Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/060207

Symmetric Decomposition of $f \in L^2(\mathbb{R})$ Via Fractional Riemann-Liouville Operators

Yulong Li[1,2]

[1]Science and Mathematics, Singapore University of Technology and Design, 8 Somapah Road Singapore 487372, Singapore
[2]Department of Mathematics and Statistics, University of Wyoming, 1000 E. University Avenue, Dept. 3036, Laramie, Wyoming, USA

Received: 12 Oct. 2018, Revised: 13 Oct. 2019, Accepted: 19 Oct. 2019
Published online: 1 Apr. 2020

Abstract: The present paper proves that given $-1/2 < s < 1/2$, for any $f \in L^2(\mathbb{R})$, there is a unique $u \in \widehat{H}^{|s|}(\mathbb{R})$ such that

$$f = \boldsymbol{D}^{-s}u + \boldsymbol{D}^{s*}u,$$

where $\boldsymbol{D}^{-s}, \boldsymbol{D}^{s*}$ are fractional Riemann-Liouville operators and the fractional derivatives are understood in the weak sense. Furthermore, regularity of u is addressed, and other versions of the results are established. Consequently, the Fourier transform of elements of $L^2(\mathbb{R})$ is characterized.

Keywords: Riemann-Liouville fractional operators, weak fractional derivative, Fourier transform, regularity, decomposition, symmetric.

1 Motivation

Fourier analysis is a large branch of mathematics whose point of departure is the study of Fourier series and integrals, which is one of most fruitful achievements of mathematics. Fourier series may be seen as the decomposition of a periodic function into sinusoid waves of varying frequencies. Application of such decompositions is naturally abundant and the importance of these compositions is well-known and incontestable (see for example [1], [2], [3], etc.). Therefore, it is natural to ask: Is this the unique way to obtain such interesting and useful decompositions?

With this motivation, author discovers other ways to obtain such interesting decompositions from the perspective of fractional Riemann-Liouville operators, which are somewhat similar to the classical Fourier decompositions despite showing some new features. The present study proves that every function $f \in L^2(\mathbb{R})$ could be written as a sum of fractional R-L integral and fractional R-L derivative of a certain function u belonging to classical Sobolev space. The fractional R-L derivative is understood in the weak sense defined in section 4. Also other versions of this kind of decomposition are pointed out and their Fourier transforms are characterized. These results help the author investigate a series of questions under the context of fractional calculus theory in the subsequent work.

On the other hand, in the analysis of fractional-order differential equations involving both left and right mixed-type fractional derivatives (see for example [4], [5], [6], [7] and the references therein), it is difficult to establish the existence of solutions because of the simultaneous presence of the left and right fractional derivatives. The present results provide an innovative means to prove the existence of solutions to some special cases of the fractional differential equations involving R-L or Caputo derivatives in $\mathbb{R}$. It is author's plan that their applications in fractional-order differential equations and the potential relations between the classical Fourier decompositions and these new decompositions will be extensively explored in future papers. Review of the present paper indicated that some techniques and results were presented in the author's dissertation [9] and archive [8] and were also applied to other works [10].

The present paper is outlined as follows. Section 2 presents notations and conventions. Section 3 addresses the preliminary knowledge on fractional R-L operators. Section 4 involves the characterization of $\widehat{H}^s(\mathbb{R})$ via R-L derivatives,

* Corresponding author e-mail: liyulong0807101@gmail.com, yulong_li@sutd.edu.sg

which has been obtained in our previous work [5]. Section 5 establishes the main results. Section 6 comprises some questions.

2 Notations

The present study adopts the following convention:

– All the functions considered in this paper are default to be real valued unless otherwise specified.
– (f,g) and $\int_{\mathbb{R}} fg$ shall be used interchangeably. Also, we denote integration $\int_A f$ on set A without pointing out the variable unless it is necessary to specify.
– $C_0^\infty(\mathbb{R})$ is the set of smooth functions with compact supports in $\mathbb{R}$.
– $\mathscr{F}(u)$ represents the Fourier transform of u defined in Definition 6, and $\widehat{u}$ represents the Plancherel transform of u defined in Theorem 3, which is well known that $\widehat{u}$ is an isometry map from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$ and coincides with $\mathscr{F}(u)$ if $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.
– $u^\vee$ represents the inverse of Plancherel transform, and $*$ denotes convolution.

3 Preliminary

3.1 Fractional R-L integrals and associated properties

Definition 1. *Let $\sigma > 0$ and $u(x) : \mathbb{R} \to \mathbb{R}$. The left and right Riemann-Liouville fractional integrals of order σ are defined as*

$$(\boldsymbol{D}^{-\sigma}u)(x) := \frac{1}{\Gamma(\sigma)} \int_{-\infty}^{x} (x-s)^{\sigma-1} u(s)\,\mathrm{d}s, \tag{1}$$

$$(\boldsymbol{D}^{-\sigma *}u)(x) := \frac{1}{\Gamma(\sigma)} \int_{x}^{\infty} (s-x)^{\sigma-1} u(s)\,\mathrm{d}s, \tag{2}$$

where $\Gamma(\sigma)$ is Gamma function.

Property 1([11], p. 96). Given $0 < \sigma$,

$$(\phi, \boldsymbol{D}^{-\sigma}\psi) = (\boldsymbol{D}^{-\sigma *}\phi, \psi), \tag{3}$$

for $\phi \in L^p(\mathbb{R})$, $\psi \in L^q(\mathbb{R})$, $p > 1, q > 1$, $1/p + 1/q = 1 + \sigma$.

