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A New Integrable Equation Constructed via Combining the Recursion Operator of the Calogero-Bogoyavlenskii-Schiff (CBS) Equation and its Inverse Operator

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Abstract: In this work we construct a new integrable equation via combining the recursion operator of the Calogero-Bogoyavlenskii-Schiff (CBS) equation and its inverse recursion operator. We show that this equation nicely passes the Painlevé property to emphasize its complete integrability. We formally derive multiple soliton solutions by using the simplified Hirota's direct method. We also use other techniques to obtain more solutions of distinct physical structures.

Keywords: Calogero-Bogoyavlenskii-Schiff equation; recursion operator; multiple soliton solutions, Painlevé analysis.

1 Introduction

The recursion operator, an integro-differential operator which connects two consecutive symmetries [1]- [9], plays an important role in constructing integrable systems. It also supports constructing families of symmetries for equations, which admit recursion operators [3], and in giving insight into the features of the integrable equations [3]. The recursion operator of a nonlinear equation indicates that this equation has infinitely many higher-order symmetries, which is a strong feature of its complete integrability as discovered by Olver [4].

The hereditary symmetry $\Phi(u(x,t))$ is a recursion operator of the following hierarchy of evolution equations

$$u_t + \Phi(u)u_x = 0, \tag{1}$$

which gives rise to a variety of (1+1)-dimensional equations. The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2}$$

is obtained by using the KdV recursion operator $\Phi(u)$

$$\Phi(u) = \partial_x^2 + 4u + 2u_x\partial_x^{-1}, \tag{3}$$

where ∂_x and ∂_x^{-1} denote the total derivative and its integration operator with respect to x respectively.

However, the integrable Calogero-Bogoyavlenskii-Schiff (CBS) equation

$$u_t + u_{xxy} + 4uu_y + 2u_x\partial_x^{-1}(u_y) = 0, \tag{4}$$

is obtained by using the following hierarchy of evolution equations

$$u_t + \Phi(u)u_y = 0, \tag{5}$$

where $\Phi(u)$ has the same form (3) as that for the KdV equation with argument x. Note that u_x in (3) is replaced by u_y to give a (2+1)-dimensional CBS equation. In a like manner, we can replace u_x in (3) by $u_y + u_z$ to develop (3+1)-dimensional CBS equation. The CBS equation (4) was studied thoroughly in the literature due to its variety of solutions and significant scientific features.

Verosky [8] extended the Olver work in [4], and admitted the use of the negative direction to obtain a sequence of equations of increasingly negative orders. Verosky [8] elaborated that the hierarchy of evolution equation (1)

$$u_t = -\Phi(u_x), \tag{6}$$

can be used in the negative order hierarchy in the form

$$u_t = -\Phi^{-1}u_x, \tag{7}$$

where the power of Φ goes to the opposite direction [7–9] In other words, the negative order equation can be reported

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as

$$\Phi(u_t) = -u_x. \tag{8}$$

The goals of this work are two fold. We aim first to combine the sense of the CBS recursion operator (3) and the sense of the negative-order recursion operator (8) to establish a new integrable equation. We will show that this newly developed equation nicely passes the Painlevé property to emphasize its complete integrability. Second, we plan to derive multiple soliton solutions for the newly developed integrable equation. More travelling wave solutions of distinct physical structures will be determined as well.

2 Formulation of the new integrable combined equation

We will combine the sense of the CBS recursion operator (1), used earlier for the derivation of the KdV and the CBS equations, and the sense of the negative-order recursion operator (3) to construct a new integrable equation. In other words, we introduce

$$v_t + \Phi(v(x,y,t))v_y + \Phi(v(x,y,t))v_t = 0, \tag{9}$$

or equivalently

$$v_t + \Phi(v(x,y,t))\,(v_y + v_t) = 0. \tag{10}$$

Using the recursion operator as defined in (3) gives the combined CBS equation with it negative-order form as

$$v_t + v_{xxy} + v_{xxt} + 4vv_y + 4vv_t + 2v_x\partial_x^{-1}(v_y + v_t) = 0, \tag{11}$$

which will be termed the combined CBS-negative-order CBS equation (CBS-nCBS).

