Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/110303

On $(h_1,h_2,m)-GA-$Convex Stochastic Processes

Miguel J. Vivas-Cortez [1,2,*] *and Jorge E. Hernández H.*[3]

[1] Departamento de Matemáticas, Decanato de Ciencias y Tecnología, Universidad Centroccidental Lisandro Alvarado. Barquisimeto, Venezuela.
[2] Departamento de Matemáticas, Facultad de Ciencias Naturales y Matemáticas, Escuela Superior Politécnica del Litoral (ESPOL), Campus Gustavo Galindo, Km 30.5, Ecuador.
[3] Departamento de Técnicas Cuantitativas, Decanato de Ciencias Económicas y Empresariales, Universidad Centroccidental Lisandro Alvarado, Barquisimeto, Venezuela.

Received: 2 Jan. 2017, Revised: 27 Feb. 2017, Accepted: 5 Mar. 2017
Published online: 1 May 2017

Abstract: In this paper we propose the $(h1;h2;m)-$GA-Convexity for stochastic processes and give some new generalized Hermite-Hadamard and Jensen type inequalities.

Keywords: $(h_1,h_2)-$convexity, Stochastic processes, GA-convexity, Hermite-Hadamard inequality, Jensen inequality

1 Introduction

The study on convex stochastic processes began in 1974 when B. Nagy in [19], applied a characterization of measurable stochastic processes to solving a generalization of the (additive) Cauchy functional equation.
In 1980, Nikodem [22] introduced the convex stochastic processes in his article.
Later in 1995, A. Skrowronski in [36] presented some further results on convex stochastic processes. In 2014 Maden et. al. [16] introduced the convex stochastic processes in the first sense and proved Hermite-Hadamard type inequalities to these processes. In the year 2014, E. Set et. al. in [33] investigated Hermite-Hadamard type inequalities for stochastic processes in the second sense.
They investigated a relation between s-convex stochastic processes in the second sense and convex stochastic processes.
For other results related to stochastic processes see [2], [3], [9], [17] where further references are given.
Convexity is one of the hypotheses often used in optimization theory. It is generally used to give global validity for certain propositions, which otherwise would only be true locally. A function $f:I\to\mathbb{R}$, where $I\subset\mathbb{R}$ is an interval, is said to be a convex function on I if the inequality

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y) \tag{1}$$

Hold for all $x,y\in I$ and $t\in[0,1]$. If the reversed inequality in 1 holds, then f is concave.

The concept of convexity has been generalized depending on the problem and applications studied. Some of these generalizations are midconvex, t-convex, quasi convex, pseudo convex, Invex, k-convex, e-convex, h-convex, (k,h) -convex, Wright-convex , E-convex, strongly convex and p-convex.

In 2005, the croatian mathematician Sanja Varošanec generalizes The notion of p-convexity giving the notion of h-convexity. (See [38]).

Imdat Iscan in year 2013 introduces a new kind of convex functions class, called harmonically convex function. In his work [13] obtains a Hermite-Hadamard inequality type for this kind of generalized convex functions.

For others recent results, see the books [4],[5],[6],[11],[21], [28], [30], where further references are given.

* Corresponding author e-mail: mvivas@ucla.edu.ve

2 Preliminaries

The following definitions are well know in the literature. C. Nicolescu in [21] wrote about the geometric-arithmetic convexity.

Definition 1. *A function $f:I\subset\mathbb{R}_+=(0,+\infty)\to\mathbb{R}$ is said to be GA-convex if*

$$f(x^t\cdot y^{(1-t)})\leq tf(x)+(1-t)f(y) \qquad (2)$$

holds for all $x,y\in I$ and $t\in[0,1]$.

G. Toader introduced in [37] the concept of $m-$convex function.

Definition 2. *For $f:[0,b]\to\mathbb{R}, b>0$ and $m\in(0,1]$, if*

$$f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y) \qquad (3)$$

is valid for all $x,y\in[0,b]$ and $t\in[0,1]$, then we say that f is an $m-$convex function.

In the year 2007, S. Varosanec in [38], introduce the following definition.

Definition 3. *Let $I,J\subseteq\mathbb{R}$ be intervals, $(0,1)\subseteq J$, and $h:J\to\mathbb{R}$ be a non-negative function such that $h\not\equiv 0$.*
A function $f:I\to\mathbb{R}$ is called $h-$convex ,or that f belongs to the class $SX(h,I)$, if f is non-negative and for all $x,y\in I$ and $t\in(0,1)$ we have

$$f(tx+(1-t)y)\leq h(t)f(t)+h(1-t)f(y). \qquad (4)$$

If the inequality 4 is reversed, then f is said to be h-concave, i.e. $f\in SV(h,I)$.

The concept of $(h,m)-$*convex function* has been introduced by Özdemir et al. in [27], as follow.

Definition 4. *Sea $h:J\subseteq\mathbb{R}\to\mathbb{R}$ be a non-negative function. We say that $f:[0,b]\to\mathbb{R}$ is $(h,m)-$convex function, if f is non-negative and for all $x,y\in[0,b], m\in[0,1]$, and $t\in(0,1)$, we have:*

$$f(tx+m(1-t)y)\leq h(t)f(x)+mh(1-t)f(y) \qquad (5)$$

In the year 2016, Bo-Yaw Xi and Fend Qi in [40], introduced the following definition:

Definition 5. *Sea $h_i:[0,1]\to\mathbb{R}_0, m:[0,1]\to(0,1]$ such that $h_i\not\equiv 0$ for i=1,2, and $f:(0,b]\to\mathbb{R}_0$.*
If

$$f(x^t\cdot y^{(1-t)m(t)})\leq h_1(t)f(x)+m(t)h_2(1-t)f(y) \qquad (6)$$

for $x,y\in[0,b)$ and $t\in[0,1]$, then f is said to be an $(h_1,h_2,m)-$ geometric-arithmetically convex function or, simply speaking an $(h_1,h_2,m)-GA$-convex function.