Property 2([11], Theorem 7.1, p.138). Assume $0 < \sigma < 1$ and $u \in L^1(\mathbb{R})$, then

$$\mathscr{F}(\boldsymbol{D}^{-\sigma *}u) = (-2\pi i\xi)^{-\sigma}\mathscr{F}(u) \text{ and } \mathscr{F}(\boldsymbol{D}^{-\sigma}u) = (2\pi i\xi)^{-\sigma}\mathscr{F}(u), \quad \xi \neq 0, \tag{4}$$

where $\mathscr{F}(\cdot)$ is the Fourier Transform as defined in Definition 6 .

Remark. The complex power functions are understood as $(\pm i\xi)^\sigma = |\xi|^\sigma e^{\pm\sigma\pi i\cdot \mathrm{sign}(\xi)/2}$.

Property 3([11], pp. 95, 96). Let $\mu > 0$. For a fixed $h \in \mathbb{R}$, τ_h is defined as $\tau_h u(x) = u(x-h)$; similarly for a fixed $\kappa > 0$, Π_κ is defined as $\Pi_\kappa u(x) = u(\kappa x)$. Under the assumption that $\boldsymbol{D}^{-\mu}u$ and $\boldsymbol{D}^{-\mu *}u$ are well-defined, the following is true:

$$\begin{aligned} \tau_h(\boldsymbol{D}^{-\mu}u) = \boldsymbol{D}^{-\mu}(\tau_h u), \quad \tau_h(\boldsymbol{D}^{-\mu *}u) = \boldsymbol{D}^{-\mu *}(\tau_h u), \\ \Pi_\kappa(\boldsymbol{D}^{-\mu}u) = \kappa^\mu \boldsymbol{D}^{-\mu}(\Pi_\kappa u), \quad \Pi_\kappa(\boldsymbol{D}^{-\mu *}u) = \kappa^\mu \boldsymbol{D}^{-\mu *}(\Pi_\kappa u). \end{aligned} \tag{5}$$

3.2 Fractional R-L derivatives and associated properties

Definition 2. *Let $u : \mathbb{R} \to \mathbb{R}$. Assume $\mu > 0$, n is the smallest integer greater than μ (i.e., $n-1 \le \mu < n$), and $\sigma = n - \mu$. The left and right Riemann-Liouville fractional derivatives of order μ are defined as*

$$(\boldsymbol{D}^{\mu} u)(x) := \frac{1}{\Gamma(\sigma)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_{-\infty}^{x} (x-s)^{\sigma-1} u(s)\,\mathrm{d}s, \tag{6}$$

$$(\boldsymbol{D}^{\mu*} u)(x) := \frac{(-1)^n}{\Gamma(\sigma)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_{x}^{\infty} (s-x)^{\sigma-1} u(s)\,\mathrm{d}s. \tag{7}$$

Property 4([5]). Let $0 < \mu$ and $u \in C_0^\infty(\mathbb{R})$, then $\boldsymbol{D}^{\mu} u, \boldsymbol{D}^{\mu*} u \in L^p(\mathbb{R})$ for any $1 \le p < \infty$.

Property 5([11], p. 137). Let $\mu > 0, u \in C_0^\infty(\mathbb{R})$, then

$$\mathscr{F}(\boldsymbol{D}^{\mu} u) = (2\pi i\xi)^{\mu} \mathscr{F}(u) \text{ and } \mathscr{F}(\boldsymbol{D}^{\mu*} u) = (-2\pi i\xi)^{\mu} \mathscr{F}(u), \quad \xi \neq 0, \tag{8}$$

where $\mathscr{F}(\cdot)$ is the Fourier Transform as defined in Definition 6 and as in Remark 2, the complex power functions are understood as $(\mp i\xi)^{\sigma} = |\xi|^{\sigma} e^{\mp \sigma \pi i \cdot sign(\xi)/2}$.

Property 6([5]). Consider τ_h and Π_κ defined in Property 3 and let $\mu > 0$, $n-1 \le \mu < n$, where n is a positive integer, then

$$\begin{aligned} \tau_h(\boldsymbol{D}^{\mu} u) = \boldsymbol{D}^{\mu}(\tau_h u), \quad \tau_h(\boldsymbol{D}^{\mu*} u) = \boldsymbol{D}^{\mu*}(\tau_h u), \\ \Pi_\kappa(\boldsymbol{D}^{\mu} u) = \kappa^{-\mu} \boldsymbol{D}^{\mu}(\Pi_\kappa u), \quad \Pi_\kappa(\boldsymbol{D}^{\mu*} u) = \kappa^{-\mu} \boldsymbol{D}^{\mu*}(\Pi_\kappa u). \end{aligned} \tag{9}$$

Now we unify the notations using $\boldsymbol{D}^{\mu} u$ and $\boldsymbol{D}^{\mu*} u$ for $\mu \in \mathbb{R}$. Namely, if $\mu \le 0$, they are understood as left and right fractional integrals, and if $0 < \mu$, as left and right derivatives. We adopt this convention throughout other sections of the paper.

4 Characterization of Sobolev space $\widehat{H}^s(\mathbb{R})$

This section involves the results of our previous work [5], which characterize the classical Sobolev space $\widehat{H}^s(\mathbb{R})$ defined in 5. This benefits the results of this study.