To eliminate the integral operator, the potential

$$v(x,y,t) = u_x(x,y,t), \tag{12}$$

will transform (11) to

$$u_{xt} + u_{xxxy} + u_{xxxt} + 4u_x(u_{xy} + u_{xt}) + 2u_{xx}(u_y + u_t) = 0. \tag{13}$$

The combined CBS-nCBS equation includes four nonlinear terms and two linear dispersive terms. The complete integrability of the combined CBS-nCBS equation (13) will be investigated in the following section, by showing it nicely passes the Painlevé test.

3 The Painlevé test

To prove the integrability of the combined CBS-nCBS equation (13), we use the Painlevé test in the sense of the Weiss–Tabor–Carnevale (WTC) method [5]. The Painlev e analysis is a powerful method for identifying the integrable properties of nonlinear partial differential equations [1]- [25]. The WTC method [5] and Kruskal's simplification method are the most widely tools used to examine the Painlevé property [5]. The WTC-Kruskal algorithm [5] is employed in three steps:
(i) leading-order analysis,
(ii) finding resonances, and
(iii) determining compatibility conditions. More details about these three steps are given in Refs. [5]- [10].

The combined CBS–nCBS equation (13) is said to possess the Painlevé property if its solutions are single-valued about arbitrary non characteristic, movable singularity manifolds. In other words, this means that its solutions can be expressed as Laurent series in the form

$$u(x,y,t) = \sum_{j=0}^{\infty} u_j(x,y,t)\,\phi^{j+\alpha}, \tag{14}$$

with a sufficient number of arbitrary functions among $u_j(x,y,t)$ in addition to $\phi(x,y,t)$. The Painlevé property is characterized by the fact that α is a negative integer and all resonances occur at positive integer values of j and are compatible.

First, the leading order α and the leading coefficient $u_0(x,y,t)$ should be determined. To obtain this, we substitute

$$u(x,y,t) = u_0(x,y,t)\phi^{\alpha}, \tag{15}$$

into Eq. (13). Balancing the nonlinear and dispersive terms, we get

$$\alpha = -1,\ u_0(x,y,t) = 2\,\phi_x. \tag{16}$$

It is well known that the resonance at $j = -1$ corresponds to the arbitrariness of the singular manifold $\phi(x,y,t) = 0$.

The next step is to check the existence of a sufficient number of arbitrary functions and to find the resonance points. To achieve this goal we insert the Laurent series

$$u(x,t) = u_0\phi^{-1} + u_j\phi^{j-1}, j \geq 1, \tag{17}$$

into Eq. (13), together with (16), to obtain the following characteristic equation for resonances

$$(j+1)(j-1)(j-4)(j-6) = 0. \tag{18}$$

The solutions (resonances j) of the characteristic equation occur at j =-1, 1, 4, and 6. Recall that the resonance at $j = -1$ corresponds to the to the arbitrariness of the singular manifold $\phi(x,y,t) = 0$.

To check the existence of sufficient number of arbitrary functions at the other resonances at $j = 1,4,6$, we substitute the Laurent series

$$u = \sum_{j=0}^{6} u_j\,\phi^{j-1}. \tag{19}$$

in (13), and collect the coefficients of ϕ^{-n}, where:
(i) From the coefficients of ϕ^{-5}, we find $u_0(x,t) = 2\,\phi_x$, which corresponds to the resonance $j = -1$.

(ii) From the coefficients of ϕ^{-4},ϕ^{-1}, and ϕ^{1}, we find equations that do not include u_1,u_4, and u_6 respectively. This absence of u_1,u_4, and u_6 proves that u_1,u_4, and u_6, which correspond to the resonances $j=1,4,6$ are arbitrary constants.
(iii) Otherwise, we find explicit expressions for u_2,u_3,u_5.

This shows that the combined CBS-nCBS equation (13) justifies the three criteria presented before. This leads to the conclusion that the combined CBS-nCBS equation (13) admits sufficient number of arbitrary functions and hence integrable in the sense of possessing the Painlevé property.

It is worth noting that the CBS equation gives only three resonances, namely $-1,4,6$, whereas the combined CBS-nCBS equation (13) gives four resonances, namely $-1,1,4,6$. In addition, we will show later that the dispersion relation of the interaction of solitons of the combined CBS-nCBS equation (13) is not the same dispersion relation of the CBS equation.

4 Multiple soliton solutions

It is well known that integrable equations describe some nonlinear phenomena in science, technology, and engineering, such as plasmas, solid state materials, fluid dynamics, and many others. Among the intriguing features of these equations are the multiple soliton solutions and an infinite number of conserved quantities they possess. In this section we will concern ourselves with finding multiple soliton solutions for this new equation.

