0);(x,y)]}{(x-x_0)(y-y_0)}$$

exists finitely.
The limit is called the B-differential of f at the point (x_0,y_0) and is denoted by $D_B(f;x_0,y_0)$ and the space of all B-differentiable functions is denoted by $D_b(\Delta_2)$.

The space of all continuous fuctions on Δ_2 is denoted by $C(\Delta_2)$ where the norm is the sup norm $||.||_\infty$.

From the definition of Bögel continuity it easily follows that $C(\Delta_2)\subset C_b(\Delta_2)$ ([2], page 52).
In what follows, let $\delta_1(x)$ and $\delta_2(y)$ be defined as

$$\delta_1{}^2(x) = \{\mathscr{K}_m((t-x)^2;x,y)\}$$

and

$$\delta_2{}^2(y) = \{\mathscr{K}_m((t-y)^2;x,y)\}.$$

3 Basic Results

Lemma 1.[12]
Let $e_{i,j} : \Delta_2 \to \mathbb{R}, e_{i,j}(x,y) = x^i y^j$ $i, j = 0,1,2$ *be the test functions. Then the following equalities hold for the operators* $\mathscr{K}_m$ *given by :*

1. $\mathscr{K}_m(e_{00};x,y) = 1;$
2. $\mathscr{K}_m(e_{10};x,y) = \frac{2mx+1}{2(m+1)};$
3. $\mathscr{K}_m(e_{01};x,y) = \frac{2my+1}{2(m+1)};$
4. $\mathscr{K}_m((t-x);x,y) = \frac{1-2x}{2(m+1)};$
5. $\mathscr{K}_m((s-y);x,y) = \frac{1-2y}{2(m+1)};$
6. $\mathscr{K}_m((t-x)^2;x,y) = \frac{3(m-1)x(1-x)+1}{3(m+1)^2};$
7. $\mathscr{K}_m((s-y)^2;x,y) = \frac{3(m-1)y(1-y)+1}{3(m+1)^2}.$

Lemma 2.[12] *For any* $(x,y) \in \Delta_2$, $(\mathscr{K}_m)_{m>1}$ *verify the following estimation:*

1. $\mathscr{K}_m((t-x)^2;x,y) \le \frac{1}{(m+1)};$
2. $\mathscr{K}_m((s-x)^2;x,y) \le \frac{1}{(m+1)};$
3. $\mathscr{K}_m((t-x)^4;x,y) \le \frac{1}{(m+1)^2};$
4. $\mathscr{K}_m((s-x)^4;x,y) \le \frac{1}{(m+1)^2};$
5. $\mathscr{K}_m((t-x)^4(s-y)^2;x,y) \le \frac{1}{(m+1)^3};$
6. $\mathscr{K}_m((t-x)^2(s-y)^4;x,y) \le \frac{1}{(m+1)^3}.$

Lemma 3.*For every* $f \in C(\Delta_2)$, *we have*

$$||\mathscr{K}_m(f)|| \le ||f||.$$

*Proof.*From the definition of operator (2) and Lemma 1, the proof of Lemma easily follows. Hence details are omitted.□

Theorem 1.[12] *If* $f \in C(\Delta_2)$, *then*

$$\lim_{m\to\infty} (\mathscr{K}_m f;x,y) = f(x,y)$$

uniformly on Δ_2.

4 Main results

In the following theorem we obtain the rate of convergence of the operators given by (2) in terms of the partial modulus of continuity.

Theorem 2.*Let* $f \in C(\Delta_2)$ *and* $(x,y) \in \Delta_2$. *Then we have the inequality*

$$|\mathscr{K}_m(f;x,y) - f(x,y)| \le 2(\omega_1(f;\delta_1(x)) + \omega_2(f;\delta_2(y))) \quad . \tag{7}$$

*Proof.*Using equation (2.1),(2.2), Lemma 1 and applying Cauchy-Schwarz inequality we have

$$\begin{aligned}
|\mathscr{K}_m(f;x,y) - f(x,y)| &\le \mathscr{K}_m(|f(t,s) - f(x,y)|;x,y) \\
&\le \mathscr{K}_m(|f(t,s) - f(t,y)|;x,y) \\
&\quad + \mathscr{K}_m(|f(t,y) - f(x,y)|;x,y) \\
&\le \mathscr{K}_m\left(\omega(f;\delta_2)(1 + \frac{|s-y|}{\delta_2});x,y\right) \\
&\quad + \mathscr{K}_m\left(\omega(f;\delta_1)(1 + \frac{|s-y|}{\delta_1});x,y\right) \\
&\le \omega_2(f;\delta_2)\left[1 + \frac{1}{\delta_2}\mathscr{K}_m(|s-y|;x,y)\right] \\
&\quad + \omega_1(f;\delta_1)\left[1 + \frac{1}{\delta_1}\mathscr{K}_m(|t-x|;x,y)\right] \\
&\le \omega_2(f;\delta_2)\left[1 + \frac{1}{\delta_2}\left(\mathscr{K}_m((s-y)^2;x,y)\right)^{1/2}\right] \\
&\quad + \omega_1(f;\delta_1)\left[1 + \frac{1}{\delta_1}\left(\mathscr{K}_m((t-x)^2:x,y)\right)^{1/2}\right].
\end{aligned}$$

Now, choosing $\delta_1 = \delta_1(x)$ and $\delta_2 = \delta_2(y)$, we obtain the required result. □

4.1 Degree of approximation

In our next result we obtain an estimate of the error in the approximation for the operators (2) with the aid of the Lipschitz class.