Example 1. Let $f(x)=|Ln(x)|$ for $x\in(0,1]$, $m(t)=c(1-t)^{l_0}$ for $t\in(0,1)$ and $0<c\leq 1$, and some $l_0\in\mathbb{R}$. Let $h_1(t)=t^{l_1}$ and $h_2(t)=t^{l_2}$ for $t\in(0,1)$ and $l_1,l_2\in\mathbb{R}$ if $l_1,l_2\leq 1$, then f is an decreasing and (h_1,h_2,m)-GA-convex function on $(0,1]$. And f is not an (h,m)-convex function on $(0,1]$.

In this paper we propose the generalization of convexity of this kind for stochastic processes.

Definition 6. *Let (Ω,F,P) be an arbitrary probability space. A function $X:\Omega\to\mathbb{R}$ is called a random variable if it is F-measurable. Let (Ω,F,P) be an arbitrary probability space and let $T\subset\mathbb{R}$ be time. A collection of random variable $X(t,w),t\in T$ with values in $\mathbb{R}$ is called a stochastic processes.*

1. *If $X(t,w)$ takes values in $S=\mathbb{R}^d$ if is called vector-valued stochastic process.*
2. *If the time T can be a discrete subset of $\mathbb{R}$, then $X(t,w)$ is called a discrete time stochastic process.*
3. *If the time T is an interval, $\mathbb{R}^+$ or $\mathbb{R}$, it is called a stochastic process with continuous time*

Throughout the paper we restrict our attention stochastic process with continuous time, i.e, index set $T=[0,+\infty)$.

Definition 7. *Set (Ω,A,P) be a probability space and $T\subset\mathbb{R}$ be an interval. We say that a stochastic process $X:T\times\Omega\to\mathbb{R}$ if*

1. *Convex if*

$$X(\lambda u+(1-\lambda)v,\cdot)\leq\lambda X(u,\cdot)+(1-\lambda)X(v,\cdot) \qquad (7)$$

for all $u,v\in T$ and $\lambda\in[0,1]$.
This class of stochastic process are denoted by C.

2. *m-convex if*

$$X(tu+m(1-t)v,\cdot)\leq tX(u,\cdot)+m(1-t)X(v,\cdot) \qquad (8)$$

for all $u,v\in T$ and $t\in[0,1], m\in(0,1]$.

Definition 8. *Let (Ω,A,P) be a probability space and $T\subset\mathbb{R}$ be an interval. We say that the stochastic process $X:\Omega\to\mathbb{R}$ is called*

1. *Continuous in probability in interval T if for all $t_0\in T$*

$$P-\lim_{t\to t_0}(t,\cdot)=X(t_0,\cdot)$$

where $P-\lim$ denotes the limit in probability;

2. *Mean-square continuous in the interval T if for all $t_0\in T$*

$$P-\lim_{t\to t_0}E(X(t,\cdot)-X(t_0,\cdot))=0$$

where $E(X(t,\cdot))$ denotes the expectation value of the random variable $X(t,\cdot)$;

3.*Increasing (decreasing) if for all $u,v \in T$ such that $t < s$,*

$$X(u,\cdot) \le X(v,\cdot), \quad (X(u,\cdot) \ge X(v,\cdot))(respectively)$$

4.*Monotonic if it's increasing or decreasing;*

5.*Differentiable at a point $t \in T$ if there is a random variable*

$$X'(t,\cdot): T \times \Omega \to \mathbb{R}$$

$$X'(t,\cdot) = P - \lim_{t \to t_0} \frac{X(t,\cdot) - X(t_0,\cdot)}{t - t_0}$$

We say that a stochastic process $X: T \times \Omega \to \mathbb{R}$ is continuous (differentiable) if it is continuous (differentiable) at every point of the interval T. [15], [35], [36], [22].

Definition 9.*Let (Ω, A, P) be a probability space $T \subset \mathbb{R}$ be an interval with $E(X(t)^2) < \infty$ for all $t \in T$.*
Let $[a,b] \subset T, a = t_0 < t_1 < \ldots < t_n = b$ be a partition of $[a,b]$ and $\theta_k \in [t_{k-1}, t_k]$ for $k = 1,2,\ldots,n$.
A random variable $Y: \Omega \to \mathbb{R}$ is called mean-square integral of the process $X(t,\cdot)$ on $[a,b]$ if the following identity holds:

$$\lim_{n\to\infty} E[X(\theta_k(t_k - t_{k-1}) - Y)^2] = 0$$

Then we can write

$$\int_a^b X(t,\cdot)dt = Y(\cdot)(a.e.).$$

Also, mean square integral operator is increasing, that is,

$$\int_a^b X(t,\cdot)dt \le \int_a^b Z(t,\cdot)dt(a.e.)$$

where $X(t,\cdot) \le Z(t,\cdot)$ in $[a,b]$ [34].
In throughout paper, we will consider the stochastic processes that is with continuous time and mean-square continuous.
Now, we give the well-known Hermite-Hadamard integral inequality for convex stochastic processes [15]:

Theorem 1.*If $X: T \times \Omega \to \mathbb{R}$ is Jensen-convex and mean square continuous in the interval $T \times \Omega$, then for any $u,v \in T$, we have*

$$X\left(\frac{u+v}{2},\cdot\right) \le \frac{1}{u-v}\int_u^v X(t,\cdot)dt \le \frac{X(u,\cdot)+X(v,\cdot)}{2}$$

Definition 10.*Let (Ω, A, P), be a, probability space $T \subset \mathbb{R}$ be an interval, we say that a stochastic processes $X: [0,b) \times \Omega \to [0,+\infty)$ is (h_1,h_2,m)-GA convex if*

$$X(u^{\lambda} v^{(1-\lambda)m(\lambda)},\cdot) \le h_1(\lambda)X(u,\cdot) + m(\lambda)h_2(1-\lambda)X(v,\cdot)$$

for all $u,v \in [0,1]$, with $h_i: [0,1] \to \mathbb{R}_0$ and $m: [0,1] \to (0,1]$ such that $h_i \not\equiv 0$ for i=1,2.

The main subject of this paper is to extend some results concerning (h_1,h_2,m)-GA-convex functions to (h_1,h_2,m)-GA-convex stochastic process.

