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Inequalities Associated with Invariant Harmonically h-Convex Functions

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Abstract: In this paper, we introduce a new class of harmonically convex functions which is called invariant harmonically h-convex function. We show that the class of invariant harmonically h-convex functions unifies several other new and classes of invariant harmonically convex functions. Some associated integral inequalities of Hermite-Hadamard type are obtained. Several special cases are also discussed.

Keywords: Convex functions, harmonically convex functions, invariant harmonically h-convex functions, hypergeometric, Hermite-Hadamard inequality.
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1 Introduction and Preliminaries

In recent years, the classical concept of convexity has been extended in different dimensions, see [1,2,3,4,5,6,7,8,11,12,13,14,15,17,20,21,22]. Iscan [9] introduced the notion of harmonic convex set and harmonic convex functions. These new classes inspired many researchers and as result several new generalizations of harmonic convexity came into the literature, for example see [16]. An interesting aspect of theory of convexity is its close relationship with theory of inequalities. Numerous inequalities have been obtained via convex functions and via its variant forms, see [6]. Recently Mishra et al. [11] introduces the class of invariant harmonic convex set and invariant harmonic convex functions. They also derived new Hermite-Hadamard type inequalities via this new class of harmonic convex functions. Inspired by this, we in this paper introduce a new unifying class of invariant harmonic convex function which is called as invariant harmonically h-convex functions. As special case we also introduce other classes of invariant harmonic convex function. We also derive some new integral inequalities of Hermite-Hadamard type associated with invariant harmonically h-convex functions. This is the main motivation of writing this paper.

We now recall some previously known concepts. Let $\mathscr{X}$ be a topological vector space. Let $\mathscr{K} \subset \mathscr{X} \setminus \{0\}$ be a set satisfying the following conditions. For $x,y \in \mathscr{K}$, let $I[y,x]$ be a path joining y and x contained in $\mathscr{K}$ and the map $\gamma_{xy};[0,1] \to I[y,x]$ be continuous. The set $\mathscr{K}$ has the invariant harmonic convex combination property in a given direction $v \in \mathscr{K}$, if the following conditions are satisfied:

(P1) $y+tv \in K$ for all $t \in [0,1]$, $v \in \mathscr{X}$ and $y \in \mathscr{K}$.

(P2) $y+tv = \begin{cases} y, & if\ t=0; \\ x, & if\ t=1. \end{cases}$ and $y+tv = \frac{x+y}{2}$ if $t=\frac{1}{2}$.

(P3) For any $z \in I[y,x] \subset \mathscr{K}$, we have $z = y+tv = x+(1-t)v$.

(P4) $\frac{xy}{y+tv} \in I[y,x]$ for all $x,y \in \mathscr{K}$.

Harmonic convex sets and harmonic convex functions are defined as:

Definition 1([20]). *A set $\mathscr{K} \subset \mathbb{R} \setminus \{0\}$ is said to be harmonic convex, if*

$$\frac{xy}{tx+(1-t)y} \in \mathscr{K} \quad \forall x,y \in \mathscr{K}, t \in [0,1].$$

Definition 2([9]). *Let $\mathscr{K}$ be a harmonic convex set. A function $f : \mathscr{K} \to \mathbb{R}$ is said to be harmonic convex*

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function, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t)f(x)+tf(y), \quad \forall x,y\in\mathscr{K}, t\in[0,1].$$

Iscan [9] proved following Hermite-Hadamard type inequality for harmonically convex functions.

Theorem 1.*Let $f:\mathscr{K}\to\mathbb{R}$ be harmonically convex function and $a,b\in\mathscr{K}$ with $a<b$. If $f\in L[a,b]$, then*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

Definition 3([11]). *A set $\mathscr{K}\subset\mathbb{R}\setminus\{0\}$ is said to be invariant harmonic convex set in the direction $v\in\mathbb{R}\setminus\{0\}$, if $\mathscr{K}$ has the invariant harmonic combination properties (P1) to (P4).*

Definition 4([11]). *A function $f:\mathscr{K}\to\mathbb{R}$ is said to be invariant harmonically convex function, if*

$$f\left(\frac{xy}{y+tv}\right) \leq (1-t)f(x)+tf(y), \quad \forall x,y\in\mathscr{K}, t\in[0,1].$$

Mishra et al. [11] proved following Hermite-Hadamard type of inequality via invariant harmonically convex function.