In the bivariate case, for $0 < \xi \le 1$, and $f \in C(\Delta_2)$ we define the Lipschitz class as:

$$|f(t,s) - f(x,y)| \le M\{(t-x)^2 + (s-y)^2\}^{\frac{\xi}{2}}.$$

Theorem 3.*Let* $f \in Lip_M(\xi)$. *Then, for every* $(x,y) \in \Delta_2$, *we have*

$$|\mathscr{K}_m(f;x,y) - f(x,y)| \le M\{\delta_1^2(x) + \delta_2^2(y)\}^{\frac{\xi}{2}}.$$

*Proof.*By our hypothesis, for $(x,y) \in \Delta_2$ we may write

$$\begin{aligned}
|(\mathscr{K}_m f)(x,y) - f(x,y)| &\le \mathscr{K}_m(|f(t,s) - f(x,y)|;x,y) \\
&\le M\mathscr{K}_m(\{|t-x|^2 + |s-y|^2\}^{\frac{\xi}{2}};x,y).
\end{aligned}$$

Now, applying Hölder's inequality with $u_1 = \dfrac{2}{\xi}$, $v_1 = \dfrac{2}{2-\xi}$ we have

$$\begin{aligned}
|(\mathscr{K}_m f)(x,y) - f(x,y)| &\le M\{\mathscr{K}_m((t-x)^2;x,y) \\
&\quad + \mathscr{K}_m((s-y)^2;x,y)\}^{\xi/2} \\
&= M\{\delta_1^2(x) + \delta_2^2(y)\}^{\frac{\xi}{2}}.
\end{aligned}$$

Hence, the proof is completed.□

Next, we consider
$C^1(\Delta_2)=\{f\in C(\Delta_2): f'_x, f'_y\in C(\Delta_2)\}$.

Theorem 4. *Let $f\in C^1(\Delta_2)$ and $(x,y)\in\Delta_2$. Then, we have*
$|\mathscr{K}_m(f;x,y)-f(x,y)|\leq \|f'_x\|\delta_1(x)+\|f'_y\|\delta_2(y)$.

Proof. Let $(x,y)\in\Delta_2$ be a fixed point. Then, by Taylor's formula we may write

$$f(t,s)-f(x,y)=\int_x^t f'_u(u,s)du+\int_y^s f'_v(x,v)dv.$$

Operating $\mathscr{K}_m(.;x,y)$ on both sides,

$$|\mathscr{K}_m(f;x,y)-f(x,y)| = \mathscr{K}_m\left(\int_x^t f'_u(u,s)du;x,y\right) +\mathscr{K}_m\left(\int_y^s f'_v(x,v)dv,x,y\right).$$

Since,

$$\left|\int_x^t f'_u(u,s)d_qu\right|\leq \|f'_x\||t-x|$$

and

$$\left|\int_y^s f'_v(x,v)d_qv\right|\leq \|f'_y\||s-y|.$$

Therefore,

$$|\mathscr{K}_m(f;x,y)-f(x,y)| \leq \|f'_x\|\mathscr{K}_m(|t-x|;x,y) +\|f'_y\|\mathscr{K}_m(|s-y|;x,y).$$

Now, using Cauchy-Schwarz inequality,

$$\begin{aligned}|\mathscr{K}_m(f;x,y)-f(x,y)| &\leq \|f'_x\|\mathscr{K}_m((t-x)^2;x,y)^{1/2} +\|f'_y\|\mathscr{K}_m((s-y)^2;x,y)^{1/2}\\ &= \|f'_x\|\delta_1(x)+\|f'_y\|\delta_2(y).\end{aligned}$$

This completes the proof of the theorem.

□

Theorem 5. *For the function $f\in C(\Delta_2)$, we have the following inequality*

$$|\mathscr{K}_m(f;x,y)-f(x,y)| \leq M\left\{\bar{\omega}_2\Big(f;\frac{\sqrt{C_m(x,y)}}{2}\Big) +min\{1,C_m\}||f||_{C(\Delta_2)}\right\}+\omega(f;\psi_m(x,y)),$$

where

$$\psi_m(x,y)=\sqrt{\left\{\left(\frac{2mx+1}{2(m+1)}-x\right)^2+\left(\frac{2my+1}{2(m+1)}-y\right)^2\right\}},$$

$$C_m(x,y)=\delta_1^2(x)+\delta_2^2(y)+\psi_m^2(x,y)$$

and the constant $M>0$, is independent of f and $C_m(x,y)$.

