Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/110304

Discrete Spline Solution for General Type Variational Problems

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Received: 6 Mar. 2016, Revised: 24 Feb. 2017, Accepted: 26 Feb. 2017
Published online: 1 May 2017

Abstract: In this paper, a discrete quintic spline method is developed for the solution of general type variational problem with the Lagrangian that depends on the minimizer and its higher derivatives. The proposed technique depends on an arbitrary parameter μ which enables us to construct discrete splines with high order accuracy. These methods are shown to be of fourth order. Numerical illustrations are given to demonstrate the validity and the effectiveness of the proposed approach.

Keywords: Calculus of variations, Discrete spline, Finite Difference, Boundary value problem, Error bound

1 Introduction

Nowadays, there is a need for finding an optimal function which arises in many fields such as elasticity, optimal control problem, optics, vibrations, beam problems and economics, see [13].
In the following, we consider the simplest form of a variational problem as:

$$J[u(x)] = \int_a^b P(x,u,u')dx, \tag{1}$$

where J is the functional that its extremum must be found. To find the maximal or minimal value of J, the boundary points have the form:

$$u(a) - A_0 = u(b) - B_0 = 0. \tag{2}$$

The solution $u(x)$ of the minimizer $J[u(x)]$ should satisfy the following Euler-Lagrange equation

$$\frac{\partial P}{\partial u} - \frac{dP}{dx}(\frac{\partial P}{\partial u'}) = 0, \tag{3}$$

with conditions (2).

In this paper, we consider the more general type variational problem with the Lagrangian that depends on the minimizer and its first and second derivative,

$$J[u(x)] = \int_a^b P(x,u,u',u'')dx, \tag{4}$$

the Euler equation can be derived as the simplest case: the variation of the goal functional is

$$\delta J = \int_a^b (\frac{\partial P}{\partial u}\delta u + \frac{\partial P}{\partial u'}\delta u' + \frac{\partial P}{\partial u''}\delta u'')dx, \tag{5}$$

integrating by parts, we obtain

$$\delta J = \int_a^b (\frac{\partial P}{\partial u} - \frac{d}{dx}\frac{\partial P}{\partial u'} + \frac{d^2}{dx^2}\frac{\partial P}{\partial u''})\delta u dx$$

$$+[\frac{\partial P}{\partial u'}\delta u + \frac{\partial P}{\partial u''}\delta u' - \frac{d}{dx}\frac{\partial P}{\partial u''}\delta u]_{x=a}^{x=b}, \tag{6}$$

the stationary condition becomes the differential equation:

$$\frac{d^2}{dx^2}\frac{\partial P}{\partial u''} - \frac{d}{dx}\frac{\partial P}{\partial u'} + \frac{\partial P}{\partial u} = 0, \tag{7}$$

together with two boundary conditions on each end,

$$\delta u'\frac{\partial P}{\partial u''} = 0, \frac{\partial P}{\partial u'}\delta u - \frac{d}{dx}\frac{\partial P}{\partial u''}\delta u = 0, \tag{8}$$

at $x = a$ and $x = b$.

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Or by the correspondent main conditions posed on the minimizer u and the derivatives $u^{(j)}, j=1,2,3$ at the end points as given below:

$$u(a)-A_1=u^{(1)}(a)-A_2=0, u^{(2)}(b)-B_1=u^{(3)}(b)-B_2=0, \quad (9)$$

or

$$u(a)-A_3=u^{(1)}(a)-A_4=0, u(b)-B_3=u^{(1)}(b)-B_4=0, \quad (10)$$

where A_i and B_i, $i=1,2,3,4$, are finite real constants.

Similarly, the stationary equations for Lagrangian $P(x,u,u',u'',....,u^{(n)})$ depending on first n derivatives of u is

$$\sum_{k=1}^{n}(-1)^n\frac{d^k}{dx^k}\frac{\partial P}{\partial u^{(k)}}+\frac{\partial P}{\partial u}=0, \quad (11)$$

in this formula u can be replaced by a vector for several minimizers.