Theorem 2([11]). *For $v\in\mathbb{R}$, let $\mathscr{K}=\mathscr{K}_v\subset\mathbb{R}\setminus\{0\}$ be a invariant harmonically convex set. For $a,b\in\mathscr{K}$ with $a<b$, let there exist vectors $v,w\in\mathbb{R}\setminus\{0\}$ with $v+w=0$, such that*
$$a+tv=\begin{cases} a, \text{ if } t=0;\\ b, \text{ if } t=1,\end{cases} \text{ and } b+tw=\begin{cases} b, \text{ if } t=0;\\ a, \text{ if } t=1.\end{cases}$$
Suppose $f:\mathscr{K}\to\mathbb{R}$ is invariant harmonically convex function with respect to the direction $v\in\mathbb{R}$. If $f\in\mathscr{L}[a,b]$, then

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

For the existence of Theorem 2 authors [11] have derived following auxiliary result.

Lemma 1([11]). *For $v\in\mathbb{R}$, let $\mathscr{K}=\mathscr{K}_v\subset\mathbb{R}\setminus\{0\}$ be a invariant harmonically convex set. Let $f:\mathscr{K}\to\mathbb{R}$ be an invariant harmonically convex function with respect to the direction $v\in\mathbb{R}\setminus\{0\}$, then for all $x,y\in\mathscr{K}$, the following inequality holds:*

$$f\left(\frac{2xy}{x+y}\right) \leq \frac{f(x)+f(y)}{2}.$$

The following auxiliary result [11] plays an important role in the development of some of our man results.

Lemma 2([11]). *Let For $v\in\mathbb{R}$, let $\mathscr{K}=\mathscr{K}_v\subset\mathbb{R}\setminus\{0\}$ be a invariant harmonically convex set. For $a,b\in\mathscr{K}$ with $a<b$, let there exist vectors $v\in\mathbb{R}\setminus\{0\}$, such that* $a+tv=\begin{cases} a, \text{ if } t=0;\\ b, \text{ if } t=1.\end{cases}$ *Suppose $f:\mathscr{K}\to\mathbb{R}$ is differentiable function on K° with respect to the direction $v\in\mathbb{R}\setminus\{0\}$. If $f'\in\mathscr{L}[a,b]$, then*

$$\frac{f(a)+f(b)}{2}-\frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x = \frac{abv}{2}\int_0^1 \frac{1-2t}{(a+tv)^2}f'\left(\frac{ab}{a+tv}\right)\mathrm{d}t$$

2 Some new classes

In this section, we define new class of invariant harmonically h-convex functions. We also discuss some special cases.

Definition 5. *Let $h:\mathscr{J}=(0,1)\subset\mathbb{R}\to\mathbb{R}$ a non-negative function. A function $f:\mathscr{K}\to\mathbb{R}$ is said to be invariant harmonically h-convex function, if*

$$f\left(\frac{xy}{y+tv}\right) \leq h(1-t)f(x)+h(t)f(y), \quad \forall x,y\in\mathscr{K}, t\in[0,1].$$

I. Under the assumptions of Definition 5, if $h(t)=t$ in Definition 5, then the class of invariant harmonically h-convex functions reduces to the class of of invariant harmonically convex functions introduced and studied by Mishra et al. [11].
II. Under the assumptions of Definition 5, if $h(t)=t^s$ in Definition 5, then the class of invariant harmonically h-convex functions reduces to a new class of Breckner type of invariant harmonically s-convex function.

$$f\left(\frac{xy}{y+tv}\right) \leq (1-t)^s f(x)+t^s f(y), \quad \forall x,y\in\mathscr{K}, t\in[0,1], s\in(0,1].$$

III. Under the assumptions of Definition 5, if $h(t)=t^{-s}$ in Definition 5, then the class of invariant harmonically h-convex functions reduces to a new class of Godunova-Levin-Dragomir type of invariant harmonically s-convex function.

$$f\left(\frac{xy}{y+tv}\right) \leq (1-t)^{-s} f(x)+t^{-s} f(y), \quad \forall x,y\in\mathscr{K}, t\in(0,1), s\in[0,1].$$

IV. Under the assumptions of Definition 5, if $h(t)=t^{-1}$ in Definition 5, then the class of invariant harmonically h-convex functions reduces to a new class of Godunova-Levin type of invariant harmonically function.

$$f\left(\frac{xy}{y+tv}\right) \leq \frac{1}{1-t}f(x)+\frac{1}{t}f(y), \ \forall x,y\in\mathscr{K}, t\in(0,1).$$

V. Under the assumptions of Definition 5, if $h(t) = 1$ in Definition 5, then the class of invariant harmonically h-convex functions reduces to a new class of invariant harmonically P function.

$$f\left(\frac{xy}{y+tv}\right) \le f(x) + f(y), \quad \forall x,y \in \mathscr{K}, t \in [0,1].$$

It is evident from the above discussed special cases that the class of invariant harmonically h-convex functions is quite unifying one. We now define the class of invariant harmonically log-convex function.