Proof. We introduce the auxiliary operators as follows:

$$\mathscr{K}^*_m(f;x,y) = \mathscr{K}_m(f;x,y)-f\left(\frac{2mx+1}{2(m+1)},\frac{2my+1}{2(m+1)}\right) +f(x,y),$$

then using Lemma 1, we have
$\mathscr{K}^*_m(1;x,y)=1$, $\mathscr{K}^*_m((t-x);x,y)=0$ and $\mathscr{K}^*_m((s-y);x,y)=0$.
From Taylor's theorem, for $g\in C^2(\Delta_2)$ and $(t,s)\in\Delta_2$,

$$\begin{aligned}g(t,s)-g(x,y) &= g(t,y)-g(x,y)+g(t,s)-g(t,y)\\ &= \frac{\partial g(x,y)}{\partial x}(t-x)+\int_x^t(t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du +\frac{\partial g(x,y)}{\partial y}(s-y) +\int_y^s(s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv.\end{aligned}$$

Applying the operator $\mathscr{K}^*_m(.,x,y)$ on both sides, we get

$$\begin{aligned}\mathscr{K}^*_m(g;x,y)-g(x,y) &= \mathscr{K}^*_m\left(\int_x^t(t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;x,y\right) +\mathscr{K}^*_m\left(\int_y^s(s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv;x,y\right)\\ &= \mathscr{K}_m\left(\int_x^t(t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;x,y\right) -\int_x^{\frac{2mx+1}{2(m+1)}}\left(\frac{2mx+1}{2(m+1)}-u\right)\frac{\partial^2 g(u,y)}{\partial u^2}du\\ &\quad+\mathscr{K}_m\left(\int_y^s(s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv;x,y\right) -\int_x^{\frac{2my+1}{2(m+1)}}\left(\frac{2my+1}{2(m+1)}-v\right)\frac{\partial^2 g(x,v)}{\partial v^2}dv.\end{aligned}$$

Hence,

$$\begin{aligned}|\mathscr{K}_m(g;x,y)-g(x,y)| &\leq \mathscr{K}_m\left(\left|\int_x^t|t-u|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du\right|;x,y\right) +\left|\int_x^{\frac{2mx+1}{2(m+1)}}\left|\frac{2mx+1}{2(m+1)}-u\right|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du\right|\\ &\quad+\mathscr{K}_m\left(\left|\int_y^s|s-v|\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv\right|;x,y\right) +\left|\int_x^{\frac{2my+1}{2(m+1)}}\left|\frac{2my+1}{2(m+1)}-v\right|\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv\right|\\ &\leq \left\{\mathscr{K}_m((t-x)^2;x,y)+\left(\frac{2mx+1}{2(m+1)}-x\right)^2 +\mathscr{K}_m((s-y)^2;x,y)+\left(\frac{2my+1}{2(m+1)}-y\right)^2\right\}\times||g||_{C^2(\Delta_2)}\\ &\leq (\delta_1^2(x)+\delta_2^2(y)+\psi_m^2(x,y))||g||_{C^2(\Delta_2)}\\ &= C_m(x,y)||g||_{C^2(\Delta_2)}.\end{aligned} \tag{8}$$

Also, using Lemma 2

$$|\mathcal{K}_m^*(f;x,y)| \leq |\mathcal{K}_m(f;x,y)| + \left| f\left(\frac{2mx+1}{2(m+1)}, \frac{2my+1}{2(m+1)}\right)\right| + |f(x,y)| \leq 3||f||. \quad (9)$$

Hence in view of (8) and (9), for any $g \in C^2(\Delta_2)$, we get $|\mathcal{K}_m(f;x,y) - f(x,y)| \leq$

$$|\mathcal{K}_m^*(f-g;x,y)| + |\mathcal{K}_m^*(g;x,y) - g(x,y)| + |g(x,y) - f(x,y)| + \left| f\left(\frac{2mx+1}{2(m+1)}, \frac{2my+1}{2(m+1)}\right) - f(x,y)\right|$$
$$\leq 4||f-g|| + |\mathcal{K}_m(g;x,y) - g(x,y)| + \left| f\left(\frac{2mx+1}{2(m+1)}, \frac{2ny+1}{2(m+1)}\right) - f(x,y)\right|$$
$$\leq \left(4||f-g|| + C_m(x,y)||g||_{C^2(\Delta_2)}\right) + \omega(f;\psi_m(x,y)).$$

Now, taking the infimum on the right side of the above inequality over all $g \in C^2(\Delta_2)$, we obtain

$$|\mathcal{K}_m(f;x,y) - f(x,y)| \leq 4K_2(f;\frac{C_m(x,y)}{4}) + \omega(f;\psi_m(x,y)).$$

Finally, using (6), the required result follows.□

4.2 Voronovskaja type theorem

In this section, we obtain a Voronovskaja type asymptotic theorem for the bivariate operators $\mathcal{K}_m(f;x,y)$.

Theorem 6. *Let $f \in C^2(\Delta_2)$ and $(x,y) \in \Delta_2$. Then, we have*
$\lim_{m\to\infty} m(\mathcal{K}_m(f;x,y) - f(x,y)) =$

$$\tfrac{1}{2}(1-2x)f_x'(x,y) + \tfrac{1}{2}(1-2y)f_y'(x,y) + \tfrac{1}{2}x(1-x)f_{xx}'' - xyf_{xy}'' + \tfrac{1}{2}y(1-y)f_{yy}'',$$

uniformly on Δ_2.