In [1], Galerkin method is used for solving the variational problems ((1), (2)). A walsh series, Shifted Legendre, Laguerre series, Shifted Chebyshev, Legendre and Haar wavelets, Adomian decomposition method, variational iteration method, Chebyshev finite difference method , Bernstein direct method and exponential spline method are used for the solution of variational problems in [2]-[13].
While (to our knowledge) there is no methods developed for the variational problems ((7) and (9) or (10)). For boundary value problems, finite difference, polynomial spline and exponential spline methods are developed in Khan et. al. [14], Usmani [15], Usmani [16] Van Daele [17] and Zahra [18,19].

The goal of this article is to establish a new discrete quintic spline method for the solution of variational problem (((7),(9)) or ((7),(10))).

The polynomial splines deal with pieces that are connected together by the continuity of certain derivatives at the knots but in the discrete spline, the connections will involve differences instead of derivatives.

Discrete splines were first proposed by Mangasarian and Schumaker [23] as solutions to certain minimization problems involving differences. Thereafter, Schumaker [24] and Lyche [25,26] discussed cubic discrete splines involving central differences. There are many papers appeared in this area but there a few papers in the field of solving boundary value problems. A discrete cubic spline is proposed for solving obstacle problems in [22]. Also, Zahra and Van Daele [20] developed a discrete spline with nonuniform mesh for solving singularly perturbed problems.

The main merits of the method in this article over other methods is the introduction of a simple technique for constructing discrete splines. Since the discrete quintic spline contains a parameter, high order accuracy can be achieved.

The article is organized as follows: In section 2, we briefly mention the main definitions of the discrete quintic spline function. Boundary conditions and convergence analysis are discussed in sections 3 and 4. Numerical results are presented to show the applicability and the accuracy in section 5. Finally, in section 6 we conclude the results of the proposed methods.

In following section, we introduce the principle of discrete quintic spline [20,21,25,26,27,28].

2 Discrete quintic spline

Let $\boldsymbol{\omega}: a=x_0<x_1<x_2<...<x_n=b$ be a uniform mesh of $[a,b]$ with $\omega=x_i-x_{i-1}, i=1,2,...,n$. Let $h\in(0,\omega]$ be a given constant and the action of the central difference operators is given by:

$$D_h^{\{0\}}r(x)=r(x),\ D_h^{\{1\}}r(x)=\frac{r(x+h)-r(x-h)}{2h},$$

$$D_h^{\{2\}}r(x)=\frac{r(x+h)-2r(x)+r(x-h)}{h^2},$$

$$D_h^{\{3\}}r(x)=\frac{r(x+2h)-2r(x+h)+2r(x-h)-r(x-h)}{2h^3},$$

$$D_h^{\{4\}}r(x)=\frac{r(x+2h)-4r(x+h)+6r(x)-4r(x-h)+r(x-2h)}{h^4},$$

and for sufficiently smooth $r(x)$: $D_0^{\{k\}}r(x)=r^{(k)}(x)$, $k=0,1,2$. Using Lyche [25], we use the basic polynomial $x^{\{k\}}$ as: $x^{\{k\}}=x^k, k=0,1,2$, $x^{\{3\}}=x(x^2-h^2)$, $x^{\{4\}}=x^2(x^2-h^2), x^{\{5\}}=x(x^2-h^2)(x^2-4h^2)$, such that $D_h^{\{1\}}x^{\{k\}}=kx^{\{k-1\}}, k=0,1,2,3,5$ and $D_h^{\{1\}}x^{\{4\}}=2x(2x^2+h^2)$.

Using [20,21,25], we can give the following theorem.

Theorem 1. Let $s(x,\omega,h)$ be a piecewise continuous function over $[a,b]$ with with the mesh $\boldsymbol{\omega}$ and $s_i(x,\omega,h)$ is its restriction on $[x_{i-1},x_i]$ connecting the points (x_{i-1},s_{i-1}) and (x_i,s_i), $i=1,2,...,n$. Then we have a unique discrete quintic spline $s(x,\omega,h)$, satisfying:

$$D_h^{\{k\}}(s_{i+1}-s_i)(x_i)=0,\ \ k=0,1,2,3,4,\ \ i=1,2,...,n-1. \quad (12)$$