3 Hermite-Hadamard Type Inequalities

Theorem 2.*Let $h_i: [0,1] \to \mathbb{R}_0$, where $h_i \not\equiv 0$ for $i = 1,2$; $m: [0,1] \to (0,1]$, and $X: [0,+\infty) \times \Omega \to \mathbb{R}_0$ be an (h_1,h_2,m)-GA-convex function on $(0, \frac{b}{m(\frac{1}{2})}] \times \Omega$ and $h_i \in L_i[a,b]$ for $0 < a < b$.*
Then

$$X(\sqrt{ab},\cdot) \le \frac{h_1(\frac{1}{2})}{\ln(b)-\ln(a)}\int_a^b X(t,\cdot)dt + \frac{m(\frac{1}{2})h_2(\frac{1}{2})}{\ln(b)-\ln(a)}\int_a^b X(\frac{t}{m(\frac{1}{2})},\cdot)dt$$

*Proof.*Since

$$\sqrt{ab} = (a^t.b^{1-t})^{\frac{1}{2}}.(a^{1-t}.b^t)^{\frac{1}{2}}$$

for $0 \le t \le 1$, from the (h_1,h_2,m)-GA-convexity of the stochastic process X on $\left(0, \frac{b}{m(\frac{1}{2})}\right] \times \Omega$, we obtain

$$X(\sqrt{ab},\cdot) \le h_1\left(\frac{1}{2}\right)X(a^t b^{1-t},\cdot) + m\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)X\left(\frac{a^{1-t}b^t}{m(\frac{1}{2})},\cdot\right).$$

Integrating both sides of the above inequality and replacing the argument, in the right side, $a^{1-t}.b^t$ and $a^t b^{1-t}$ for $0 \le t \le 1$ by s, then

$$\int_0^1 X(a^{1-t}b^t,\cdot)dt = \frac{1}{\ln(b)-\ln(a)}\int_a^b X(s,\cdot) \tag{9}$$

and

$$\int_0^1 X\left(\frac{a^{1-t}b^t}{m(\frac{1}{2})},\cdot\right)dt = \frac{1}{\ln(b)-\ln(a)}\int_a^b X\left(\frac{s}{m(\frac{1}{2})},\cdot\right)ds \tag{10}$$

The proof of theorem is complete.

Theorem 3.*Let $h_i: [0,1] \to \mathbb{R}_0$, where $h_i \not\equiv 0$ for $i = 1,2$; $m: [0,1] \to (0,1]$, and $X: [0,+\infty) \times \Omega \to \mathbb{R}_0$ be an (h_1,h_2,m)-GA-convex stochastic process on $(0, \frac{b}{m}]$ such that X is an integrable stochastic process in $[a, \frac{b}{m}] \times \Omega$ and $h_1, h_2 \in L_1([0,1])$ for $0 < a < b$, then*

$$\frac{1}{\ln(b)-\ln(a)}\int_a^b X(t,\cdot)dt \le min\{A,B\},$$

where

$$A=X(a,\cdot)\int_0^1 h_1(t)dt+mX\left(\frac{b}{m},\cdot\right)\int_0^1 h_2(t)dt$$

and

$$B=X(b,\cdot)\int_0^1 h_1(t)dt+mX\left(\frac{a}{m},\cdot\right)\int_0^1 h_2(t)dt.$$

If $h_1(t)=h_2(t)=h(t)$ *for all* $t\in[0,1]$, *we have*

$$\frac{1}{\ln(b)-\ln(a)}\int_a^b X(t,\cdot)dt\le min\{C,D\},$$

where

$$C=\left(X(a,\cdot)+mX\left(\frac{b}{m},\cdot\right)\right)\int_0^1 h(t)dt$$

and

$$D=\left(X(b,\cdot)+mX\left(\frac{a}{m},\cdot\right)\right)\int_0^1 h(t)dt.$$

*Proof.*Letting $x=a^{1-t}b^t$ for $0\le t\le 1$, by the (h_1,h_2,m)-GA-convexity of X and (9), we obtain

$$\frac{1}{\ln(b)-\ln(a)}\int_a^b X(t,\cdot)dt=\int_0^1 X(a^{1-t}b^t,\cdot)dt \le \min\{A,B\},$$

where

$$A=X(a,\cdot)\int_0^1 h_1(t)dt+mX\left(\frac{b}{m},\cdot\right)\int_0^1 h_2(t)dt$$

and

$$B=X(b,\cdot)\int_0^1 h_1(t)dt+mX\left(\frac{a}{m},\cdot\right)\int_0^1 h_2(t)dt$$

The proof of Theorem is complete.

Theorem 4.*Let* $h_i:[0,1]\to\mathbb{R}_0$,$h_i\not\equiv 0$ *for* $i=1,2;$ $m\in(0,1]$,$X:[0,+\infty)\times\Omega\to\mathbb{R}_0$ *be an* (h_1,h_2,m)*-GA-stochastic process on* $\left(0,\frac{b}{m^2}\right]\times\Omega$ *such that* X *is integrable in* $[a,\frac{b}{m}]\times\Omega$ *and* $h_1,h_2\in L_1([0,1])$ *for* $0<a<b$ *then*

$$X(\sqrt{ab},\cdot)$$

$$\le\frac{h_1(\frac{1}{2})}{\ln(a)-\ln(b)}\int_a^b X(t,\cdot)dt$$

$$+m\frac{h_2\left(\frac{1}{2}\right)}{\ln(b)-\ln(a)}\int_a^b X\left(\frac{t}{m},\cdot\right)dt$$

$$\le\min\left\{\left(A\int_0^1 h_1(t)dt+mB\int_0^1 h_2(t)dt\right),\right.$$

$$\left.\left(C\int_0^1 h_1(t)dt+mD\int_0^1 h_2(t)dt\right)\right\}$$

where

$$A=\left[h_1\left(\frac{1}{2}\right)X(a,\cdot)+mh_2\left(\frac{1}{2}\right)X\left(\frac{a}{m},\cdot\right)\right],$$

$$B=\left[h_1\left(\frac{1}{2}\right)X\left(\frac{b}{m},\cdot\right)+mh_2\left(\frac{1}{2}\right)X\left(\frac{b}{m^2},\cdot\right)\right],$$

$$C=\left[h_1\left(\frac{1}{2}\right)X(b,\cdot)+mh_2\left(\frac{1}{2}\right)X\left(\frac{b}{m},\cdot\right)\right]$$

and

$$D=\left[h_1\left(\frac{1}{2}\right)X\left(\frac{a}{m},\cdot\right)+mh_2\left(\frac{1}{2}\right)X\left(\frac{a}{m^2},\cdot\right)\right].$$