Definition 3([5]). *Assume $u, w \in L^1_{loc}(\mathbb{R})$, $\mu > 0$. We say function $w(x)$ is a weak μ-order left fractional derivative of u, denoted as $\boldsymbol{D}^{\mu} u = w$, if*

$$(u, \boldsymbol{D}^{\mu*}\psi) = (w, \psi), \quad \textit{for any} \quad \psi \in C_0^\infty(\mathbb{R}). \tag{10}$$

Analogously, $w(x)$ is called a weak μ-order right fractional derivative of u, denoted as $\boldsymbol{D}^{\mu*} u = w$, if

$$(u, \boldsymbol{D}^{\mu}\psi) = (w, \psi), \quad \textit{for any} \quad \psi \in C_0^\infty(\mathbb{R}). \tag{11}$$

Definition 4([5]). *Let $s \ge 0$. Define spaces*

$$\widetilde{W}_R^s(\mathbb{R}) = \{u(x) \in L^2(\mathbb{R}), \boldsymbol{D}^{s*} u \in L^2(\mathbb{R})\}, \tag{12}$$

$$\widetilde{W}_L^s(\mathbb{R}) = \{u(x) \in L^2(\mathbb{R}), \boldsymbol{D}^{s} u \in L^2(\mathbb{R})\}, \tag{13}$$

where $\boldsymbol{D}^{s*} u$ and $\boldsymbol{D}^{s} u$ are in the weak fractional derivative sense as defined in Definition 3. A semi-norm

$$|u|_L := \|\boldsymbol{D}^{s} u\|_{L^2(\mathbb{R})} \textit{ for } \widetilde{W}_L^s(\mathbb{R}) \textit{ and } |u|_R := \|\boldsymbol{D}^{s*} u\|_{L^2(\mathbb{R})} \textit{ for } \widetilde{W}_R^s(\mathbb{R}), \tag{14}$$

is given with the corresponding norm

$$\|u\|_\star := (\|u\|^2_{L^2(\mathbb{R})} + |u|^2_\star)^{1/2}, \ \star = L, R. \tag{15}$$

Remark. Notice that, by convention, $\widetilde{W}_L^0(\mathbb{R}) = \widetilde{W}_R^0(\mathbb{R}) = \widehat{H}^0(\mathbb{R}) = L^2(\mathbb{R})$.

Now we can characterize $\widehat{H}^s(\mathbb{R})$ as follows.

Theorem 1([5]). *Given* $s \geq 0$, $\widetilde{W}_R^s(\mathbb{R}) = \widetilde{W}_L^s(\mathbb{R}) = \widehat{H}^s(\mathbb{R})$ *with equal semi-norms and norms.*

We also use the following fact.

Corollary 1([5]). *Given* $s \geq 0$. $u \in \widehat{H}^s(\mathbb{R})$ *if and only if there exists a Cauchy sequence* $\{u_n\} \subset C_0^\infty(\mathbb{R})$ *in* $L^2(\mathbb{R})$ *so that* $\lim_{n\to\infty} u_n = u$ *and* $\{\boldsymbol{D}^s u_n\}$ *is also a Cauchy sequence in* $L^2(\mathbb{R})$. *Consequently, we have* $\lim_{n\to\infty} \boldsymbol{D}^s u_n = \boldsymbol{D}^s u$.
Likewise,
$u \in \widehat{H}^s(\mathbb{R})$ *if and only if there exists a Cauchy sequence* $\{u_n\} \subset C_0^\infty(\mathbb{R})$ *in* $L^2(\mathbb{R})$ *so that* $\lim_{n\to\infty} u_n = u$ *and* $\{\boldsymbol{D}^{s*} u_n\}$ *is also a Cauchy sequence in* $L^2(\mathbb{R})$. *Consequently, we have* $\lim_{n\to\infty} \boldsymbol{D}^{s*} u_n = \boldsymbol{D}^{s*} u$.

5 Main results

In this section, under weak fractional derivative sense defined in Section 4, the following result is established:

Theorem 2.*(1). Given* $-1/2 < s < 1/2$, *for* $\forall\ f \in L^2(\mathbb{R})$, *there is a unique* $u \in \widehat{H}^{|s|}(\mathbb{R})$ *such that the following decomposition holds:*

$$f = \boldsymbol{D}^{-s} u + \boldsymbol{D}^{s*} u. \tag{16}$$

Furthermore, $f \in \widehat{H}^t(\mathbb{R})$ *iff* $u \in \widehat{H}^{s+t}(\mathbb{R})$, *where* $0 < t$.
(2). Given $-1/2 < s < 1/2$, *for any* $f \in L^2(\mathbb{R})$, *there exists a unique* $u \in \widehat{H}^{|s|}(\mathbb{R})$ *such that the following decomposition holds:*

$$f = \boldsymbol{D}^{-s} u + \boldsymbol{D}^{s} u. \tag{17}$$

Furthermore, $f \in \widehat{H}^t(\mathbb{R})$ *iff* $u \in \widehat{H}^{s+t}(\mathbb{R})$, *where* $0 < t$.

*Remark.*If s is positive, $\boldsymbol{D}^{s*} u, \boldsymbol{D}^s u$ are understood as the weak fractional derivative of u and $\boldsymbol{D}^{-s} u, \boldsymbol{D}^{-s*} u$ are understood in the usual sense, namely, the R-L integrals of u. Also, we could derive other variants or generalizations of Theorem 2, such as:

$$f = p\boldsymbol{D}^{-s} u + q\boldsymbol{D}^{s*} u, \tag{18}$$

where $p, q \in \mathbb{R}$ are suitable numbers (for example, $p > 0, q > 0$).
In the following, we focus only on Theorem 2.

5.1 Several lemmas

Toward the proof of Theorem 2 several necessary lemmas are first established. First, we point out that the proof for the case $0 < s < 1/2$ and the case $-1/2 < s \leq 0$ in Theorem 2 have no essential differences, because the sign change of s results only in the exchange of notations of derivatives and integrals. For simplicity and preservation of generality, we establish the proof of Theorem 2 only for the case $0 < s < 1/2$, and the proof for the case $-1/2 < s \leq 0$ follows analogously without essential obstacle.