Our main purpose is to solve the problem (7) with the boundary conditions given by Eq.(9) or Eq.(10). So we need first to obtain explicit expression to $s_i(x,\omega,h)$. Clearly $s_i(x,\omega,h)$ should pass through the points (x_{i-1},s_{i-1}) and (x_i,s_i). Let

$$s_i(x_i,\omega,h)=s_i,\qquad D_h^{\{4\}}s_i(x_i,\omega,h)=M_i, \quad (13)$$

then since $D_h^{\{4\}}s(x,\omega,h)$ is linear in the interval $[x_{i-1},x_i]$, we have:

$$D_h^{\{4\}}s_i(x,\omega,h)=\frac{x_i-x}{\omega}M_{i-1}+\frac{x-x_{i-1}}{\omega}M_i, \quad (14)$$

It follows from Eq.(14) that

$$s_i(x,\omega,h)=\frac{(x_i-x)^{\{5\}}}{120\omega}M_{i-1}+\frac{(x-x_{i-1})^{\{5\}}}{120\omega}M_i$$
$$+\frac{(x_i-x)^{\{3\}}}{6\omega}p_i+\frac{(x-x_{i-1})^{\{3\}}}{6\omega}q_i$$
$$+\frac{(x_i-x)}{\omega}d_i+\frac{(x-x_{i-1})}{\omega}e_i, x\in[x_{i-1},x_i], \quad (15)$$

where p_i, q_i , d_i and e_i are constants. we can determine these constants by the conditions:

$$s_i(x_{i-1},\omega,h)=s_{i-1},\ \ s_i(x_i,\omega,h)=s_i,$$

$$D_h^{\{2\}}s_i(x_{i-1},\omega,h)=F_{i-1} \quad \text{and} \quad D_h^{\{2\}}s_i(x_i,\omega,h)=F_i. \quad (16)$$

Then we have the following expressions

$$p_i=F_{i-1}-\frac{(\omega^2-h^2)}{6}M_{i-1},\ \ q_i=F_i-\frac{(\omega^2-h^2)}{6}M_i,$$
$$d_i=s_{i-1}-\frac{(\omega^2-h^2)(\omega^2-4h^2)}{120}M_{i-1}-\frac{(\omega^2-h^2)}{6}p_i,$$
$$e_i=s_i-\frac{(\omega^2-h^2)(\omega^2-4h^2)}{120}M_i-\frac{(\omega^2-h^2)}{6}q_i, \quad (17)$$

Then, we get the discrete quintic spline

$$s_i(x,\omega,h)=\frac{(x_i-x)^{\{5\}}}{120\omega}M_{i-1}+\frac{(x-x_{i-1})^{\{5\}}}{120\omega}M_i$$
$$+q_i[\frac{(x-x_{i-1})^{\{3\}}-(\omega^2-h^2)(x-x_{i-1})}{6\omega}]$$
$$+p_i[\frac{(x_i-x)^{\{3\}}-(\omega^2-h^2)(x_i-x)}{6\omega}]$$
$$+[s_i-\frac{(\omega^2-h^2)(\omega^2-4h^2)}{120}M_i]\frac{(x-x_{i-1})}{\omega}$$
$$+[s_{i-1}-\frac{(\omega^2-h^2)(\omega^2-4h^2)}{120}M_{i-1}]\frac{(x_i-x)}{\omega}. \quad (18)$$

For $h\to 0$, Eq.(18) reduces to the ordinary quintic spline:

$$s_i(x,\omega)=\frac{(x_i-x)^5}{120\omega}M_{i-1}+\frac{(x-x_{i-1})^5}{120\omega}M_i$$
$$+[F_i-\frac{\omega^2}{6}M_i][\frac{(x-x_{i-1})^3}{6\omega}-\frac{\omega(x-x_{i-1})}{6}]$$
$$+[F_{i-1}-\frac{\omega^2}{6}M_{i-1}][\frac{(x_i-x)^{\{3\}}}{6\omega}-\frac{\omega(x_i-x)}{6}]$$
$$+[s_i-\frac{\omega^4}{120}M_i]\frac{(x-x_{i-1})}{\omega}+[s_{i-1}-\frac{\omega^4}{120}M_{i-1}]\frac{(x_i-x)}{\omega}. \quad (19)$$