Proof. Let $(x,y) \in \Delta_2$. By Taylor's theorem, we have

$$f(t,s) = f(x,y) + f_x'(x,y)(t-x) + f_y'(x,y)(s-y) + \frac{1}{2}\{f_{xx}''(x,y)(t-x)^2 + 2f_{xy}''(x,y)(t-x)(s-y) + f_{yy}''(x,y)(s-y)^2\} + \varepsilon(t,s;x,y)\sqrt{(t-x)^4+(s-y)^4}, \quad (10)$$

for $(t,s) \in \Delta_2$, where $\varepsilon(t,s;x,y) \in C(\Delta_2)$ and $\varepsilon(t,s;x,y) \to 0$, as $(t,s) \to (x,y)$.

Applying $\mathcal{K}_m(.;x,y)$ on both sides of (10), we obtain

$$\mathcal{K}_m(f(t,s);x,y) = f(x,y) + f_x'(x,y)\mathcal{K}_m((t-x);x,y) + f_y'(x,y)\mathcal{K}_m((s-y);x,y) + \frac{1}{2}\{f_{xx}''(x,y)\mathcal{K}_m((t-x)^2;x,y) + 2f_{xy}''(x,y)\mathcal{K}_m((t-x)(s-y);x,y) + f_{yy}''(x,y)\mathcal{K}_m((s-y)^2;x,y)\} + \mathcal{K}_m(\varepsilon(t,s;x,y)\sqrt{(t-x)^4+(s-y)^4};x,y).$$

Applying Lemma 1, we may write,

$$\lim_{m\to\infty} m\mathcal{K}_m(f;x,y) - f(x,y)) = \frac{1}{2}(1-2x)f_x'(x,y) + \frac{1}{2}(1-2y)f_y'(x,y) + \frac{1}{2}x(1-x)f_{xx}'' - xyf_{xy}'' + \frac{1}{2}y(1-y)f_{yy}'' + \lim_{m\to\infty} m\mathcal{K}_m(\varepsilon(t,s;x,y)\sqrt{(t-x)^4+(s-y)^4};x,y).$$

Now, applying Cauchy-Schwarz inequality,

$$\left|\mathcal{K}_m(\varepsilon(t,s;x,y)\sqrt{(t-x)^4+(s-y)^4};x,y)\right| \leq \{\mathcal{K}_m(\varepsilon^2(t,s;x,y);x,y)\}^{1/2} \times \{\mathcal{K}_m((t-x)^4+(s-y)^4;x,y)\}^{1/2}$$
$$\leq \{\mathcal{K}_m(\varepsilon^2(t,s;x,y),x,y)\}^{1/2} \times \{\mathcal{K}_m((t-x)^4;x,y) + \mathcal{K}_m((s-y)^4;x,y)\}^{1/2}. \quad (11)$$

Applying Theorem 1, since $\varepsilon^2(t,s;x,y) \to 0$ as $m \to \infty$, we obtain

$$\lim_{m\to\infty} \mathcal{K}_m(\varepsilon^2(t,s;x,y);x,y) = 0.$$

Further, in view of Lemma 1

$$m\left\{\mathcal{K}_m((t-x)^4;x,y) + \mathcal{K}_m((s-y)^4;x,y)\right\}^{1/2} = O(1),$$

as $m \to \infty$ uniformly in $(x,y) \in \Delta_2$.

Hence,

$$\lim_{m\to\infty} m\mathcal{K}_m(\varepsilon(t,s;x,y)\sqrt{(t-x)^4+(s-y)^4};x,y) = 0,$$

uniformly in $(x,y) \in \Delta_2$.

Thus the proof is completed.□

Theorem 7. ***(Simultaneous approximation)*** *For any $(x,y) \in \mathring{\Delta}_2$(interior of Δ_2) and $f \in C(\Delta_2)$, $\mathcal{K}_m$ satisfies*

$$\lim_{m\to\infty}\left(\frac{\partial}{\partial\omega}(\mathcal{K}_m f)(\omega,y)\right)_{\omega=x} = \frac{\partial f}{\partial x}(x,y), \quad (12)$$

and

$$\lim_{m\to\infty}\left(\frac{\partial}{\partial\omega}(\mathcal{K}_m f)(x,v)\right)_{v=y} = \frac{\partial f}{\partial y}(x,y). \quad (13)$$