Lemma 1.*Given* $0 < s < 1/2$, *then the sets* $M = \{w : w = \boldsymbol{D}^{-s}\psi + \boldsymbol{D}^{s*}\psi,\ \forall \psi \in C_0^\infty(\mathbb{R})\}$ *and* $\widetilde{M} = \{w : w = \boldsymbol{D}^{-s}\psi + \boldsymbol{D}^{s}\psi,\ \forall \psi \in C_0^\infty(\mathbb{R})\}$ *are dense in* $L^2(\mathbb{R})$, *respectively.*

*Proof.*The proof is established by invoking the theorem in 5. First we check the conditions for the application of the theorem, and consider the set M.

Since $0 < s < 1/2$, for $\psi \in C_0^\infty(\mathbb{R})$, it is true that $\boldsymbol{D}^{-s}\psi \in L^2(\mathbb{R})$ by applying Theorem 5.3 ([11], p. 103). By Property 4 we have $\boldsymbol{D}^{s*}\psi \in L^p(\mathbb{R}), p \geq 1$. Thus $M \subset L^2(\mathbb{R})$. Then it could be directly verified that M is a subspace of $L^2(\mathbb{R})$ by checking closeness of addition and scalar multiplication.

Consequently, all conditions are met to allow the utilization of 5. Suppose now $g \in L^2(\mathbb{R})$ satisfies $(g, w) = 0\ \forall w \in M$, then the set M is dense in $L^2(\mathbb{R})$ and proved if this last equation implies that $g = 0$, namely g is a zero function on $\mathbb{R}$.

Pick a non-zero function $\varphi \in C_0^\infty(\mathbb{R})$, which is possible. Then Plancherel Theorem in 3 guarantees that $\widehat{\varphi}$ is also a non-zero function. Taking into account the continuity of $\widehat{\varphi}$, we know $\widehat{\varphi}(\xi) \neq 0$ on a certain non-empty open $(a, b) \subset \mathbb{R}$.

Now set $v(x) = \varphi(\varepsilon x)$, with any fixed $\varepsilon > 0$. It is clear that $v \in C_0^\infty(\mathbb{R})$. Using Property 2 and Property 5 (Fourier transform properties) for $w = \boldsymbol{D}^{-s}v + \boldsymbol{D}^{s*}v$ gives

$$\widehat{w}(\xi) = \widehat{v}(\xi)\left((2\pi i\xi)^{-s} + (-2\pi i\xi)^{s}\right) = \frac{1}{\xi}\widehat{\psi}(\frac{\xi}{\varepsilon})\left((2\pi i\xi)^{-s} + (-2\pi i\xi)^{s}\right). \tag{19}$$

Following the practice stated in Remark 2, it is easy to see that $(2\pi i\xi)^{-s} + (-2\pi i\xi)^{s} \neq 0$ a.e. on $\mathbb{R}$ by observing that $|(2\pi i\xi)^{-s}| \neq |(-2\pi i\xi)^{s}|$ a.e.. This implies $\widehat{w} \neq 0$ a.e. on $(\varepsilon a, \varepsilon b)$ since $\widehat{\psi}(\frac{\xi}{\varepsilon}) \neq 0$ a.e. on $(a\varepsilon, b\varepsilon)$.

Now for $y \in \mathbb{R}$, we set the cross-correlation function

$$G(-y) := \int_{\mathbb{R}} g(x)w(x-y)\,\mathrm{d}x = \int_{\mathbb{R}} g(x)\,\tau_y w(x)\,\mathrm{d}x.$$

Using Property 3 and 6 gives $\tau_y w \in M$. Hence $G(-y) = 0\ \forall y \in \mathbb{R}$ by our assumption. Therefore by Plancherel Theorem $\widehat{G} = 0$. Notice $G(y) = g(-x) * w(x)$. Since $g, w \in L^2(\mathbb{R})$, using convolution theorem ([12], Theorem 1.2, p.12) gives $\widehat{G} = \widehat{g(-x)} \cdot \widehat{w(x)} = 0$.

Since $\overline{\widehat{w}} \neq 0$ a.e. on $(\varepsilon a, \varepsilon b)$, it is concluded that $\widehat{g(-x)} = 0$ a.e. on $(\varepsilon a, \varepsilon b)$. Notice $\varepsilon > 0$ is arbitrary, i.e. $\widehat{g} = 0$ a.e. on $\mathbb{R}$. In view of Plancherel Theorem in 3 we see that $g(x) = 0$ a.e. on $\mathbb{R}$, which confirms that M is dense in $L^2(\mathbb{R})$.

Finally, the same conclusion is true for $\widetilde{M}$ by repeating the similar foregoing calculations.

Lemma 2. *Given* $0 < s < 1/2$, $\psi \in C_0^\infty(\mathbb{R})$, *then*

$$(\boldsymbol{D}^{-s}\psi, \boldsymbol{D}^{s*}\psi) = \|\psi\|^2_{L^2(\mathbb{R})}, \quad (\boldsymbol{D}^{-s}\psi, \boldsymbol{D}^{s}\psi) = \cos(s\pi)\|\psi\|^2_{L^2(\mathbb{R})}. \tag{20}$$

Proof. Consider the first equality.