Definition 6. *A function $f : \mathscr{K} \subset \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+$ is said to be invariant harmonically log-convex function, if*

$$f\left(\frac{xy}{y+tv}\right) \le f^{1-t}(x) f^t(y), \quad \forall x,y \in \mathscr{K}, t \in [0,1].$$

Taking log on both sides of above inequality, we have

$$\log f\left(\frac{xy}{y+tv}\right) \le (1-t)\log f(x) + t\log f(y).$$

The class of invariant harmonically quasi convex functions can be defined as:

Definition 7. *A function $f : \mathscr{K} \to \mathbb{R}$ is said to be invariant harmonically quasi convex function, if*

$$f\left(\frac{xy}{y+tv}\right) \le \max\{f(x), f(y)\}, \quad \forall x,y \in \mathscr{K}, t \in [0,1].$$

3 Main Results

In this section, we prove our main results.

Definition 8. *Two functions f and g are said to be similarly ordered, if*

$$(f(x) - f(y))(g(x) - g(y)) \ge 0, \quad \forall x,y \in \mathbb{R}.$$

Proposition 1. *Let f and g be two invariant harmonically h-convex functions. If f and g are similarly ordered functions and $h(t) + h(1-t) \le 1$, then the product fg is also harmonically convex function.*

Proof. Let f and g be invariant harmonically convex functions. Then

$$\begin{aligned}
& f\left(\frac{ab}{b+tv}\right) g\left(\frac{ab}{b+tv}\right) \\
& \le [h(1-t)f(a) + h(t)f(b)][h(1-t)g(a) + h(t)g(b)] \\
& = [h(1-t)]^2 f(a)g(a) \\
& \quad + h(t)h(1-t)[f(a)g(b) + f(b)g(a)] + [h(t)]^2 f(b)g(b) \\
& \le [h(1-t)]^2 f(a)g(a) \\
& \quad + h(t)h(1-t)[f(a)g(a) + f(b)g(b)] + [h(t)]^2 f(b)g(b) \\
& = [h(1-t)f(a)g(a) + h(t)f(b)g(b)][h(t) + h(1-t)] \\
& \le h(1-t)f(a)g(a) + h(t)f(b)g(b). \qquad (1)
\end{aligned}$$

This shows that the product of two invariant harmonically h-convex functions is again invariant harmonically h-convex function. □

Theorem 3. *For $v \in \mathbb{R}$, let $\mathscr{K} = \mathscr{K}_v \subset \mathbb{R} \setminus \{0\}$ be a invariant harmonically convex set. For $a, b \in \mathscr{K}$ with $a < b$, let there exist vectors $v, w \in \mathbb{R} \setminus \{0\}$ with $v + w = 0$, such that*

$$a + tv = \begin{cases} a, & if\ t=0; \\ b, & if\ t=1, \end{cases} \quad and \quad b + tw = \begin{cases} b, & if\ t=0; \\ a, & if\ t=1. \end{cases}$$

Suppose $f : \mathscr{K} \to \mathbb{R}$ is invariant harmonically h-convex function with respect to the direction $v \in \mathbb{R}$. If $f \in \mathscr{L}[a,b]$ and $h(\frac{1}{2}) \ne 0$, then

$$\begin{aligned}
\frac{1}{2h(\frac{1}{2})} f\left(\frac{2ab}{a+b}\right) & \le \frac{ab}{v} \int_a^b \frac{f(x)}{x^2} \mathrm{d}x \\
& \le [f(a) + f(b) \int_0^1 h(t) \mathrm{d}t.
\end{aligned}$$

Proof. Since f is harmonically h-convex function, so utilizing Lemma 1, we have

$$f\left(\frac{2xy}{x+y}\right) \le h(\frac{1}{2})[f(x) + f(y)].$$

Let $x = \frac{ab}{b+tv}$ and $y = \frac{ab}{a+tw}$, then, we have

$$\frac{2xy}{x+y} = \frac{2ab}{a+b+t(v+w)} = \frac{2ab}{a+b}, \quad \because w = -v.$$

This implies

$$f\left(\frac{2ab}{a+b}\right) \le h(\frac{1}{2}) \left[f\left(\frac{ab}{b+tv}\right) + f\left(\frac{ab}{a+tw}\right) \right].$$

Integrating both sides of above inequality with respect to t on [0,1], we have

$$\begin{aligned}
& \int_0^1 f\left(\frac{2ab}{a+b}\right) \mathrm{d}t \\
& \le h(\frac{1}{2}) \left[\int_0^1 f\left(\frac{ab}{b+tv}\right) \mathrm{d}t + \int_0^1 f\left(\frac{ab}{a+tw}\right) \mathrm{d}t \right].
\end{aligned}$$