In the previous section, we showed that the combined KdV-nKdV equation (13)

$$u_{xt}+u_{xxxy}+u_{xxxt}+4u_x(u_{xy}+u_{xt})+2u_{xx}(u_y+u_t)=0, \tag{20}$$

is completely integrable in the sense of possessing the Painlevé property.

Substituting

$$u(x,y,t)=e^{\theta_i}=e^{k_ix+r_iy-c_it}, \tag{21}$$

into the linear terms of (20) gives the dispersion relation as

$$c_i=\frac{k_i^2r_i}{1+k_i^2}. \tag{22}$$

Consequently, the dispersion variable becomes

$$\theta_i=k_ix+r_iy-\frac{k_i^2r_i}{1+k_i^2}t. \tag{23}$$

We next use the transformation

$$u(x,y,t)=R\,(\ln f(x,y,t))_x, \tag{24}$$

where R is a constant that will be determined. The auxiliary function $f(x,y,t)$ for the single soliton solution is given by

$$f(x,y,t)=1+e^{k_1x+r_1y-\frac{k_1^2r_1}{1+k_1^2}t}. \tag{25}$$

Substituting (24) and (25) in (20), and solving for R, we find that single soliton solution exists only if

$$R=2. \tag{26}$$

which gives the following soliton solution

$$u(x,y,t)=\frac{2k_1\,e^{k_1x+r_1y-\frac{k_1^2r_1}{1+k_1^2}t}}{1+e^{k_1x+r_1y-\frac{k_1^2r_1}{1+k_1^2}t}}. \tag{27}$$

The single soliton solution $v(x,y,t)$ is obtained by using the potential (12).

For the two soliton solutions, we set the auxiliary function $f(x,t)$ as

$$f(x,y,t)=1+e^{\theta_1}+e^{\theta_2}+a_{12}e^{\theta_1+\theta_2}, \tag{28}$$

where the wave variables $\theta_i, i=1,2$ are given above in (23), and a_{12} is the phase shift that will be determined. Using (28) in (20), and solving, we find that the phase shift is given by the Hirota type as

$$a_{12}=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{29}$$

and this can be generalized to

$$a_{ij}=\frac{(k_i-k_j)^2}{(k_i+k_j)^2}, 1\le i<j\le 3, \tag{30}$$

which is the same phase shift of the standard CBS equation. The obtained results will give the two-soliton solutions for (20). It is to be noted that the phase shifts a_{ij} do not depend on the coefficients r_i of the space variable y.

For the three soliton solutions, the auxiliary function is given by

$$f(x,y,t)=1+e^{\theta_1}+e^{\theta_2}+e^{\theta_3}+a_{12}e^{\theta_1+\theta_2}+a_{13}e^{\theta_1+\theta_3}+a_{23}e^{\theta_2+\theta_3}+b_{123}e^{\theta_1+\theta_2+\theta_3}, \tag{31}$$

where the wave variable θ_i is given above in (23), and a_{12} is the phase shift. Proceeding as before, we find

$$b_{123}=a_{12}a_{23}a_{13}, \tag{32}$$

and this gives three soliton solutions (20).

It is interesting to point out that the dispersion relations of the standard CBS equation and the combined CBS-nCBS equation (13) are derived as k_i^3 and $\frac{k_i^2r_i}{1+k_i^2}$ respectively, for $1\le i\le N, N$ is finite. However, the phase shifts remain the same for both the CBS and the combined CBS-nCBS equations.

In the forth coming sections, we will use a set of distinct ansatze to determine other exact solutions, with distinct physical features. The techniques which we will use are mostly used in the literature, where detailed description can be found in [11]- [13].

5 Other solutions

In this section, we will use some of the well known techniques to determine other solutions of the newly developed equation (20)

5.1 Using the tanh/coth method

The tanh method admits the use of the the solution as

$$u(x,y,t) = a_0 + a_1 \tanh(kx+ry-ct). \qquad (33)$$

Substituting this assumption into Eq. (20), collecting the coefficients of $\tanh^i(kx+ry-ct), i = 0,2,4$, and solving the resulting system we find the following set of solutions

$$\begin{aligned} a_0 &= a, a \,\text{is any nonzero constant},\\ a_1 &= 2k,\\ c &= \tfrac{4k^2 r}{1+4k^2}. \end{aligned} \qquad (34)$$

where $a_0 = a$ is left as a free parameter. This in turn gives the soliton solution

$$u(x,y,t) = a + 2k \tanh(kx+ry-\frac{4k^2 r}{1+4k^2}t). \qquad (35)$$

In a like manner, we can show that

$$u(x,y,t) = a + 2k \coth(kx+ry-\frac{4k^2 r}{1+4k^2}t), \qquad (36)$$

is a singular solution of the same equation.