From the proof of Lemma 1, we already know, for $0 < s < 1/2$, $\boldsymbol{D}^{-s}\psi \in L^2(\mathbb{R})$ and $\boldsymbol{D}^{s*}\psi, \boldsymbol{D}^{s}\psi \in L^p(\mathbb{R}), p \geq 1$. This allows us to use Parseval Formula 4 and Fourier Transform Properties 2, 5 to obtain

$$\begin{aligned}(\boldsymbol{D}^{-s}\psi, \boldsymbol{D}^{s*}\psi) &= (\widehat{\boldsymbol{D}^{-s}\psi}, \overline{\widehat{\boldsymbol{D}^{s*}\psi}}) \\ &= (\mathscr{F}(\boldsymbol{D}^{-s}\psi), \overline{\mathscr{F}(\boldsymbol{D}^{s*}\psi)}) \\ &= ((2\pi i\xi)^{-s}\widehat{\psi}, \overline{(-2\pi i\xi)^{s}\widehat{\psi}}).\end{aligned} \tag{21}$$

Invoking Remark 3.1, we find

$$((2\pi i\xi)^{-s}\widehat{\psi}, \overline{(-2\pi i\xi)^{s}\widehat{\psi}}) = \|\psi\|^2_{L^2(\mathbb{R})}. \tag{22}$$

For the second equality, similarly we have

$$\begin{aligned}(\boldsymbol{D}^{-s}\psi, \boldsymbol{D}^{s}\psi) &= (\widehat{\boldsymbol{D}^{-s}\psi}, \overline{\widehat{\boldsymbol{D}^{s}\psi}}) \\ &= (\mathscr{F}(\boldsymbol{D}^{-s}\psi), \overline{\mathscr{F}(\boldsymbol{D}^{s}\psi)}) \\ &= ((2\pi i\xi)^{-s}\widehat{\psi}, \overline{(2\pi i\xi)^{s}\widehat{\psi}}).\end{aligned} \tag{23}$$

By invoking Remark 3.1, this becomes

$$\begin{aligned}&((2\pi i\xi)^{-s}\widehat{\psi}, \overline{(2\pi i\xi)^{s}\widehat{\psi}}) \\ &= e^{is\pi}\int_{-\infty}^{0}(-2\pi\xi)^{-s}\widehat{\psi}\cdot(-2\pi\xi)^{s}\overline{\widehat{\psi}}\,\mathrm{d}\xi + e^{-is\pi}\int_{-\infty}^{0}(2\pi\xi)^{-s}\widehat{\psi}\cdot(2\pi\xi)^{s}\overline{\widehat{\psi}}\,\mathrm{d}\xi \\ &= e^{is\pi}\int_{-\infty}^{0}|\widehat{\psi}|^2\,\mathrm{d}\xi + e^{-is\pi}\int_{-\infty}^{0}|\widehat{\psi}|^2\,\mathrm{d}\xi \\ &= \cos(s\pi)\|\psi\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{24}$$

In the last step we have used the fact that $\overline{\widehat{\psi}(-\xi)} = \widehat{\psi}(\xi)$ for real valued function ψ.

Lemma 3. *Given $0 < s < 1/2$, $\psi \in C_0^\infty(\mathbb{R})$, then*

$$\|\boldsymbol{D}^{-s}\psi + \boldsymbol{D}^{s*}\psi\|^2_{L^2(\mathbb{R})} = \|\boldsymbol{D}^{-s}\psi\|^2_{L^2(\mathbb{R})} + \|\boldsymbol{D}^{s*}\psi\|^2_{L^2(\mathbb{R})} + 2\|\psi\|^2_{L^2(\mathbb{R})}, \tag{25}$$

$$\|\boldsymbol{D}^{-s}\psi + \boldsymbol{D}^{s}\psi\|^2_{L^2(\mathbb{R})} = \|\boldsymbol{D}^{-s}\psi\|^2_{L^2(\mathbb{R})} + \|\boldsymbol{D}^{s}\psi\|^2_{L^2(\mathbb{R})} + 2\cos(s\pi)\|\psi\|^2_{L^2(\mathbb{R})}. \tag{26}$$

Proof. This is a direct consequence of Lemma 2.

Now we are in the position to prove Theorem 2 for $0 < s < 1/2$.

Proof. The proof is shown only for Case (1) in Theorem 2, since Case (2) could be established analogously without obstacle. It will be convenient to keep in mind the fact of Theorem 1 in the following.
Step 1.
Given $0 < s < 1/2, f \in L^2(\mathbb{R})$. By Lemma 1, there is a sequence $\{\psi_n\} \subset C_0^\infty(\mathbb{R})$ so that

$$f = \lim_{n\to\infty} (\boldsymbol{D}^{-s}\psi_n + \boldsymbol{D}^{s*}\psi_n), \quad \text{in } L^2(\mathbb{R}). \tag{27}$$

It is clear that $\{\boldsymbol{D}^{-s}\psi_n + \boldsymbol{D}^{s*}\psi_n\}$ is a Cauchy sequence in $L^2(\mathbb{R})$. Equation (25) implies that $\{\boldsymbol{D}^{-s}\psi_n\}, \{\boldsymbol{D}^{s*}\psi_n\}$ and $\{\psi_n\}$ are Cauchy sequences separately because each term on the right hand side has a positive coefficient, namely, $1, 1, 2$.

Denote limit function $u = \lim_{n\to\infty}\psi_n$ and by invoking Corollary 1, we know $u \in \widehat{H}^s(\mathbb{R})$ and $\lim_{n\to\infty}\boldsymbol{D}^{s*}\psi_n = \boldsymbol{D}^{s*}u$ in the weak fractional derivative sense.