Using the change of variable technique, we have

$$\frac{1}{h(\frac{1}{2})} f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{v} \int_a^b \frac{f(x)}{x^2} \mathrm{d}x - \frac{ab}{w} \int_a^b \frac{f(x)}{x^2} \mathrm{d}x.$$

This implies

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{v} \int_a^b \frac{f(x)}{x^2} \mathrm{d}x. \qquad (2)$$

Also, since f is invariant harmonically h-convex function, then

$$f\left(\frac{ab}{b+tv}\right) \le h(1-t)f(a) + h(t)f(b).$$

Integrating both sides of above inequality with respect to t on $[0,1]$, we have

$$\frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq [f(a)+f(b)]\int_0^1 h(t)\mathrm{d}t. \tag{3}$$

Combining inequalities (2) and (3) completes the proof. □

We now discuss some special cases of Theorem 3.

I. Under the assumptions of Theorem 3, if $h(t)=t$, then we have Theorem 3.3 [].

II. Under the assumptions of Theorem 3, if $h(t)=t^s$, then we have Hermite-Hadamard type of inequality associated with Breckner type of invariant harmonically s-convex functions.

$$2^{s-1}f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{s+1}.$$

III. Under the assumptions of Theorem 3, if $h(t)=t^{-s}$, then we have Hermite-Hadamard type of inequality associated with Godunova-Levin-Dragomir type of invariant harmonically s-convex functions.

$$\frac{1}{2^{s+1}}f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{1-s}.$$

IV. Under the assumptions of Theorem 3, if $h(t)=1$, then we have Hermite-Hadamard type of inequality associated with invariant harmonically P-functions.

$$\frac{1}{2}f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq f(a)+f(b).$$

We now derive Hermite-Hadamard type inequality via product of two invariant harmonucally convex functions.

Theorem 4. *For $v\in\mathbb{R}$, let $\mathscr{K}=\mathscr{K}_v\subset\mathbb{R}\setminus\{0\}$ be a invariant harmonically convex set. For $a,b\in\mathscr{K}$ with $a<b$, let there exist vectors $v,w\in\mathbb{R}\setminus\{0\}$ with $v+w=0$, such that*

$$a+tv=\begin{cases} a, \text{ if } t=0;\\ b, \text{ if } t=1,\end{cases} \quad \text{and} \quad b+tw=\begin{cases} b, \text{ if } t=0;\\ a, \text{ if } t=1.\end{cases}$$

Suppose $f,g:\mathscr{K}\to\mathbb{R}$ is invariant harmonically h-convex function with respect to the direction $v\in\mathbb{R}$. If $fg\in\mathscr{L}[a,b]$, then

$$\frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right)\mathrm{d}x$$
$$\leq M(a,b)\int_0^1 h^2(t)\mathrm{d}t + N(a,b)\int_0^1 h(t)h(1-t)\mathrm{d}t,$$

where

$$M(a,b)=f(a)g(a)+f(b)g(b), \tag{4}$$

and

$$N(a,b)=f(a)g(b)+f(b)g(a). \tag{5}$$

Proof. Let f,g be two invariant harmonically h-convex functions, then we have

$$\frac{ab}{v}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right)\mathrm{d}x$$
$$=\int_0^1 f\left(\frac{ab}{b+tv}\right)g\left(\frac{ab}{b+tv}\right)\mathrm{d}t$$
$$\leq \int_0^1 (h(1-t)f(a)+h(t)f(b))$$
$$\times(h(1-t)g(a)+h(t)w(b))\mathrm{d}t$$
$$=M(a,b)\int_0^1 (h(t))^2\mathrm{d}t + N(a,b)\int_0^1 h(t)h(1-t)\mathrm{d}t.$$

This completes the proof. □

Theorem 5. *Under the assumptions of Theorem 4, if f and g are similarly ordered functions, then, we have*

$$\frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right)\mathrm{d}x \leq M(a,b)\int_0^1 h(t)\mathrm{d}t,$$

where $M(a,b)$ is given by (4).

Proof. The proof is obvious. □

Now using Lemma 2, we prove our next results.