*Proof.*Since $f \in C^1(\Delta_2)$, therefore, by Taylor's formula, we have

$$f(t,s) = f(x,y) + f_x'(x,y)(t-x) + f_y'(x,y)(s-y)$$
$$+\phi(t,s;x,y)\sqrt{(t-x)^2+(s-y)^2} \text{ for } (t,s) \in \mathring{\Delta}_2,$$

where $\phi(t,s;x,y) \in C(\Delta_2)$ and $\phi(x,y) = 0$. Operating $\mathscr{K}_m$ to the above equation and then differentiating partially with respect to x and using Lemma 1, we get

$$\left(\frac{\partial}{\partial\omega}(\mathscr{K}_m f;w,y)\right)_{\omega=x} = f(x,y)\left(\frac{\partial}{\partial\omega}(\mathscr{K}_m 1;w,y)\right)_{\omega=x}$$
$$+f_x'(x,y)\left(\frac{\partial}{\partial\omega}(\mathscr{K}_m(t-x);w,y)\right)_{\omega=x}$$
$$+f_y'(x,y)\left(\frac{\partial}{\partial\omega}(\mathscr{K}_m(s-y);w,y)\right)_{\omega=x}$$
$$+\left(\frac{\partial}{\partial\omega}(\mathscr{K}_m\phi(t,s;x,y)\sqrt{(t-x)^2+(s-y)^2};\omega,y)\right)_{\omega=x},$$
$$= f_x'(x,y)\left\{\frac{\partial}{\partial\omega}\left(\frac{2m\omega}{2(m+1)}\right)\right\}_{\omega=x}$$
$$+f_y'(x,y)\left\{\frac{\partial}{\partial\omega}\left(\frac{2my}{2(m+1)}\right)\right\}_{\omega=x}$$
$$+\left(\frac{\partial}{\partial\omega}(\mathscr{K}_m\phi(t,s;x,y)\sqrt{(t-x)^2+(s-y)^2};\omega,y)\right)_{\omega=x}$$
$$= f_x'(x,y)\left(\frac{m}{m+1}\right) + T,$$

where

$$T = \left(\frac{\partial}{\partial\omega}(\mathscr{K}_m\phi(t,s;x,y)\sqrt{(t-x)^2+(s-y)^2};\omega,y)\right)_{\omega=x}.$$

Now we will show that for every $(x,y) \in \mathring{\Delta}_2$, $T \to 0$, as $m \to \infty$.

$$T = (m+1)^2 \sum_{k,j=0,k+j\le m}\left(\frac{\partial}{\partial\omega}p_{m,k,j}(w,y)\right)_{\omega=x}$$
$$\times\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}}\int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}}\phi(t,s;x,y)\sqrt{(t-x)^2+(s-y)^2}dtds$$
$$= (m+1)^2 \sum_{k,j=0,k+j\le m}\frac{(k-mx)(1-y)-x(j+my)(x,y)}{x(1-x-y)}$$
$$\times p_{m,k,j}\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}}\int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}}\phi(t,s;x,y)$$
$$\times\sqrt{(t-x)^2+(s-y)^2}dtds$$
$$= \frac{(m+1)^2(1-y)}{x(1-x-y)}\sum_{k,j=0,k+j\le m}(k-mx)p_{m,k,j}(x,y)$$
$$\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}}\int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}}\phi(t,s;x,y)\sqrt{(t-x)^2+(s-y)^2}dtds$$
$$-\frac{(m+1)^2}{(1-x-y)}\sum_{k,j=0,k+j\le m}(j+my)p_{m,k,j}(x,y)$$
$$\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}}\int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}}\phi(t,s;x,y)\sqrt{(t-x)^2+(s-y)^2}dtds$$
$$= T_1 + T_2, \text{ (say)}.$$

First, we estimate T_1. Using Cauchy-Schwarz inequality, we have

$$T_1 \le \frac{m(1-y)}{x(1-x-y)}\left(\sum_{k,j=0,k+j\le m}p_{m,k,j}(x,y)\left(\frac{k}{m}-x\right)^2\right)^{1/2}$$
$$\times\left((\mathscr{K}_m(\phi^2(t,s;x,y)\{(t-x)^2+(s-y)^2\};x,y))^{1/2}\right)$$
$$\le \frac{m(1-y)}{x(1-x-y)}\left(\sum_{k,j=0,k+j\le m}p_{m,k,j}(x,y)\left(\frac{k}{m}-x\right)^2\right)^{1/2}$$
$$\times\Big\{(\mathscr{K}_m\phi^2(t,s;x,y)(t-x)^2;x,y)$$
$$+(\mathscr{K}_m\phi^2(t,s;x,y;s-y)^2)(x,y)\Big\}^{1/2}$$
$$\le \frac{m(1-y)}{x(1-x-y)}\left((B_m(t-x)^2)(x,y)\right)^{1/2}$$
$$\times\left(\left\{(\mathscr{K}_m\phi^4(t,s;x,y))\right\}\right)^{1/4}\Big\{((\mathscr{K}_m(t-x)^4;x,y))^{1/2}$$
$$+((\mathscr{K}_m(s-y)^4;x,y))^{1/2}\Big\}^{1/2}.$$

Now, using Lemma 2.2 of [12], we have $(B_m(s-x))^2(x,y) = O(\frac{1}{m})$ as $m \to \infty$ and applying Lemma 2, we get

$$|T_1| \le N\left\{(\mathscr{K}_m\phi^4(t,s;x,y);x,y)\right\}^{1/4},$$

for some constant $N > 0$.
Now, from Theorem 1, for any $(x,y) \in \mathring{\Delta}_2$, we obtain,

$$\lim_{m\to\infty}\left\{(\mathscr{K}_m\phi^4(t,s;x,y);x,y)\right\} = \phi^4(x,y) = 0.$$

Hence, for every $(x,y) \in \mathring{\Delta}_2$, $T_1 \to 0$, as $m \to \infty$.