Using the continuity condition $D_h^{\{1\}}s_i(x_i,\omega,h)=D_h^{\{1\}}s_{i+1}(x_i,\omega,h)$, we get:

$$\alpha_1M_{i-1}+\alpha_2M_i+\alpha_1M_{i+1}=-6(s_{i-1}-2s_i+s_{i+1})+(\omega^2-h^2)F_{i-1}$$
$$+(4\omega^2+2h)F_i+(\omega^2-h^2)F_{i+1}\ i=1,2,...,n-1, \quad (20)$$

where

$$\alpha_1=\frac{(\omega^2-h^2)(7\omega^2+2h^2)}{60} \quad \text{and} \quad \alpha_2=\frac{(\omega^2-h^2)(16\omega^2-4h^2)}{60}.$$

Also, from the continuity condition $D_h^{\{3\}}s_i(x_i,\omega,h)=D_h^{\{3\}}s_{i+1}(x_i,\omega,h)$, we have:

$$\frac{(\omega^2-h^2)}{6}M_{i-1}+\frac{(4\omega^2+2h^2)}{6}M_i+\frac{(\omega^2-h^2)}{6}M_{i+1}$$
$$=F_{i-1}-2F_i+F_{i+1}, i=1,2,...,n-1 \quad (21)$$

Now we get expressions for F_{i-1}, F_i and F_{i+1} from Eq.(20) and Eq.(21) by the operation (Eq.(20) - (ω^2-h^2) Eq.(21))

$$F_i=\frac{(s_{i-1}-2s_i+s_{i+1})}{\omega^2}+\frac{(h^2-\omega^2)}{120\omega^2}[(\omega^2-4h^2)M_{i-1}$$
$$+(8\omega^2+8h^2)M_i+(\omega^2-4h^2)M_{i+1}], \quad (22)$$

Increase the indices of Eq.(22) by one and then reduce the indices by one, we get:

$$F_{i+1}=\frac{(s_i-2s_{i+1}+s_{i+2})}{\omega^2}+\frac{(h^2-\omega^2)}{120\omega^2}[(\omega^2-4h^2)M_i$$
$$+(8\omega^2+8h^2)M_{i+1}+(\omega^2-4h^2)M_{i+2}], \quad (23)$$
$$F_{i-1}=\frac{(s_{i-2}-2s_{i-1}+s_i)}{\omega^2}+\frac{(h^2-\omega^2)}{120\omega^2}[(\omega^2-4h^2)M_{i-2}$$
$$+(8\omega^2+8h^2)M_{i-1}+(\omega^2-4h^2)M_i], \quad (24)$$

Now from Eqns (22),(23) and (24) into Eq.(21),one have

$$s_{i-2}-4s_{i-1}+6s_i-4s_{i+1}+s_{i+2}=\alpha M_{i-2}+\beta M_{i-1}$$
$$+\gamma M_i+\beta M_{i+1}+\alpha M_{i+2}, i=2,3,...,n-2, \quad (25)$$

where

$$\alpha=\frac{(\omega^2-h^2)(\omega^2-4h^2)}{120},$$
$$\beta=\frac{(\omega^2-h^2)(13\omega^2+8h^2)}{60},\gamma=\frac{(11\omega^4+5h^2\omega^2+4h^4)}{20}.$$

For $h\to 0$, Eq.(25) reduces to the relation of ordinary quintic spline, see [16,17,22]:

$$s_{i-2}-4s_{i-1}+6s_i-4s_{i+1}+s_{i+2}=\frac{\omega^4}{120}[M_{i-2}$$
$$+26M_{i-1}+66M_i+26M_{i+1}+M_{i+2}], i=2,3,...,n-2. \quad (26)$$

For the direct computation of the unknowns, the relation (25) needs three extra conditions in case of the boundary conditions (9) and two equations for the boundary conditions (10). These equations are constructed in the following section.