*Proof.*From the (h_1,h_2,m)-GA-convexity of X on $\left(0,\frac{b}{m^2}\right]$, we obtain

$$X\left(\sqrt{ab},\cdot\right)$$

$$\le h_1\left(\frac{1}{2}\right)X\left(a^tb^{1-t},\cdot\right)+mh_2\left(\frac{1}{2}\right)X\left(\frac{a^{1-t}b^t}{m},\cdot\right)$$

$$\le\min\left\{h_1\left(\frac{1}{2}\right)\left[h_1(t)X(a,\cdot)+mh_2(1-t)X\left(\frac{b}{m},\cdot\right)\right]\right.$$

$$+mh_2\left(\frac{1}{2}\right)\left[h_1(1-t)X(\frac{a}{m},\cdot)+mh_2(t)X(\frac{b}{m^2},\cdot)\right],$$

$$h_1\left(\frac{1}{2}\right)\left[h_1(1-t)X(b,\cdot)+mh_2(t)X\left(\frac{a}{m},\cdot\right)\right]$$

$$\left.+mh_2\left(\frac{1}{2}\right)\left[h_1(t)X\left(\frac{b}{m},\cdot\right)+mh_2(1-t)X\left(\frac{a}{m^2},\cdot\right)\right]\right\}$$

Substituting $a^{1-t}b^t$ and a^tb^{1-t} for $0\le t\le 1$ by u an integrating on both sides of the above inequality with respect to $t\in[0,1]$ lead to

$$X\left(\sqrt{ab},\cdot\right)$$

$$\le\frac{h_1\left(\frac{1}{2}\right)}{\ln(b)-\ln(a)}\int_a^b X(u,\cdot)du$$

$$+m\frac{h_2\left(\frac{1}{2}\right)}{\ln(b)-\ln(a)}\int_a^b X\left(\frac{u}{m},\cdot\right)du$$

$$\le\min\left\{h_1\left(\frac{1}{2}\right)\left[h_1(t)X(a,\cdot)+mh_2(1-t)X\left(\frac{b}{m},\cdot\right)\right]\right.$$

$$+mh_2\left(\frac{1}{2}\right)\left[h_1(1-t)X(\frac{a}{m},\cdot)+mh_2(t)X\left(\frac{b}{m^2},\cdot\right)\right],$$

$$h_1\left(\frac{1}{2}\right)\left[h_1(1-t)X(b,\cdot)+mh_2(t)X\left(\frac{a}{m},\cdot\right)\right]$$

$$\left.+mh_2\left(\frac{1}{2}\right)\left[h_1(t)X\left(\frac{b}{m},\cdot\right)+mh_2(1-t)X\left(\frac{a}{m^2},\cdot\right)\right]\right\}.$$

the theorem is proved.

4 Other inequalities for products of $(h_1,h_2,m)-GA-$convex stochastic processes

In this section we give some results about Hermite-Hadamard type inequalities for the product of stochastic processes.

Theorem 5. *Let $h_i:[0,1]\to\mathbb{R}_0$ such that $h_i\not\equiv 0$ for $i=1,2$, $m:[0,1]\to(0,1]$, and $X,Y:[0,+\infty)\times\Omega\to\mathbb{R}_0$ are (h_1,h_2,m)-GA-convex stochastic processes on $\left(0,\frac{b}{m(\frac{1}{2})}\right]\times\Omega$ such that $X\cdot Y$ is an integrable stochastic processes on $\left[a,\frac{b}{m(\frac{1}{2})}\right]\times\Omega$ for $0<a<b$ then*

$$X\left(\sqrt{ab},\cdot\right)\cdot Y\left(\sqrt{ab},\cdot\right) \tag{11}$$

$$\leq\frac{h_1\left(\frac{1}{2}\right)}{\ln(b)-\ln(a)}\int_a^b X(s,\cdot)Y(s,\cdot)ds$$

$$+\frac{m\left(\frac{1}{2}\right)h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{\ln(b)-\ln(a)}$$

$$\times\int_a^b\left[X\left(\frac{s}{m\left(\frac{1}{2}\right)},\cdot\right)\cdot Y(s,\cdot)+X(s,\cdot)\cdot Y\left(\frac{s}{m\left(\frac{1}{2}\right)},\cdot\right)\right]ds$$

$$+\frac{[m\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)]^2}{\ln(b)-\ln(a)}\int_a^b X\left(\frac{s}{m\left(\frac{1}{2}\right)},\cdot\right)Y\left(\frac{s}{m\left(\frac{1}{2}\right)},\cdot\right)ds$$

Proof. Using the (h_1,h_2,m)-GA-convexity of X and Y on $\left(0,\frac{b}{m(\frac{1}{2})}\right]\times\Omega$, we obtain

$$X\left(\sqrt{ab},\cdot\right)\cdot Y\left(\sqrt{ab},\cdot\right) \tag{12}$$

$$\leq\left[h_1\left(\frac{1}{2}\right)X\left(a^tb^{1-t},\cdot\right)+m\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)X\left(\frac{a^{1-t}b^t}{m\left(\frac{1}{2}\right)},\cdot\right)\right]$$

$$\times\left[h_1\left(\frac{1}{2}\right)Y\left(a^tb^{1-t},\cdot\right)+m\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)Y\left(\frac{a^{1-t}b^t}{m\left(\frac{1}{2}\right)},\cdot\right)\right]$$

Letting $s=a^{1-t}b^t$ and $s=a^tb^{1-t}$ for $t\in[0,1]$ and integrating the inequality 12 on $[0,1]$ with respect to t, we arrive at the inequality 11. The proof is completed.