In a similar manner, we can show that for every $(x,y) \in \mathring{\Delta}_2$, $T_2 \to 0$, as $m \to \infty$.
On combining the estimates of T_1 and T_2, for every $(x,y) \in \mathring{\Delta}_2$, we get $T \to 0$, as $m \to \infty$. Hence, we get the result (12). By a similar reasoning, we can prove (13). Thus, the proof is completed.□

5 Approximation properties of GBS operators of Bernstein-Kantorovich type

For $m \in \mathbb{N}$ and $f \in C_b(\Delta_2)$, the GBS operators associated with the operators $\mathscr{K}_m$ is defined by

$$U\mathscr{K}_m(f(t,s);(x,y))$$
$$= (m+1)^2\sum_{k,j=0,k+j\le m}p_{m,k,j}(x,y)$$
$$\times\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}}\int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}}[f(t,y)+f(x,s)-f(t,s)]dtds,$$

for any $(x,y) \in \Delta_2$. More details on GBS operators can be found in [6], [9], [13].

For $f \in C_b(\Delta_2)$, the Lipschitz class $Lip_M(\xi)$ with $0 < \xi \leq 1$ is defined as:

$$Lip_M(\xi) = \left\{ f \in C_b(\Delta_2) : |\Delta f[(t,s);(x,y)]| \leq M\{(t-x)^2 + (s-y)^2\}^{\frac{\xi}{2}} \right\},$$

for $(t,s),(x,y) \in \Delta_2$.
In the following theorem, we obtain the degree of approximation for the operators $(U\mathscr{K}_m f)(x,y)$ by means of the Lipschitz class of Bögel continuous functions.

Theorem 8. *For $f \in Lip_M(\xi)$, we have*

$$|U\mathscr{K}_m f(x,y) - f(x,y)| \leq M\{\delta_1^2(x) + \delta_2^2(y)\}^{\frac{\xi}{2}},$$

for $M > 0$, $\xi \in (0,1]$.

Proof. By the definition of the operators $U\mathscr{K}_m f(x,y)$ and by linearity of the operators $\mathscr{K}_m$ given by (2), we can write

$$\begin{aligned}(U\mathscr{K}_m f)(x,y) &= \mathscr{K}_m(f(x,s) + f(t,y) - f(t,s);x,y) \\ &= \mathscr{K}_m(f(x,y) - \Delta f[(t,s);(x,y)];x,y) \\ &= f(x,y)\mathscr{K}_m(e_{00};x,y) \\ &\quad - \mathscr{K}_m(\Delta f[(t,s);(x,y)];x,y).\end{aligned}$$

By the hypothesis, we get

$$\begin{aligned}&|(U\mathscr{K}_m)f(x,y) - f(x,y)| \\ &\leq \mathscr{K}_m(|\Delta f[(t,s);(x,y)]|;x,y) \\ &\leq M\mathscr{K}_m(\{(t-x)^2 + (s-y)^2\}^{\frac{\xi}{2}};x,y).\end{aligned}$$

Now, using the Hölder's inequality with $u_1 = 2/\xi, u_2 = 2/(2-\xi)$ and Lemma 1, we have

$$\begin{aligned}&|(U\mathscr{K}_m f)(x,y) - f(x,y)| \\ &\leq M\{\mathscr{K}_m((t-x)^2;x,y) + \mathscr{K}_m((s-y)^2;x,y)\}^{\frac{\xi}{2}} \\ &\quad \times \{\mathscr{K}_m(e_{00};x,y)\}^{(2-\xi)/2} \\ &= M\{\delta_1^2(x) + \delta_2^2(y)\}^{\frac{\xi}{2}}.\end{aligned}$$

This completes the proof.□

In order to improve the measure of smoothness, for $f \in C_b(\Delta_2)$, the mixed $K-$ functional ([3], [5])is defined by

$$K_{mixed}(f;t_1,t_2) = \inf_{g_1,g_2,h}\left\{ \|f - g_1 - g_2 - h\|_\infty + t_1\left\|D_B^{2,0}g_1\right\|_\infty + t_2\left\|D_B^{0,2}g_2\right\|_\infty + t_1t_2\left\|D_B^{2,2}h\right\|_\infty \right\},$$

where $g_1 \in C_B^{2,0}, g_2 \in C_B^{0,2}, h \in C_B^{2,2}$ and, for $i,j = 0,1,2, C_B^{i,j}$ is the space of functions $f \in C_b(\Delta_2)$ such that the mixed partial derivatives $D_B^{p,q}f,\ p = 0,1,\cdots,i, q = 0,1,\cdots,j$ is continuous in Δ_2. The partial derivatives are defined as follows:

$$D_x f(x_0,y_0) := D_B^{1,0}(f;x_0,y_0) = \lim_{x\to x_0}\frac{\Delta_x f\{(x_0,x);y_0\}}{x - x_0},$$

and

$$D_y f(x_0,y_0) := D_B^{0,1}(f;x_0,y_0) = \lim_{y\to y_0}\frac{\Delta_y f\{x_0;(y_0,y)\}}{y - y_0},$$

where $\Delta_x f\{(x_0,x);y_0\} = f(x,y_0) - f(x_0,y_0)$ and $\Delta_y f\{x_0;(y_0,y)\} = f(x_0,y) - f(x_0,y_0)$. Similarly the second order partial derivatives can be shown to be same as the ordinary derivatives. For instance, the derivative of $D_x f(x_0,y_0)$ with respect to the variable y at the point (x_0,y_0) is defined by
$D_yD_x f(x_0,y_0) :=$

$$D_B^{0,1}D_B^{1,0}(f;x_0,y_0) = \lim_{y\to y_0}\frac{\Delta_y(D_x f)\{x_0;(y_0,y)\}}{y - y_0}.$$