Then denote limit function $v = \lim_{n\to\infty}\boldsymbol{D}^{-s}\psi_n$, and we claim $v = \boldsymbol{D}^{-s}u$, where $\boldsymbol{D}^{-s}u$ is to be understood in usual sense, namely, R-L integral of u (which is well-defined). To see this, the condition $0 < s < 1/2$ allows to use Property 1 to obtain

$$(\boldsymbol{D}^{-s}u, \phi) = (u, \boldsymbol{D}^{-s*}\phi), \quad \forall \phi \in C_0^\infty(\mathbb{R}). \tag{28}$$

On the other hand,

$$(v, \phi) = \lim_{n\to\infty}(\boldsymbol{D}^{-s}\psi_n, \phi) = \lim_{n\to\infty}(\psi_n, \boldsymbol{D}^{-s*}\phi) = (u, \boldsymbol{D}^{-s*}\phi). \tag{29}$$

Thus, $(\boldsymbol{D}^{-s}u - v, \phi) = 0, \forall \phi \in C_0^\infty(\mathbb{R})$, which deduces $v = \boldsymbol{D}^{-s}u$ a.e.. Therefore, $f = \boldsymbol{D}^{-s}u + \boldsymbol{D}^{s*}u$.
Step 2.
Uniqueness of u is from the norm estimate in Equation (25). Suppose that $f = \boldsymbol{D}^{-s}u_1 + \boldsymbol{D}^{s*}u_1 = \boldsymbol{D}^{-s}u_2 + \boldsymbol{D}^{s*}u_2$, $u_1, u_2 \in \widehat{H}^s(\mathbb{R})$, then it is deduced that $\|u_1 - u_2\|^2_{L^2(\mathbb{R})} = 0$. Thus $u_1 = u_2$ in $L^2(\mathbb{R})$.
Step 3.
Now suppose $f \in \widehat{H}^t(\mathbb{R})$, where $0 < t$, we intend to show $u \in \widehat{H}^{s+t}(\mathbb{R})$. Suppose $t \le 2s$. First note the fact that $\boldsymbol{D}^{-s}u \in \widehat{H}^{2s}(\mathbb{R})$. Using Property 1, which is permissible here, gives

$$(\boldsymbol{D}^{-s}u, \boldsymbol{D}^{2s*}\phi) = (u, \boldsymbol{D}^{-s*}\boldsymbol{D}^{2s*}\phi) = (u, \boldsymbol{D}^{s*}\phi) = (\boldsymbol{D}^{s}u, \phi), \forall \phi \in C_0^\infty(\mathbb{R}). \tag{30}$$

The last equality above was by $u \in \widetilde{W}^s_L(\mathbb{R})$. Therefore, by definition, $\boldsymbol{D}^{-s}u \in \widetilde{W}^{2s}_L(\mathbb{R})$, namely $\boldsymbol{D}^{-s}u \in \widehat{H}^{2s}(\mathbb{R})$ by Theorem 1. Then $\boldsymbol{D}^{s*}u = f - \boldsymbol{D}^{-s}u \in \widetilde{W}^t_L(\mathbb{R})$ by noticing our assumption $t \le 2s$ and the fact that $\widehat{H}^{2s}(\mathbb{R}) \subset \widehat{H}^t(\mathbb{R})$. Another use of Theorem 1 concludes $\boldsymbol{D}^{s*}u \in \widetilde{W}^t_R(\mathbb{R})$, which implies by definition of weak fractional derivative that

$$(\boldsymbol{D}^t(\boldsymbol{D}^{s*}u), \phi) = (\boldsymbol{D}^{s*}u, \boldsymbol{D}^t\phi), \quad \forall \phi \in C_0^\infty(\mathbb{R}). \tag{31}$$

Observe that

$$(\boldsymbol{D}^{s*}u, \boldsymbol{D}^t\phi) = \lim_{n\to\infty}(\boldsymbol{D}^{s*}\psi_n, \boldsymbol{D}^t\phi) = \lim_{n\to\infty}(\psi_n, \boldsymbol{D}^{s+t}\phi), \quad \forall \phi \in C_0^\infty(\mathbb{R}), \tag{32}$$

which concludes $u \in \widetilde{W}^{s+t}_R(\mathbb{R})$, namely $u \in \widehat{H}^{s+t}(\mathbb{R})$. Thus we actually raise the regularity of u to $\widehat{H}^{s+t}(\mathbb{R})$ from $\widehat{H}^s(\mathbb{R})$.

For $t > 2s$, we just need to rewrite $t = 2s + \textbf{\textit{Residue}}$, and repeat the above-mentioned procedure to gradually raise the regularity of u from $\widehat{H}^s(\mathbb{R})$ to $\widehat{H}^{2s}(\mathbb{R})$, and repeat the same procedure again for ***Residue***, all the way to $\widehat{H}^{s+t}(\mathbb{R})$.
Step 4.
Now suppose $u \in \widehat{H}^{s+t}(\mathbb{R})$, where $0 < t$, we intend to show $f \in \widehat{H}^t(\mathbb{R})$. By definition, it is easy to verify that $f - \boldsymbol{D}^{-s}u = \boldsymbol{D}^{s*}u \in \widetilde{W}^t_R(\mathbb{R})$, and $\boldsymbol{D}^{-s}u \in \widetilde{W}^{2s+t}_L(\mathbb{R})$. Hence, $\boldsymbol{D}^{-s}u + \boldsymbol{D}^{s*}u = f \in \widehat{H}^t(\mathbb{R})$ using Theorem 1 and the fact that $\widehat{H}^{t_1}(\mathbb{R}) \subset \widehat{H}^{t_2}(\mathbb{R})$ for $t_1 \ge t_2$. This completes the proof for the case $0 < s < 1/2$.