Theorem 6. *Let $h_i:[0,1]\to\mathbb{R}_0,h_i\not\equiv 0$ for $i=1,2$; $m_1,m_2\in[0,1]$, and $X,Y:[0,+\infty)\times\Omega\to\mathbb{R}$. If X is on (h_1,h_2,m_1)-GA-convex stochastic processes on $(0,\frac{b}{m_1}]\times\Omega,Y$ is an (h_1,h_2,m)-GA-convex stochastic process on $(0,\frac{b}{m_2}]\times\Omega$, and $X\cdot Y$ is integrable stochastic process on $[a,b]\times\Omega$ and $h_1^2\cdot h_2^2\in L_1([0,1])$ for $0<a<b$, then*

$$\frac{1}{\ln(a)-\ln(b)}\int_a^b X(s,\cdot)Y(s,\cdot)ds\leq A+B+C \tag{13}$$

Where

$$A=X(a,\cdot)Y(a,\cdot)\int_0^1 h_1^2(t)dt$$

$$B=m_1m_2X(\frac{b}{m_1},\cdot)Y(\frac{b}{m_2},\cdot)\int_0^1 h_2^2(t)dt$$

$$C=[m_2X(a,\cdot)\cdot Y(\frac{b}{m_2},\cdot)$$

$$+m_1X(\frac{b}{m_1},\cdot)Y(a,\cdot)]\int_0^1 h_1(t)h_2(1-t)dt$$

Proof. Let $s=a^tb^{1-t}$ for $t\in[0,1]$. By the (h_1,h_2,m)-GA-convexity in stochastic processes of X and Y, we have

$$\frac{1}{\ln(b)-\ln(a)}\int_a^b X(s,\cdot)Y(s,\cdot)ds \tag{14}$$

$$=\int_0^1 X\left(a^tb^{1-t},\cdot\right)Y\left(a^tb^{1-t},\cdot\right)dt$$

$$\leq\int_0^1[h_1(t)X(a,\cdot)+m_1h_2(1-t)X\left(\frac{b}{m_1},\cdot\right)]$$

$$\times[h_1(t)Y(a,\cdot)+m_2h_2(1-t)X\left(\frac{b}{m_2},\cdot\right)]dt$$

$$=X(a,\cdot)Y(a,\cdot)\int_0^1 h_1^2(t)dt$$

$$+m_1m_2X\left(\frac{b}{m_1},\cdot\right)Y\left(\frac{b}{m_2},\cdot\right)\int_0^1 h_2^2(t)dt$$

$$+[m_2X(a,\cdot)Y\left(\frac{b}{m_2},\cdot\right)+m_1X\left(\frac{b}{m_1},\cdot\right)Y(a,\cdot)]$$

$$\times\int_0^1 h_1(t)h_2(1-t)dt.$$

The proof of theorem is complete.

5 Jensen type inequalities for $(h_1,h_2,m)-GA-$convex stochastic processes

The following result is related with the discrete version of the classical Jensens's inequality for $(h_1,h_2,m)-GA-$convex stochastic processes.

Theorem 7. *Let $h_i:[0,1]\to\mathbb{R}_0, h_i\not\equiv 0$ for $i=1,2$; and $h_1(t_1)h_2(t_2)\le h_2(t_1t_2)$ for all $t_1,t_2\in[0,1]$ and h_2 be a supermultiplicative function. Let $m:[0,1]\to(0,1]$ and $X:[0,+\infty)\times\Omega\to\mathbb{R}_0$ be a $(h_1,h_2,m)-GA-$convex stochastic process. Then*

$$X\left(\prod_{i=1}^{n} t_i^{w_i\prod_{j=0}^{i-1}m(w_j)},\cdot\right) \tag{15}$$

$$\le h_1(w_1)X(t_1,\cdot)+\sum_{i=2}^{n}\left(\prod_{j=1}^{i-1}m(w_j)\right)h_2(w_i)X(t_i,\cdot)$$

holds for all $t_i\in(0,b], w_i>0$ with $\sum_{i=1}^{n}w_i=1$ and $m(w_0)=1$

Proof. Using induction over n . When $n=2$ taking $t=w_1$ and $1-t=w_2$ in Definition 10 we obtain 15. Suppose that for $n=k$ the inequality 15 holds, that is

$$X\left(\prod_{i=1}^{k} t_i^{w_i\prod_{j=0}^{i-1}m(w_j)},\cdot\right) \tag{16}$$

$$\le h_1(w_1)X(t_1,\cdot)+\sum_{i=2}^{k}\left(\prod_{j=1}^{i-1}m(w_j)\right)h_2(w_i)X(t_i,\cdot)$$

When $n=k+1$, letting $S_k=\sum_{i=1}^{k+1}w_i$, by Definition 10 and the induction hypothesis we have

$$X\left(\prod_{i=1}^{n} t_i^{w_i\prod_{j=0}^{i-1}m(w_j)},\cdot\right)$$

$$=X\left(t_1^{w_1}\left(\prod_{i=2}^{k+1} t_i^{w_i/S_k\prod_{j=0}^{i-1}m(w_j)}\right)^{m(w_1)S_k},\cdot\right)$$

$$\le h_1(w_1)X(t_1,\cdot)$$

$$+m(w_1)h_2(S_k)X\left(t_2^{w_2/S_k}\prod_{i=3}^{k+1}t_i^{w_i/S_k\prod_{j=2}^{i-1}m(w_j)},\cdot\right)$$

$$\le h_1(w_1)X(t_1,\cdot)$$

$$+m(w_1)h_2(S_k)\left[h_1\left(\frac{w_2}{S_k}\right)X(t_2,\cdot)\right.$$

$$\left.+\sum_{i=3}^{k+1}\left(\prod_{j=2}^{i-1}m(w_j)\right)h_2\left(\frac{w_i}{S_k}\right)X(t_i,\cdot)\right]$$

Since h_2 is a supermultiplicative function, we have $h_2(S_k)h_2(w_i/S_k)\le h_2(w_i)$ for $i=1,2,...,n$. This implies that when $n=k+1$ the inequality 15 holds. The proof is complete.