In our next result, we study the order of approximation of $\{U\mathscr{K}_m(f)\}$ to the function $f \in C_b(\Delta_2)$.

Theorem 9. *Let $(U\mathscr{K}_m f)(x,y)$ be the GBS operator of $(\mathscr{K}_m f)(x,y)$. Then*

$$|(U\mathscr{K}_m f)(x,y) - f(x,y)| \leq 2K_{mixed}\left(f;\frac{1}{2(m+1)},\frac{1}{2(m+1)}\right),$$

for each $f \in C_b(\Delta_2)$.

Proof. For $g_1 \in C_B^{2,0}(\Delta_2)$, by Taylor formula we may write

$$g_1(t,s) = g_1(x,y) + (t-x)D_B^{1,0}g_1(x,y) + \int_x^t (t-u)D_B^{2,0}g_1(u,y)\,du$$

([4], page 67-69). Since the operators $U\mathscr{K}_m f(x,y)$ is linear,

$U\mathscr{K}_m(1;x,y) = 1$, $U\mathscr{K}_m(t;x,y) = x$ and $U\mathscr{K}_m(s;x,y) = y$.

We have,

$$(U\mathscr{K}_m g_1)(x,y) = g_1(x,y) + U\mathscr{K}_m\left(\int_x^t (t-u)D_B^{2,0}g_1(u,y)\,du;x,y\right).$$

By the definition of $U\mathscr{K}_m$ and Lemma 2
$|U\mathscr{K}_m(g_1;x,y)-g_1(x,y)|=$

$$\left|\mathscr{K}_m\left(\int_x^t (t-u)\left[D_B^{2,0}g_1(u,y)-D_B^{2,0}g_1(u,s)\right]du;x,y\right)\right|$$

$$\leq \mathscr{K}_m\left(\left|\int_x^t |t-u|\left|D_B^{2,0}g_1(u,y)-D_B^{2,0}g_1(u,s)\right|du;x,y\right|\right)$$

$$\leq \frac{1}{2}\left\|D_B^{2,0}g_1\right\|_\infty \mathscr{K}_m\left((t-x)^2;x,y\right)$$

$$\leq \frac{1}{2(m+1)}\left\|D_B^{2,0}g_1\right\|.$$

Similarly, we can write
$|U\mathscr{K}_m(g_2;x,y)-g_2(x,y)|$

$$\leq \frac{1}{2}\left\|D_B^{0,2}g_2\right\|_\infty \mathscr{K}_m\left((s-y)^2;x,y\right)$$

$$\leq \frac{1}{2(m+1)}\left\|D_B^{0,2}g_2\right\|_\infty, \text{ for } g_2\in C_B^{0,2}(\Delta_2).$$

Again, for $h\in C_B^{2,2}(\Delta_2)$,

$$h(t,s)=h(x,y)+(t-x)D_B^{1,0}h(x,y)$$
$$+(s-y)D_B^{0,1}h(x,y)+\int_x^t (s-y)(t-u)D_B^{2,1}h(u,y)du$$
$$+(t-x)(s-y)D_B^{1,1}h(x,y)$$
$$+\int_x^t (t-u)D_B^{2,0}h(u,y)du+\int_y^s (s-v)D_B^{0,2}h(x,v)dv$$
$$+\int_y^s (t-x)(s-v)D_B^{1,2}h(x,v)dv$$
$$+\int_x^t\int_y^s (t-u)(s-v)D_B^{2,2}h(u,v)dvdu.$$

Since $U\mathscr{K}_m((t-x);x,y)=0=U\mathscr{K}_m((s-y);x,y)$, we have
$|U\mathscr{K}_m(h;x,y)-h(x,y)|$

$$\leq \left|\mathscr{K}_m\left(\int_x^t\int_y^s (t-u)(s-v)D_B^{2,2}h(u,v)dv\,du;x,y\right)\right|$$

$$\leq \mathscr{K}_m\left(\left|\int_x^t\int_y^s (t-u)(s-v)D_B^{2,2}h(u,v)dvdu\right|;x,y\right)$$

$$\leq \mathscr{K}_m\left(\int_x^t\int_y^s |t-u||s-v|\left|D_B^{2,2}h(u,v)\right|dvdu;x,y\right)$$

$$\leq \frac{1}{4}\left\|D_B^{2,2}h\right\|_\infty \mathscr{K}_m\left((t-x)^2(s-y)^2;x,y\right)$$