3 Boundary conditions

1.First class of boundary equations

The boundary equations are written in terms of the minimizer u and the derivatives $u^{(j)}, j=1,2,3$. We can get the first end condition near $x=a$ as:

$$-5.5s_0+9s_1-4.5s_2+s_3=3\omega s_0^{(1)}+\omega^4(a_0M_0+a_1M_1$$

$$+a_2M_2+a_3M_3+a_4M_4+a_5M_5), i=1, \quad (27)$$

The other two end conditions near $x=b$ are:

$$s_{n-3}-4s_{n-2}+5s_{n-1}-2s_n=-\omega^2F_n+\omega^4(b_5M_{n-5}$$

$$+b_4M_{n-4}+b_3M_{n-3}+b_2M_{n-2}+b_1M_{n-1}+b_0M_n), i=n-1, \quad (28)$$

and

$$s_{n-2}-2s_{n-1}+s_n=\omega^2F_n-\omega^3s_n^{(3)}+\omega^4(c_5M_{n-5}+c_4M_{n-4}$$

$$+c_3M_{n-3}+c_2M_{n-2}+c_1M_{n-1}+c_0M_n), i=n, \quad (29)$$

2.Second class of boundary equations

For this class, the boundary equations are presented in terms of the minimizer u and its first derivative. We can address the first end condition near $x=a$ as given by Eq.(27).

The second end condition near $x=b$ is given by:

$$s_{n-3}-4.5s_{n-2}+9s_{n-1}-5.5s_n=-3\omega s_n^{(1)}+\omega^4(d_5M_{n-5}$$

$$+d_4M_{n-4}+d_3M_{n-3}+d_2M_{n-2}+d_1M_{n-1}+d_0M_n), i=n-1, \quad (30)$$

where a_j, b_j, c_j and $d_j, j=0,1,.,5$ are free parameters.

Also, the local truncation errors $lte_i, i=1,n-1,n$ of the equations given by Eq. (27), Eq. (28) and Eq.(29) can be addressed as follows:

$$lte_1=-5.5u_0+9u_1-4.5u_2+u_3-3\omega u_0^{(1)}-\omega^4(a_0u_0^{(4)}$$

$$+a_1u_1^{(4)}+a_2u_2^{(4)}+a_3u_3^{(4)}+a_4u_4^{(4)}+a_5u_5^{(4)}), i=1, \quad (31)$$

$$lte_{n-1}=u_{n-3}-4u_{n-2}+5u_{n-1}-2u_n+\omega^2u_n^{(2)}-\omega^4(b_5u_{n-5}^{(4)}$$

$$+b_4u_{n-4}^{(4)}+b_3u_{n-3}^{(4)}+b_2u_{n-2}^{(4)}+b_1u_{n-1}^{(4)}+b_0u_n^{(4)}), i=n-1, \quad (32)$$

and

$$lte_n=u_{n-2}-2u_{n-1}+u_n-\omega^2u_n^{(2)}-\omega^3u_n^{(3)}+\omega^4(c_5u_{n-5}^{(4)}$$

$$+c_4u_{n-4}^{(4)}+c_3u_{n-3}^{(4)}+c_2u_{n-2}^{(4)}+c_1u_{n-1}^{(4)}+c_0u_n^{(4)}), i=n,. \quad (33)$$

Similarly, the local truncation errors for the equations given by Eq. (27) and Eq. (30) can be obtained by the same manner.

Lemma 1

Let $u\in C^8[a,b]$ and $u_i=u(x_i)$ then the local truncation errors $lte_i, i=2,3,...,n-2$ of the scheme (25) are:

$$lte_i=\frac{1}{12}(1-3\mu^2)\omega^6u_i^{(6)}+\frac{1}{240}(4-15\mu^2+8\mu^4)\omega^8u_i^{(8)}$$

$$+O(\omega^9), i=2,3,...,n-2, \quad (34)$$

and

$$lte_i=\begin{cases} c_6\omega^6u_i^{(6)}+O(\omega^8), i=2,3,...,n-2, & \mu\neq\frac{1}{\sqrt{3}}, \\ \frac{-1}{2160}\omega^8u_i^{(8)}+O(\omega^9), i=2,3,...,n-2, & \mu=\frac{1}{\sqrt{3}}, \end{cases} \quad (35)$$

where $h=\mu\omega$ is a parameter and c_6 is a constant.