Theorem 6. *Under the assumptions of Lemma 2, if $|f'|^q$, $q\geq 1$ is invariant harmonically h-convex function, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq \frac{abv}{2}\mathscr{A}_1^{1-\frac{1}{q}}\left(\mathscr{A}_2|f'(a)|^q+\mathscr{A}_3|f'(b)|^q\right)^{\frac{1}{q}},$$

where

$$\mathscr{A}_1 = a^{-2}\Big[{}_2\mathscr{F}_1(2,2;3;-\frac{v}{a})$$
$$-{}_2\mathscr{F}_1(2,1;2;-\frac{v}{a})+{}_2\mathscr{F}_1(2,1;3;-\frac{v}{2a})\Big], \tag{6}$$

$$\mathscr{A}_2 = \int_0^1 \frac{|1-2t|h(t)}{(a+tv)^2}\mathrm{d}t, \tag{7}$$

and

$$\mathscr{A}_3 = \int_0^1 \frac{|1-2t|h(1-t)}{(a+tv)^2}\mathrm{d}t, \tag{8}$$

respectively.

Proof. Using Lemma 2, power mean inequality and the fact that $|f'|^q$ is invariant harmonically h-convex function, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right| \\
&=\left|\frac{abv}{2}\int_0^1 \frac{1-2t}{(a+tv)^2} f'\left(\frac{ab}{a+tv}\right)\mathrm{d}t\right| \\
&\le \frac{abv}{2}\int_0^1 \left|\frac{1-2t}{(a+tv)^2}\right|\left|f'\left(\frac{ab}{a+tv}\right)\right|\mathrm{d}t \\
&\le \frac{abv}{2}\left(\int_0^1 \frac{|1-2t|}{(a+tv)^2}\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \left|\frac{1-2t}{(a+tv)^2}\right|\left|f'\left(\frac{ab}{a+tv}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\le \frac{abv}{2}\left(\int_0^1 \frac{|1-2t|}{(a+tv)^2}\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \frac{|1-2t|}{(a+tv)^2}\right. \\
&\left.\times\left[h(t)|f'(a)|^q+h(1-t)|f'(b)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}} \\
&=\frac{abv}{2}\mathscr{A}_1^{1-\frac{1}{q}}\left(\mathscr{A}_2|f'(a)|^q+\mathscr{A}_3|f'(b)|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

We now discuss some special cases of Theorem 6.

Corollary 1. *Under the assumptions of Theorem 6, if $q=1$, then, we have*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right| \\
&\le \frac{abv}{2}\left(\mathscr{A}_2|f'(a)|+\mathscr{A}_3|f'(b)|\right),
\end{aligned}$$

where $\mathscr{A}_2$, $\mathscr{A}_3$ are given by (7) and (8) respectively.

If $h(t)=t^s$ in Theorem 6, we have result for Breckner type of invariant harmonically s-convex functions.

Corollary 2. *Under the assumptions of Lemma 2, if $|f'|^q$, $q\ge 1$ is Breckner type of invariant harmonically s-convex function, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right| \\
&\le \frac{abv}{2}\mathscr{A}_1^{1-\frac{1}{q}}\left(\mathscr{B}_1|f'(a)|^q+\mathscr{B}_2|f'(b)|^q\right)^{\frac{1}{q}},
\end{aligned}$$

where $\mathscr{A}_1$ is given by (6), and

$$\begin{aligned}
\mathscr{B}_1 &= \int_0^1 \frac{|1-2t|t^s}{(a+tv)^2}\mathrm{d}t \\
&= a^{-2}\left[\frac{2}{s+2}\,{}_2\mathscr{F}_1\left(2,s+2;s+3;-\frac{v}{a}\right)\right. \\
&\quad -\frac{1}{s+1}\,{}_2\mathscr{F}_1\left(2,s+1;s+2;-\frac{v}{a}\right) \\
&\quad \left.+\frac{1}{2^s(s+1)(s+2)}\,{}_2\mathscr{F}_1\left(2,s+1;s+3;-\frac{v}{2a}\right)\right], \quad (9)
\end{aligned}$$

$$\begin{aligned}
\mathscr{B}_2 &= \int_0^1 \frac{|1-2t|(1-t)^s}{(a+tv)^2}\mathrm{d}t \\
&= a^{-2}\left[\frac{2}{(s+1)(s+2)}\,{}_2\mathscr{F}_1\left(2,2;s+3;-\frac{v}{a}\right)\right. \\
&\quad -\frac{1}{s+1}\,{}_2\mathscr{F}_1\left(2,1;s+2;-\frac{v}{a}\right) \\
&\quad \left.+\frac{1}{2}\,{}_2\mathscr{F}_1\left(2,1;3;-\frac{v}{2a}\right)\right], \quad (10)
\end{aligned}$$

respectively.

If $h(t)=t^{-s}$ in Theorem 6, we have result for Godunova-Levin-Dragomir type of invariant harmonically s-convex functions.