The case $-1/2 < s \le 0$ could be established analogously to complete the whole proof of Theorem 2.

Utilizing Theorem 2, Fourier transforms of the functions of $L^2(\mathbb{R})$ can be characterized as follows:

Corollary 2.*(1). Given $f \in L^2(\mathbb{R})$, $-1/2 < s < 1/2$, there exists a unique $u \in \widehat{H}^s(\mathbb{R})$ satisfying*

$$\widehat{f}(\xi) = \left((2\pi i\xi)^{-s} + (-2\pi i\xi)^s\right)\widehat{u}(\xi). \tag{33}$$

(2). Given $f \in L^2(\mathbb{R})$, $-1/2 < s < 1/2$, there exists a unique $u \in \widehat{H}^s(\mathbb{R})$ satisfying

$$\widehat{f}(\xi) = \left((2\pi i\xi)^{-s} + (2\pi i\xi)^s\right)\widehat{u}(\xi). \tag{34}$$

The complex power functions are understood as $(\pm i\xi)^\sigma = |\xi|^\sigma e^{\pm\sigma\pi i\cdot \mathrm{sign}(\xi)/2}$.

*Proof.*Again, the proof is shown only for the case $0 < s < 1/2$ and part (1), the case $-1/2 < s \leq 0$ and part (2) could be established analogously without essential difference.

Fix $f \in L^2(\mathbb{R})$, from Theorem 2 and the proof, there is a unique $u \in \widehat{H}^s(\mathbb{R})$ and a Cauchy sequence $\{\psi_n\} \subset C_0^\infty(\mathbb{R})$ such that

$$f = \boldsymbol{D}^{-s}u + \boldsymbol{D}^{s*}u, \tag{35}$$

and

$$\lim_{n\to\infty} \boldsymbol{D}^{-s}\psi_n = \boldsymbol{D}^{-s}u, \quad \lim_{n\to\infty} \boldsymbol{D}^{s*}\psi_n = \boldsymbol{D}^{s*}u, \quad \lim_{n\to\infty} \psi_n = u, \quad \text{in } L^2(\mathbb{R}). \tag{36}$$

Then we know

$$\widehat{f} = \widehat{\boldsymbol{D}^{-s}u} + \widehat{\boldsymbol{D}^{s*}u}, \text{and } \{\widehat{\boldsymbol{D}^{-s}\psi_n}\}, \{\widehat{\boldsymbol{D}^{s*}\psi_n}\}, \{\widehat{\psi_n}\} \text{ converge in } L^2(\mathbb{R}). \tag{37}$$

On one hand, there is a subsequence $\{\widehat{\psi_{n_i}}\}$ that converges pointwise almost everywhere to $\widehat{u}$ (6). Consequently, $(2\pi i\xi)^{-s}\widehat{\psi_{n_i}}$ converges pointwise to $(2\pi i\xi)^{-s}\widehat{u}$ a.e. and $(-2\pi i\xi)^s\widehat{\psi_{n_i}}$ converges pointwise to $(-2\pi i\xi)^{-s}\widehat{u}$ a.e. On the other hand, in $L^2(\mathbb{R})$

$$\begin{aligned} \widehat{\boldsymbol{D}^{-s}u} &= \lim_{n\to\infty} \widehat{\boldsymbol{D}^{-s}\psi_{n_i}} = \lim_{n\to\infty} (2\pi i\xi)^{-s}\widehat{\psi_{n_i}}, \\ \widehat{\boldsymbol{D}^{s*}u} &= \lim_{n\to\infty} \widehat{\boldsymbol{D}^{s*}\psi_{n_i}} = \lim_{n\to\infty} (-2\pi i\xi)^{s}\widehat{\psi_{n_i}}. \end{aligned} \tag{38}$$

Therefore by another use of 6, we know that there is a subsequence of $\{(2\pi i\xi)^{-s}\widehat{\psi_{n_i}}\}$ converging pointwise almost everywhere to $\widehat{\boldsymbol{D}^{-s}u}$ and that there is a subsequence of $\{(-2\pi i\xi)^s\widehat{\psi_{n_i}}\}$ converging pointwise almost everywhere to $\widehat{\boldsymbol{D}^{s*}u}$. Hence

$$\widehat{\boldsymbol{D}^{-s}u} = (2\pi i\xi)^{-s}\widehat{u} \text{ a.e.}, \quad \widehat{\boldsymbol{D}^{s*}u} = (-2\pi i\xi)^{-s}\widehat{u} \text{ a.e.}. \tag{39}$$

Thus

$$\widehat{f}(\xi) = \left((2\pi i\xi)^{-s} + (-2\pi i\xi)^s\right)\widehat{u}(\xi), \text{ a.e.}. \tag{40}$$

This completes the whole proof.

6 Conclusion

We have constructed a bunch of maps $T_s : f \mapsto u$ from $L^2(\mathbb{R})$ to $\widehat{H}^{|s|}(\mathbb{R})$ in Theorem 2, which depends on s. The main questions are as follows " Is the map T_s onto? Can we extend the results in Theorem 2 to the case of $|s| = 1/2$?", "what are the algebraic structures underlying these operators? what is the potential relationship between these decompositions and the classical Fourier decompositions?". Further discussions and applications are explored in separate work.

Acknowledgements

The author expresses gratitude to the reviewers for their careful reading and helpful comments.