6 Applications

Some applications are derived as a result of the Theorems obtained in the previous sections. These involves inequalities type $s-$GA convex and $(s,m)-$GA convex stochastic processes.

From Theorem 2 we have the following Remark.

Remark. 1.Letting $h_1(t)=h_2(t)=t$, for all $t\in[0,1]$ and $m(t)=1$ for all $t\in(0,1]$ we have

$$X(\sqrt{ab},\cdot)\le\frac{1}{\ln(b)-\ln(a)}\int_a^b X(t,\cdot)dt$$

2.Letting $h_1(t)=h_2(t)=t^s$ for all $t\in[0,1]$,$s\in(0,1]$ and $m(t)=1$ for all $t\in(0,1]$ then we obtain

$$2^{s-1}X(\sqrt{ab},\cdot)\le\frac{1}{\ln(b)-\ln(a)}\int_a^b X(t,\cdot)dt.$$

From Theorem 3 we can deduce the following.

Corollary 1. *Let $h_1(t)=t^{s_1}$ and $h_2(t)=t^{s_2}$ for all $t\in(0,1)$, and $s_1,s_2\in(-1,1)$ and $m\in(0,1]$, and $X:(0,+\infty)\times\Omega\to\mathbb{R}_0$ be an (h_1,h_2,m)-GA-convex stochastic processes on $(0,\frac{b}{m}]\times\Omega$ such that X is an integrable processes stochastic on $(a,\frac{b}{m^2}]\times\Omega$ for $0<a<b$. Then*

$$\frac{1}{\ln(b)-\ln(a)}\int_a^b X(t,\cdot)dt$$

$$\le\min\left\{\frac{X(a,\cdot)}{s_1+1}+m\frac{X(\frac{b}{m},\cdot)}{s_2+1},\frac{X(b,\cdot)}{s_1+1}+m\frac{X(\frac{a}{m},\cdot)}{s_2+1}\right\}$$

Corollary 2. *Let $h:[0,1]\to\mathbb{R}_0, h\not\equiv 0$ and $m\in[0,1]$, and $X:[0,+\infty)\times\Omega\to\mathbb{R}_0$ be an (h,m)-GA-convex stochastic process on $(0,\frac{b}{m^2}]\times\Omega$ such that X is integrable in $[a,\frac{b}{m}]\times\Omega$, and $h\in L_1([0,1])$ for $0<a<b$ Then*

$$\frac{X\left(\sqrt{ab},\cdot\right)}{h\left(\frac{1}{2}\right)}$$

$$\le\frac{1}{\ln(b)-\ln(a)}\int_a^b\left[X(s,\cdot)+mX\left(\frac{s}{m},\cdot\right)\right]ds$$

$$\le\min\{A,B,C,D\}\int_0^1 h(t)\,dt$$

where

$$A=X(a,\cdot)+mX(\frac{a}{m},\cdot)+mX(\frac{b}{m},\cdot)+m^2X(\frac{b}{m^2},\cdot)$$

$$B=2mX(\frac{a}{m},\cdot)+X(b,\cdot)+m^2X(\frac{b}{m^2},\cdot)$$

$$C=X(a,\cdot)+mX(\frac{a}{m^2},\cdot)+2mX(\frac{b}{m},\cdot)$$

$$C=mX(\frac{a}{m},\cdot)+m^2X(\frac{a}{m^2},\cdot)+X(b,\cdot)+mX(\frac{b}{m},\cdot)$$

*Proof.*This can be derived from letting $h_1(t) = h_2(t) = h(t)$ for all $t \in [0,1]$ and considering the symmetry between a and b in theorem.

Corollary 3.*Under the conditions of Corollary 2 , if $h(t) = t^s$ for $t \in (0,1)$ and $s \in (-1,1)$, then*

$$2^s X(\sqrt{ab},\cdot) \leq \frac{1}{\ln(b)-\ln(a)} \int_a^b [X(s,\cdot)+mX(\frac{s}{m},\cdot)]ds \leq \frac{1}{s+1} min\{A,B,C,D\}$$

Where

$$A = X(a,\cdot)+mX(\frac{a}{m},\cdot)+mX(\frac{b}{m},\cdot)+m^2X(\frac{b}{m^2},\cdot)$$
$$B = 2mX(\frac{a}{m},\cdot)+X(b,\cdot)+m^2X(\frac{b}{m^2},\cdot)$$
$$C = X(a,\cdot)+mX(\frac{a}{m^2},\cdot)+2mX(\frac{b}{m},\cdot)$$
$$C = mX(\frac{a}{m},\cdot)+m^2X(\frac{a}{m^2},\cdot)+X(b,\cdot)+mX(\frac{b}{m},\cdot)$$

Now, we give some applications from Theorems 5 and 6.

Corollary 4.*Under the conditions of theorem , if $h_1(t) = h_2(t) = h(t)$ for all $t \in [0,1]$, then*

$$\frac{1}{\ln(b)-\ln(a)} \int_a^b X(s,\cdot)Y(s,\cdot)ds \leq A+B$$

Where

$$A = \left[X(a,\cdot)Y(a,\cdot)+m_1m_2X(\frac{b}{m_1},\cdot)Y\left(\frac{b}{m_2},\cdot\right)\right] \int_0^1 h^2(t)dt$$

And

$$B = \left[m_1X\left(\frac{b}{m_1},\cdot\right)Y(a,\cdot)+m_2X(a,\cdot)Y\left(\frac{b}{m_2},\cdot\right)\right] \times \int_0^1 h(t)h(1-t)dt$$

In particular, if $h(t) = t^s$ for $t \in (0,1), s \in (-\frac{1}{2},1]$, and $m = m_1 = m_2$, then

$$\frac{1}{\ln(b)-\ln(a)} \int_a^b X(s,\cdot)Y(s,\cdot)ds \leq C+D$$

Where

$$C = \frac{1}{2s+1}\left[X(a,\cdot)Y(a,\cdot)+m^2X\left(\frac{b}{m},\cdot\right)Y\left(\frac{b}{m},\cdot\right)\right]$$

$$D = m\beta(s+1,s+1)\left[X(a,\cdot)Y\left(\frac{b}{m},\cdot\right)+X\left(\frac{b}{m},\cdot\right)Y(a,\cdot)\right]$$

and β denotes the well known Beta function.