Corollary 3. *Under the assumptions of Lemma 2, if $|f'|^q$, $q\ge 1$ is Godunova-Levin-Dragomir type of invariant harmonically s-convex function, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right| \\
&\le \frac{abv}{2}\mathscr{A}_1^{1-\frac{1}{q}}\left(\mathscr{C}_1|f'(a)|^q+\mathscr{C}_2|f'(b)|^q\right)^{\frac{1}{q}},
\end{aligned}$$

where $\mathscr{A}_1$ is given by (6), and

$$\begin{aligned}
\mathscr{C}_1 &= \int_0^1 \frac{|1-2t|t^{-s}}{(a+tv)^2}\mathrm{d}t \\
&= a^{-2}\left[\frac{2}{2-s}\,{}_2\mathscr{F}_1\left(2,2-s;3-s;-\frac{v}{a}\right)\right. \\
&\quad -\frac{1}{1-s}\,{}_2\mathscr{F}_1\left(2,1-s;2-s;-\frac{v}{a}\right) \\
&\quad \left.+\frac{2^s}{(1-s)(2-s)}\,{}_2\mathscr{F}_1\left(2,1-s;3-s;-\frac{v}{2a}\right)\right], \quad (11)
\end{aligned}$$

$$\mathscr{C}_2 = \int_0^1 \frac{|1-2t|(1-t)^{-s}}{(a+tv)^2}\mathrm{d}t$$

$$
\begin{aligned}
&= a^{-2}\Bigg[\frac{2}{(1-s)(2-s)}\,{}_2\mathscr{F}_1(2,2;3-s;-\frac{v}{a})\\
&\quad -\frac{1}{1-s}\,{}_2\mathscr{F}_1(2,1;2-s;-\frac{v}{a})\\
&\quad +\frac{1}{2}\,{}_2\mathscr{F}_1(2,1;3;-\frac{v}{2a})\Bigg], \qquad (12)
\end{aligned}
$$

respectively.

If $h(t)=1$ in Theorem 6, we have result for invariant harmonically P-functions.

Corollary 4. *Under the assumptions of Lemma 2, if $|f'|^q$, $q\geq 1$ is invariant harmonically P-function, then*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right|\\
&\leq \frac{abv}{2}\mathscr{A}_1\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}},
\end{aligned}
$$

where $\mathscr{A}_1$ is given by (6).

Theorem 7. *Under the assumptions of Lemma 2, if $|f'|^q$, $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$, is invariant harmonically h-convex. Then, we have*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right|\\
&\leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\mathscr{A}_4|f'(a)|^q+\mathscr{A}_5|f'(b)|^q\right)^{\frac{1}{q}},
\end{aligned}
$$

where

$$
\mathscr{A}_4=\int_0^1 \frac{h(t)}{(a+tv)^{2q}}\mathrm{d}t, \qquad (13)
$$

and

$$
\mathscr{A}_5=\int_0^1 \frac{h(1-t)}{(a+tv)^{2q}}\mathrm{d}t, \qquad (14)
$$

respectively.

Proof. Using Lemma 2, Holder's inequality and the fact that $|f'|^q$ is invariant harmonically h-convex function, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right|\\
&\leq \frac{ab(b-a)}{2}\int_0^1\left|\frac{1-2t}{(a+tv)^2}\right|\left|f'\left(\frac{ab}{a+tv}\right)\right|\mathrm{d}t\\
&\leq \frac{ab(b-a)}{2}\left(\int_0^1|1-2t|^p\mathrm{d}t\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1\frac{1}{(a+tv)^{2q}}\left|f'\left(\frac{ab}{a+tv}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\
&\leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1\frac{1}{(a+tv)^{2q}}\right.\\
&\quad\left.\times\left\{h(t)|f'(a)|^q+h(1-t)|f'(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\\
&=\frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\mathscr{A}_4|f'(a)|^q+\mathscr{A}_5|f'(b)|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

This completes the proof. □

If $h(t)=t^s$ in Theorem 7, we have result for Breckner type of invariant harmonically s-convex functions.

Corollary 5. *Under the assumptions of Lemma 2, if $|f'|^q$, $q\geq 1$ is Breckner type of invariant harmonically s-convex function, then*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right|\\
&\leq \frac{abv}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\mathscr{D}_1|f'(a)|^q+\mathscr{D}_2|f'(b)|^q\right)^{\frac{1}{q}},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathscr{D}_1&=\int_0^1\frac{t^s}{(a+tv)^{2q}}\mathrm{d}t\\
&=\frac{a^{-2q}{}_2F_1[2q,1+s,2+s,-\frac{v}{a}]}{1+s}, \qquad (15)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathscr{D}_2&=\int_0^1\frac{(1-t)^s}{(a+tv)^{2q}}\mathrm{d}t\\
&=\frac{a^{-2q}{}_2F_1[2q,1,2+s,-\frac{v}{a}]}{1+s}, \qquad (16)
\end{aligned}
$$

respectively.