A Several definitions and theorems

Definition 5(Sobolev Spaces Via Fourier Transform). *Let* $\mu \geq 0$. *Define*

$$\widehat{H}^{\mu}(\mathbb{R}) = \left\{ w \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (1 + |2\pi\xi|^{2\mu}) |\widehat{w}(\xi)|^2 \,\mathrm{d}\xi < \infty \right\}, \tag{A.1}$$

where $\widehat{w}$ *is the Plancherel transform defined in Theorem 3. The space is endowed with semi-morn*

$$|u|_{\widehat{H}^{\mu}(\mathbb{R})} := \| |2\pi\xi|^{\mu} \widehat{u} \|_{L^2(\mathbb{R})}, \tag{A.2}$$

and norm

$$\|u\|_{\widehat{H}^{\mu}(\mathbb{R})} := \left(\|u\|^2_{L^2(\mathbb{R})} + |u|^2_{\widehat{H}^{\mu}(\mathbb{R})} \right)^{1/2}. \tag{A.3}$$

Definition 6(Fourier Transform). *Given a function* $f : \mathbb{R} \to \mathbb{R}$, *the Fourier Transform of* f *is defined as*

$$\mathscr{F}(f)(\xi) := \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x) \,\mathrm{d}x \quad \forall \xi \in \mathbb{R}.$$

Theorem 3(Plancherel Theorem, [13], p. 187).
One can associate to each $f \in L^2(\mathbb{R})$ *a function* $\widehat{f} \in L^2(\mathbb{R})$ *such that each of the following is valid:*

- *If* $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, *then* $\widehat{f}$ *is the defined Fourier transform of* f *in Definition 6.*
- *For every* $f \in L^2(\mathbb{R})$, $\|f\|_2 = \|\widehat{f}\|_2$.
- *The mapping* $f \to \widehat{f}$ *is a Hilbert space isomorphism of* $L^2(\mathbb{R})$ *onto* $L^2(\mathbb{R})$.

Theorem 4([14], p. 189). *Assume* $u, v \in L^2(\mathbb{R}^n)$. *Then*

- $\int_{\mathbb{R}^n} u\bar{v} = \int_{\mathbb{R}^n} \widehat{u}\overline{\widehat{v}}$.
- $u = (\widehat{u})^{\vee}$.

Theorem 5([15], Theorem 4.3-2, p. 191). *Assume that* $(Y, (\cdot,\cdot))$ *is a Hilbert space and* X *is a non-empty subspace of* Y. *Then* X *is dense in* Y, *namely* $\overline{X} = Y$ *if and only if the following is true: if* $y \in Y$ *satisfies* $(x, y) = 0$ *for all* $x \in X$, *then* $y = 0$.

Theorem 6([16]). *If* $1 \leq p \leq \infty$ *and if* $\{f_n\}$ *is a Cauchy sequence in* $L^p(\mathbb{R})$ *with limit* $f(x)$, *then* $\{f_n\}$ *has a subsequence which converges pointwise almost everywhere to* $f(x)$.

References

[1] P. E. T. Jorgensen, *Harmonic analysis*, Smooth and non-smooth, Published for the Conference Board of the Mathematical Sciences, CBMS Regional Conference Series in Mathematics, Vol. **128**, American Mathematical Society, Providence, RI, 2018.
[2] J. Duoandikoetxea, *Fourier analysis*, Translated and revised from the 1995 Spanish original by David Cruz-Uribe, Graduate Studies in Mathematics, Vol. **29**, American Mathematical Society, Providence, RI, 2001.
[3] D. W. Kammler, *A first course in Fourier analysis*, 2nd Edition, Cambridge University Press, Cambridge, 2007.
[4] M. E. Hernández-Hernández and V. N. Kolokoltsov, On the solution of two-sided fractional ordinary differential equations of Caputo type, *Fract. Calc. Appl. Anal.* **19** (6), 1393–1413 (2016).
[5] V. Ginting and Y. Li, On the fractional diffusion-advection-reaction equation in $\mathbb{R}$, *Fract. Calc. Appl. Anal.* **22**(4), 1039–1062, (2019).
[6] M. Japundžić and D. Rajter-Ćirić, Reaction-advection-diffusion equations with space fractional derivatives and variable coefficients on infinite domain, *Fract. Calc. Appl. Anal.* **18**(4), 911–950, (2015).
[7] R. Stern, F. Effenberger, H. Fichtner and T. Schäfer, The space-fractional diffusion-advection equation: analytical solutions and critical assessment of numerical solutions, *Fract. Calc. Appl. Anal.* **17**(1), 171–190, (2014).
[8] Y. Li, Symmetric decompositions of $f \in L^2(\mathbb{R})$ via fractional Riemann-Liouville operators, arXiv:1807.01847, (2018).
[9] Y. Li, On fractional differential equations and related questions, Ph.D. thesis, ProQuest Dissertations and Theses, 2019.
[10] Y. Li, Characterizations of fractional Sobolev spaces from the perspective of Riemann-Liouville operators, *J. Fract. Calc. Appl.* **11**(2), 102–110 (2020).

[11] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives*, Gordon and Breach Science Publishers, Yverdon, 1993.
[12] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Vol. **204** of North-Holland Mathematics Studies, Elsevier Science B.V., Amsterdam, 2006.
[13] W. Rudin, *Real and complex analysis*, 3rd Edition, McGraw-Hill Book Co., New York, 1987.
[14] L. C. Evans, *Partial differential equations*, 2nd Edition, Vol. **19** of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2010.
[15] L. Tartar, *An introduction to Sobolev spaces and interpolation spaces*, Vol. **3** of Lecture Notes of the Unione Matematica Italiana, Springer, Berlin; UMI, Bologna, 2007.
[16] W. Rudin, *Functional analysis*, 2nd Edition, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.