The following are some applications of Theorem 7.

Corollary 5.*Under the conditions of Theorem 7, if $w_1 = \ldots = w_n = 1/n$ then*

$$X\left(\prod_{i=1}^n t_i^{\frac{1}{n}[m(1/n)]^{i-1}},\cdot\right) \leq h_1\left(\frac{1}{n}\right)X(t_1,\cdot)+h_2\left(\frac{1}{n}\right)\sum_{i=2}^n\left[m\left(\frac{1}{n}\right)\right]^{i-1}X(t_i,\cdot) \quad (17)$$

Corollary 6.*Let $h : [0,1] \to \mathbb{R}_0$ be a supermultiplicative function such that $h \not\equiv 0, m \in (0,1]$ and $X : [0,+\infty) \times \Omega \to \mathbb{R}_0$ be a $(h,m)-GA-$convex stochastic process. Then the inequality*

$$X\left(\prod_{i=1}^n t_i^{m^{i-1}w_i},\cdot\right) \leq \sum_{i=1}^n m^{i-1}h(w_i)X(t_i,\cdot) \quad (18)$$

holds for all $t_i \in (0,b]$ and $w_i > 0$ with $\sum_{i=1}^n w_i = 1$.

*Proof.*This follows from Theorem 7 by putting $h_1(t) = h_2(t) = h(t)$ and $m(t) = m$ for all $t \in [0,1]$ and $m \in (0,1]$.

Corollary 7.*Let $h(t) = t^s$ for $t \in (0,1), s \in [-1,1]$ and $m \in (0,1]$. Let $X : [0,+\infty) \times \Omega \to \mathbb{R}_0$ be a $(h,m)-GA-$convex stochastic process. Then the inequality*

$$X\left(\prod_{i=1}^n t_i^{m^{i-1}w_i},\cdot\right) \leq \sum_{i=1}^n m^{i-1}w_i^sX(t_i,\cdot) \quad (19)$$

holds for all $t_i \in (0,b]$ and $w_i > 0$ with $\sum_{i=1}^n w_i = 1$.

7 Conclusion

In this paper we establish new inequalities of Hermite-Hadamard type inequalitues, and others like Jensen type, for $(h_1,h_2,m)-$GA convex stochastic. We hope this research can stimulate the research of applications in this area.

Acknowledgement

The authors acknowledges to the Concejo de Desarrollo Científico, Humanístico y Tecnológico (CDCHT) from Universidad Centroccidental Lisandro Alvarado and Centro de Investigación en Matemáticas Aplicadas a Ciencia e Ingeniería (CIMACI - FCMN). from Escuela Superior Politécnica del Litoral, for the thecnical support.

References

[1] Bai R., Qi F and Xi B. *Hermite-Hadamard type inequalities for the m− and (α,m)-logarithmically convex functions.* Filomat, **27**(2013), no. 1, 1–7. **MR3243893**

[2] A. Bain, D. Crisan. *Fundamentals of Stochastic Filtering.* Stochastic Modelling and Applied Probability, 60. Springer, New York. 2009. **MR2454694**

[3] Bhattacharya, R. N.; Waymire, E.C. *Stochastic processes with applications.* Classics in Applied Mathematics, 61. Society for Industrial and Applied Mathematics (SIAM), 2009. **MR3396216.**

[4] E. Beckenbach and R. Bellman. *An Introduction to Inequalities.* New Mathematical Library, 3 Random House, New York-Toronto 1961. **MR0130141**

[5] E. Beckenbach and R. Bellman. *Inequalities.* Springer Verlag Berlin Heidelberg. 1965. **MR0192009.**

[6] B.C. Carlson. *Special Functions of Applied Mathematics.* Academic Press, New York, 1977. **MR0590943**

[7] G. Cristescu , M. Gianu. *Detecting the non-Convex sets with Youness and Noor types convexities.* Bul. Stiint. Univ. Politeh. Timis., Ser. Mat.Fiz., **55**(69) (2010), no. 1, 20-27. **MR2742048.**

[8] G. Cristescu, M. Gianu. *Shape properties of Noor's convex sets.* Proceedings of the Twelfth Symposium of Mathematics and its Applications, 91–99, Ed. Politeh., Timioara, 2010. **MR2654717.**

[9] Devolder, Pierre; Janssen, Jacques; Manca, Raimondo. Basic stochastic processes. Mathematics and Statistics Series. ISTE, London; John Wiley and Sons, Inc. 2015. **MR3558939**

[10] S S.Dragomir. *On some new inequalities of Hermite-Hadamard type for $m-$convex functions*. Tamkang Journal of Mathematics. **33**(2002), no. 1 , 55–65. **MR1885425**

[11] G.H. Hardy, J.E. Littlewood, G. Pólya. *Inequalities.* 2d ed. Cambridge, at the University Press. 1952. **MR0046395**

[12] H. Iqbal, S. Nazir . *Semi- φ_h and Strongly log-φ convexity* . Stud. Univ. Babe-Bolyai Math. 59 (2014), no. 2, 141–154. **MR3229437.**

[13] I. Işcan. *Hermite-Hadamard type inequalities for harmonically convex functions.* Hacet. J. Math. Stat. 43 (2014), no. 6, 935–942. MR3331150.

[14] Kazi H., Neuman E. *Inequalities and bounds for elliptic integrals.* J. Approx. Theory. **146**(2) 212–226, 2007. **MR2328180.**

[15] D. Kotrys. *Hermite-hadamart inequality for convex stochastic processes*, Aequationes Mathematicae 83 (2012) 143-151. **MR2885506**

[16] S. Maden, M. Tomar, E. Set. *Hermite-Hadamard Type Inequalities for s-Convex Stochastic Processes in the Second Sense.*, Turkish Journal of Analysis and Number Theory. **2**(2014), no. 6, 202-207.