If $h(t)=t^{-s}$ in Theorem 7, we have result for Godunova-Levin type of invariant harmonically s-convex functions.

Corollary 6. *Under the assumptions of Lemma 2, if $|f'|^q$, $q\geq 1$ is Godunova-Levin type of invariant harmonically s-convex function, then*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{v}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right|\\
&\leq \frac{abv}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\mathscr{D}_3|f'(a)|^q+\mathscr{D}_4|f'(b)|^q\right)^{\frac{1}{q}},
\end{aligned}
$$

where

$$\mathscr{D}_3 = \int_0^1 \frac{t^{-s}}{(a+tv)^{2q}}dt = \frac{a^{-2q}{}_2F_1[2q,1-s,2-s,-\frac{v}{a}]}{1-s}, \quad (17)$$

and

$$\mathscr{D}_4 = \int_0^1 \frac{(1-t)^{-s}}{(a+tv)^{2q}}dt = \frac{a^{-2q}{}_2F_1[2q,1,2-s,-\frac{v}{a}]}{1-s}, \quad (18)$$

respectively.

If $h(t)=1$ in Theorem 7, we have result for invariant harmonically P-functions.

Corollary 7. *Under the assumptions of Lemma 2, if $|f'|^q$, $q\ge 1$ is invariant harmonically P-function, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{v}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{abv}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \mathscr{D}^{\frac{1}{q}}\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}},$$

where

$$\mathscr{D} = \int_0^1 \frac{1}{(a+tv)^{2q}}dt = a^{-2q}{}_2\mathscr{F}_1[2q,1,2,-\frac{v}{a}]. \quad (19)$$

Now we derive results for harmonically log-convex functions.

Theorem 8. *For $v\in\mathbb{R}$, let $\mathscr{K}=\mathscr{K}_v\subset\mathbb{R}\setminus\{0\}$ be a invariant harmonically convex set. For $a,b\in\mathscr{K}$ with $a<b$, let there exist vectors $v,w\in\mathbb{R}\setminus\{0\}$ with $v+w=0$, such that*

$$a+tv=\begin{cases} a, \text{ if } t=0; \\ b, \text{ if } t=1, \end{cases} \text{ and } b+tw=\begin{cases} b, \text{ if } t=0; \\ a, \text{ if } t=1. \end{cases}$$

Suppose $f:\mathscr{K}\to\mathbb{R}_+$ is invariant harmonically log-convex function with respect to the direction $v\in\mathbb{R}$. If $f\in\mathscr{L}[a,b]$, then

$$f\left(\frac{2ab}{a+b}\right) \le \exp\left[\frac{ab}{v}\int_a^b \log\left(\frac{f(x)}{x^2}\right)dx\right] \le \sqrt{(f(a)f(b))}. \quad (20)$$

Proof. Let f be harmonically log-convex function. For $t=\frac{1}{2}$, we have

$$f\left(\frac{2xy}{x+y}\right) \le [(f(x))(f(y))]^{\frac{1}{2}}.$$

This implies that

$$f\left(\frac{2ab}{a+b}\right) \le \left[\left\{f\left(\frac{ab}{b+tv}\right)\right\}\left\{f\left(\frac{ab}{a+tw}\right)\right\}\right]^{\frac{1}{2}}.$$

Taking log on both sides, we get

$$\log f\left(\frac{2ab}{a+b}\right) \le \frac{1}{2}\left[\log f\left(\frac{ab}{b+tv}\right)+\log f\left(\frac{ab}{a+tw}\right)\right].$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\log f\left(\frac{2ab}{a+b}\right) \le \frac{1}{2}\left[\int_0^1 \log f\left(\frac{ab}{b+tv}\right)dt+\int_0^1 \log f\left(\frac{ab}{a+tw}\right)dt\right].$$

This implies that

$$\log f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{v}\int_a^b \log\left(\frac{f(x)}{x^2}\right)dx. \quad (21)$$

Also

$$f\left(\frac{ab}{b+tv}\right) \le f^{1-t}(a)f^t(b).$$

This implies that

$$\log f\left(\frac{ab}{b+tv}\right) \le (1-t)\log f(a)+t\log f(b).$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\frac{ab}{v}\int_a^b \log\left(\frac{f(x)}{x^2}\right)dx \le \frac{\log f(a)+\log f(b)}{2} = \log(f(a)f(b))^{\frac{1}{2}}. \quad (22)$$

Combining (21) and (22), we have

$$\log f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{v}\int_a^b \log\left(\frac{f(x)}{x^2}\right)dx \le \log(f(a)f(b))^{\frac{1}{2}}. \quad (23)$$

Taking antilog on both sides of (23), we have

$$f\left(\frac{2ab}{a+b}\right) \le \exp\left[\frac{ab}{v}\int_a^b \log\left(\frac{f(x)}{x^2}\right)dx\right] \le \sqrt{(f(a)f(b))}.$$

This completes the proof. □

References

[1] W. W. Breckner, Stetigkeitsaussagen fiir eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen. Pupl. Inst. Math. 23, 13-20, (1978).