$$\leq \frac{1}{4(m+1)^2}\left\|D_B^{2,2}h\right\|_\infty.$$

Therefore, for $f\in C_b(\Delta_2)$, we obtain

$$|U\mathscr{K}_m(f;x,y)-f(x,y)|$$
$$\leq |(f-g_1-g_2-h)(x,y)|+|(g_1-U\mathscr{K}_mg_1)(x,y)|$$
$$+|(g_2-U\mathscr{K}_mg_2)(x,y)|+|(h-U\mathscr{K}_mh)(x,y)|$$
$$+|U\mathscr{K}_m((f-g_1-g_2-h);x,y)|$$
$$\leq 2\|f-g_1-g_2-h\|_\infty+\frac{1}{2(m+1)}\left\|D_B^{2,0}g_1\right\|_\infty$$
$$+\frac{1}{2(m+1)}\left\|D_B^{0,2}g_2\right\|_\infty+\frac{1}{4(m+1)^2}\left\|D_B^{2,2}h\right\|_\infty.$$

Taking the infimum over all $g_1\in C_B^{2,0}, g_2\in C_B^{0,2}, h\in C_B^{2,2}$, we obtain the required result.□

*Example 1.*Let $\Delta_2=\{(x,y)\in\mathbb{R}\times\mathbb{R}|\ x,y\geq 0, x+y\leq 1\}$. For $m=5$ (orange), $m=15$ (yellow), $m=25$ (pink), the convergence of the operator $\mathscr{K}_m$ given by (2), to $f(x,y)=y^3+\sin(\pi x^2/2)$(green) is illustrated in figure 1 for $(x,y)\in\Delta_2$.

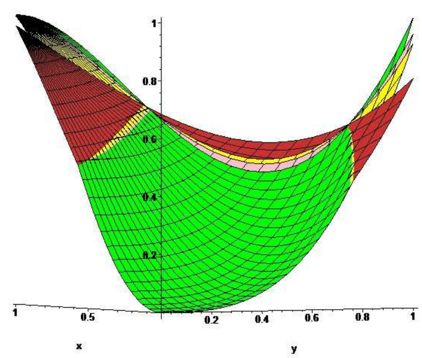

Fig. 1: The convergence of $\mathscr{K}_m(f;x,y)$ to $f(x,y)$ (green f, orange $\mathscr{K}_5$, yellow $\mathscr{K}_{15}$, pink $\mathscr{K}_{25}$).

*Example 2.*For $m=10,25,40$, respectively, the convergence of $U\mathscr{K}_m(f;x,y)$ (yellow, blue, orange) to $f(x,y)=y+\cos\left(4x^2\right)$ is illustrated in figure 2. It is seen that as the values of m increase, the convergence of $U\mathscr{K}_m(f;x,y)$ to $f(x,y)$ becomes better.
In figure 3, for $m=20$, the comparison of convergence of $\mathscr{K}_m(f;x,y)$ (orange) and its GBS operator $U\mathscr{K}_m(f;x,y)$ (pink) to $f(x,y)=y^3\sin(4x)$(green) is observed.

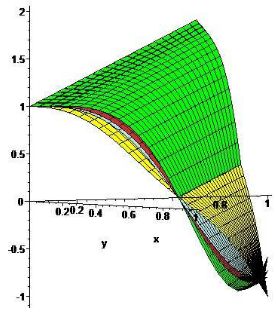

Fig. 2: The convergence of $U\mathscr{K}_m(f;x,y)$ to $f(x,y)$ (green f, yellow $U\mathscr{K}_{10}$, blue $U\mathscr{K}_{25}$, orange $U\mathscr{K}_{40}$).

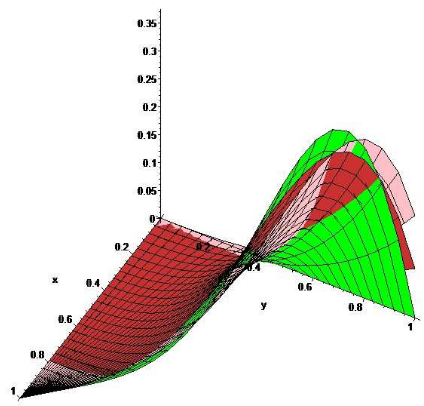

Fig. 3: The comparison of convergence of $\mathscr{K}_m(f;x,y)$ and $U\mathscr{K}_m(f;x,y)$ to $f(x,y)$ (green f, orange $\mathscr{K}_{20}$, pink $U\mathscr{K}_{20}$).

6 Conclusion

The rate of convergence of the bivariate Bernstein-Kantorovich type operators introduced by Pop and Farcas [12] is studied by means of Lipschitz class and the class of continuously differentiable functions on the triangle Δ_2. The Voronovskaja type theorem for twice differentiable functions and the simultaneous approximation property for the first order partial derivatives of these operators are established. The degree of approximation of the associated GBS operators is obtained by means of the Lipschitz class for Bögel continuous functions and the Peetre's K-functional. Furthermore, the convergence of the bivariate and the GBS operators is shown by illustrative graphics using Maple algorithms.

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