[17] Mikosch, Thomas. *Elementary stochastic calculus?with finance in view.* Advanced Series on Statistical Science and Applied Probability, 6. World Scientific Publishing Co., Inc.,2010. **MR1728093.**

[18] L. Montrucchio. *Lipschitz continuous of policy functions for strongly concave optimization problems* . J. Math. Econom.**16**(1987),no. 3, 259 - 273.**MR0910416**

[19] B. Nagy. *On a generalization of the Cauchy equation.* Aequationes Math. **11** (1974). 165–171. **MR0353429**

[20] E. Neuman. *Inequalities involving a logarithmically convex functions and their applications to special functions.* Journal of Inequalities in Pure and Applied Mathematics. **7**(2006), no. 1, Article 16. **MR2217179**

[21] C. Nicolescu., L. Peerson. *Convex Functions and Their Applications. A Contemporary Approach.* CMS Books in Mathematics. Springer, New York. 2006. **MR2178902**

[22] K. Nikodem. *On convex stochastic processes.*, Aequationes Math. **20** (1980), no. 2-3, 184–197. **MR0577487**

[23] M. A. Noor. *Advanced convex analysis.* Lecture Notes. Mathematics Department, COMSATS, Institute of Information Technology, Islamabad, Pakistan, 2010.

[24] M. A. Noor. *On some characterizations of nonconvex functions.* Nonlinear Anal. Forum 12 (2007), no. 2, 193–201. **MR2404197.**

[25] M. A. Noor. *Differentiable non-convex functions and general variational inequalities.* Appl. Math. Comput. **199**(2008),no.623–630. **MR2420590**

[26] W. Orlicz. *A note on modular spaces*. I. Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **9** 1961 157–162. MR0131763

[27] M.E. Özdemir, A. O. Akdemir, E. Set. *On $(h,m)-$Convexity and Hadamard Type Inequalities.* Transylv. J. Math. Mech. **8** (2016), no. 1, 51–58. **MR3531967.**

[28] J. E. Pečarić, F. Proschan, Y. L. Tong. *Convex functions ,partial orderings and statistical applications.* Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston.1992. **MR1162312.**

[29] B. T. Polyak. *Existence theorems and convergence of minimizing sequence extremum problems with restrictions.* Soviet Math. Dokl. **7** (1966), 72–75.

[30] A. W. Roberts, D. E. Varberg. *Convex functions.* Pure and Applied Mathemtics, Vol. 57. Academic Pres. New York - London. 1973. **MR0442824**

[31] M. Z. Sarikaya , E. Set , M. E. Ozdemir. *On some new inequalities of Hadamard type involving h-convex functions.* Acta. Math. Univ. Comenian. **79**(2010), no. 2,265–272. **MR2745175.**

[32] M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak. *Hermite-Hadamard inequalities for fractional integrals and relater fractional inequalities.* Math. Comput. Model. **57**,2403-2407 (2013)

[33] E. Set, M. Tomar, S. Maden. *Hermite Hadamard Type Inequalities for s-Convex Stochastic Processes in the Second Sense.* Turkish Journal of Analysis and Number Theory, **2**(2014, no. 6, 202-207.

[34] J.J. Shynk. *Probability, Random Variables, and Random Processes: Theory and Signal Processing Applications.* Wiley, 2013. **MR3088510**

[35] A. Skowronski. *On some properties of J-convex stochastic processes.* Aequationes Mathematicae 44 (1992) 249-258. **MR1181272**

[36] A. Skowronski. *On Wrighy-Convex Stochastic Processes*. Ann. Math. Sil. **9**(1995), 29-32. **MR186450.**

[37] G. Toader. *Some generalizations of the convexity.* Proceedings of the Colloquium on Approximation and Optimization, Univ. Cluj-Napoca, Cluj-Napoca, 1985, 329–338. **MR0847286.**

[38] S. Varošanec. *On $h-$convexity.* J. Math. Anal. Appl. **326** (2007), no. 1, 303?311. **MR2277784.**

[39] Ying Wu, Feng Qi and Da-Wei Niu. *Integral inequalities of Hermite-Hadamard type for the product of strongly logarithmically and other convex functions.*, Maejo International Journal of Science and Technology, Available online at www.mijst.mju.ac.th (2015).

[40] B. Xi, F. Qi. *Properties and Inequalities for the $(h_1,h_2)-$ and $(h_1,h_2,m)-$GA-Convex functions.* Journal Cogent Mathematics. **3**(2016)

[41] B. Xi,S. Wang and F. Qi. P*Properties and inequalities fot the h- and (h,m)-logarithmically convex functions.* Creat. Math. Inform. **23** (2014), no. 1, 123–130. **MR3288515.**

[42] E. A. Youness. *E-convex sets, E-convex functions, and E-convex programming.*, J. Optim. Theory Appl. **102**(1999), no. 2, 439–450. **MR1706810.**

Miguel J. Vivas C. earned his PhD degree from Universidad Central de Venezuela, Caracas, Distrito Capital (2014) in the field Pure Mathematics (Nonlinear Analysis). He has vast experience of teaching and research at university levels. It covers many areas of Mathematical such as Inequalities, Bounded Variation Functions and Ordinary Differential Equations. He has written and published several research articles in reputed international journals of mathematical and textbooks. He is currently Titular Professor in Decanato de Ciencias y Tecnologa of Universidad Centroccidental Lisandro Alvarado (UCLA), Barquisimeto, Lara state, Venezuela, and invited professor in Facultad de Ciencias Naturales y Matemáticas from Escuela Superior Politécnica del Litoral (ESPOL), Guayaquil, Ecuador.

Jorge E. Hernández H. earned his M.Sc. degree from Universidad Centroccidental Lisandro Alvarado, Barquisimeto, Estado Lara (2001) in the field Pure Mathematics (Harmonic Analysis). He has vast experience of teaching at university levels. It covers many areas of Mathematical such as Mathematics applied to Economy, Functional Analysis, Harmonical Analysis (Wavelets). He is currently Associated Professor in Decanato de Ciencias Económicas y Empresariales of Universidad Centroccidental Lisandro Alvarado (UCLA), Barquisimeto, Lara state, Venezuela.