[2] G. Cristescu, L. Lupsa, Non-connected Convexities and Applications, Kluwer Academic Publishers, Dordrecht, Holland, 2002.

[3] G. Cristescu, M. A. Noor, M. U. Awan, Bounds of the second degree cumulative frontier gaps of functions with generalized convexity, Carpath. J. Math, 30(2), (2014).

[4] S. S. Dragomir, Inequalities of Hermite-Hadamard type for h-convex functions on linear spaces, RGMIA Research Report Collection 16(11), (2013). Article 72.

[5] S. S. Dragomir, B. Mond, Integral inequalities of Hadamard's type for log-convex functions, Demonstration Math. 2, 354-364, (1998).

[6] S. S. Dragomir, C. E. M. Pearce, Selected topics on Hermite-Hadamard inequalities and applications, Victoria University, Australia (2000).

[7] S. S. Dragomir, J. Pečarić and L. E. Persson, Some inequalities of Hadamard type, Soochow J. Math, 21, 335-341, (1995).

[8] E. K. Godunova and V. I. Levin, Neravenstva dlja funkcii sirokogo klassa, soderzascego vypuklye, monotonnye i nekotorye drugie vidy funkii. Vycislitel. Mat. i. Fiz. Mezvuzov. Sb. Nauc. Trudov, MGPI, Moskva. 138-142, (1985).

[9] I. Iscan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacettepe J. Math. Stat., 43(6), 935-942, (2014).

[10] M. V. Mihai, M. A. Noor, K. I. Noor, M. U. Awan, Some integral inequalities for harmonic h-convex functions involving hypergeometric functions. Appl. Math. Comput. 252, 257-262, (2015).

[11] S. N. Mishra, P. K. Das, Some results associated with Hermite-Hadamard type inequalities in invariant harmonic convex set, Global J. Pure and Appl. Math., 13(4), 1241-1248, (2017).

[12] M. A. Noor, M. U. Awan, K. I. Noor, On some inequalities for relative semi-convex functions, J. Ineqal. Appl. 2013, 2013:332.

[13] M. A. Noor, K. I. Noor, M. U. Awan, Geometrically relative convex functions, Appl. Math. Infor. Sci., 8(2), 607-616, (2014).

[14] M. A. Noor, K. I. Noor, M. U. Awan, Hermite-Hadamard inequalities for relative semi-convex functions and applications, Filomat, 28(2), 221-230, (2014).

[15] M. A. Noor, K. I. Noor, M. U. Awan, Some characterizations of harmonically log-convex functions, Proc. Jangjeon Math. Soc., 17(1), 51-61, (2014).

[16] M. A. Noor, K. I. Noor, M. U. Awan, S. Costache, Some integral inequalities for harmonically h-convex functions. U. P. B. Sci. Bull., Series A. 77(1), 5-16, (2015).

[17] M.A. Noor, K.I. Noor, M.U. Awan, S. Khan, Fractional HermiteHadamard inequalities for some new classes of GodunovaLevin functions, Appl. Math. Inf. Sci. 8(6), 2865-2872, (2014).

[18] M. Z. Sarikaya, A. Saglam, H. Yildirim, On some Hadamard-type inequalities for h-convex functions. Jour. Math. Ineq. 2(3), 335-341, (2008).

[19] M. Z. Sarikaya, E. Set, M. E. Özdemir, On some new inequalities of Hadamard type involving h-convex functions. Acta Math. Univ. Comenianae. 2, 265-272, (2010).

[20] H.-N Shi, J. Zhang, Some new judgement theorems of Schur geometric and Schur harmonic convexities for a class of symmetric functions, J. Inequal. Appl. 2013, 2013:527.

[21] S. Varosanec, On h-convexity, J. Math. Anal. Appl. 326, 303-311, (2007).

[22] T.-Y Zhang, A.-P. Ji, F. Qi, Integral inequalities of Hermite-Hadamard type for harmonically quasi-convex functions, Proc. Jangjeon Math. Soc., 16(3), 399-407, (2013).

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