5.2 Using the tan/cot method

Using the tan method and the balance scheme, we set the solution as

$$u(x,t) = a_0 + a_1 \tan(kx+ry-ct), \qquad (37)$$

Substituting this assumption into the reduced equation (20), collecting the coefficients of $\tan^i(kx-ct), i = 0,2,4$, and solving the resulting system we find the following set of solutions

$$\begin{aligned} a_0 &= a, a \,\text{is any nonzero constant},\\ a_1 &= -2k,\\ c &= -\tfrac{4k^2 r}{1-4k^2}. \end{aligned} \qquad (38)$$

where $a_0 = a$ is left as a free parameter. This in turn gives the soliton solution

$$u(x,y,t) = a - 2k \tan(kx+ry+\frac{4k^2 r}{1-4k^2}t). \qquad (39)$$

In a similar manner, we can obtain the exact solution

$$u(x,y,t) = a + 2k \cot(kx+ry+\frac{4k^2 r}{1-4k^2}t). \qquad (40)$$

5.3 Using the rational tanh/coth method

Using the rational tanh method, we set the solution as

$$u(x,y.t) = \frac{1}{a_0 + a_1 \tanh(kx+ry-ct)}. \qquad (41)$$

Substituting this assumption into Eq. (20), collecting the coefficients of $\tanh^i(kx+ry-ct), 0 \le i \le 5$, and solving the resulting system we find the following set of solutions

$$\begin{aligned} a_1 &= b, b \,\text{is any nonzero constant},\\ a_0 &= \pm\sqrt{\tfrac{b(2bk-1)}{2k}},\\ c &= \tfrac{4k^2 r}{1+4k^2}. \end{aligned} \qquad (42)$$

where $a_1 = b$ is left as a free parameter. This in turn gives the soliton solution

$$u(x,y,t) = \frac{1}{\pm\sqrt{\frac{b(2bk-1)}{2k}} + b\tanh(kx+ry-\frac{4k^2 r}{1+4k^2}t)}. \qquad (43)$$

In a like manner we can obtain the singular solution

$$u(x,y,t) = \frac{1}{\pm\sqrt{\frac{b(2bk-1)}{2k}} + b\coth(kx+ry-\frac{4k^2 r}{1+4k^2}t)}. \qquad (44)$$

5.4 Using the rational tan/cot method

Using the rational tan method, we set the solution as

$$u(x,y,t) = \frac{1}{a_0 + a_1 \tan(kx-ct)}. \qquad (45)$$

Substituting this assumption into Eq. (20), collecting the coefficients of $\tanh^i(kx+ry-ct), 0 \le i \le 5$, and solving the resulting system we find the following set of solutions

$$\begin{aligned} a_1 &= b, b \,\text{is any nonzero constant},\\ a_0 &= \pm\sqrt{\tfrac{b(1-2bk)}{2k}},\\ c &= \tfrac{4k^3}{4k^2-1}. \end{aligned} \qquad (46)$$

where $a_1 = b$ is left as a free parameter. This in turn gives the soliton solution

$$u(x,y,t) = \frac{1}{\pm\sqrt{\frac{b(1-2bk)}{2k}} + b\tan(kx+ry-\frac{4k^2 r}{4k^2-1}t)}. \qquad (47)$$

In a like manner we can obtain the singular solution

$$u(x,y,t) = \frac{1}{\pm\sqrt{\frac{b(1-2bk)}{2k}} - b\cot(kx+ry-\frac{4k^2 r}{4k^2-1}t)}. \qquad (48)$$

6 Discussion

We used the recursion operator for the Calogero-Bogoyavlenskii-Schiff (CBS) equation and the inverse recursion operator to formally establish a combined CBS equation with its negative-order form. Using the MACSYMA package, we showed that the newly established equation passes nicely the Painlevé test, and this confirms its complete integrability in the Painlevé sense. Multiple soliton solutions were obtained for this equation. Moreover, we showed that this equation gives a variety of travelling wave solutions.

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