Proof

To get the local truncation errors $lte_i, i=2,3,...,n-2$ of Eq.(25), we first rewrite this equation in the following form:

$$lte_i=u_{i-2}-4u_{i-1}+6u_i-4u_{i+1}+u_{i+2}-\alpha u_{i-2}^{(4)}-\beta u_{i-1}^{(4)}$$

$$-\gamma u_i^{(4)}-\beta u_{i+1}^{(4)}-\alpha u_{i+2}^{(4)}, i=2,3,...,n-2.$$

In the above equation, Taylor series are used to expand the terms u_{i-2} and $u_{i-2}^{(4)}$,etc. around the point x_i and Lemma 1 is then derived.

Lemma 2 The local truncation errors lte_i for the boundary equations are:

(i) First class of boundary equations type

Fourth order method

$$lte_i=\begin{cases} \frac{1}{6720}\omega^8u_i^{(8)}+O(\omega^9), i=1 \\ \frac{-241}{60480}\omega^8u_i^{(8)}+O(\omega^9), i=n-1 \\ \frac{-893}{60480}\omega^8u_i^{(8)}+O(\omega^9), i=n, \end{cases} \quad (36)$$

where

$$(a_0,.,a_3)=\frac{1}{280}(8,151,52,-1),$$

$$(b_0,.,b_3)=\frac{1}{180}(49,71,-19,4),$$

$$(c_0,.,c_3)=\frac{1}{360}(28,245,56,1), a_j=d_j$$

Sixth order method

$$lte_i=\begin{cases} \frac{-1}{960}\omega^{10}u_i^{(10)}+O(\omega^{11}), i=1 \\ \frac{-504}{151200}\omega^{10}u_i^{(10)}+O(\omega^{11}),, i=n-1 \\ \frac{-5071}{604800}\omega^{10}u_i^{(10)}+O(\omega^{11}),, i=n, \end{cases} \quad (37)$$

where

$$(a_0,.,a_5)=\frac{1}{33600}(937,18240,5990,140,-135,28),$$

$$(b_0,.,b_5)=\frac{1}{60480}(4233,43274,5662,3432,-1391,230),$$

$$(c_0,.,c_5) = \frac{1}{60480}(14918, 30693, -18272, 11446, -4158, 653),$$
$$a_j = d_j.$$

(ii) Second class of boundary equations type
Fourth order method

$$lte_i = \left\{ \frac{1}{6720}\omega^8 u_i^{(8)} + O(\omega^9),\ i = 1, n-1, \right. \quad (38)$$

Sixth order method

$$lte_i = \left\{ \frac{-1}{960}\omega^{10} u_i^{(10)} + O(\omega^{11}),\ i = 1, n-1, \right. \quad (39)$$

Proof
See Lemma 1.
In the following, we propose a simple algorithm for computing discrete quintic spline.

Algorithm for discrete quintic spline solution

Clearly, we assume that the general form of the variational problem (7) at the grid point x_i as $u_i^{(4)} = f(x_i, u_i)$, then $M_i = f_i$.

Now the steps needed to evaluate the discrete quintic spline solution of the problem ((7),(9)) or the problem ((7),(10)) are listed as :

1. Determine the mesh ω and choose a value of the parameter $h (\in (0, \omega])$.
2. Solve the system (25) for $s_i,\ i = 1, 2, \ldots, n$.
3. M_i can be evaluated from $M_i = M(x_i, s_i)$.
4. From Eq.(22), F_i can be evaluated.
5. After computing s_i, M_i and F_i, it is possible to construct a discrete quintic spline function from Eq. (18).

This algorithm has its own advantage in comparison with nonpolynomial spline methods [6,13,14,18, 25-27]. For example, once the solution has been computed, the information needed to construct a discrete quintic spline function is available but it is not always possible for the nonpolynomial spline solution.

4. Convergence analysis

The discrete quintic spline solution of Eq.(7) and Eq.(9) is based on the nonlinear equations given by Eqns ((25),(27)-(29)) and on the nonlinear equations given by Eqns ((25),(27) and (30)).

Let $U = (u_i)$, $S = (s_i)$, $C = (c_i)$, $\bar{C} = (\bar{c_i})$, $T = (lte_i)$, $E = (e_i), e_i = u_i - s_i$ be n-dimensional column vectors. Then we can formulate our methods in the following matrix form:

$$\begin{array}{l} (i)\, N_0 U + B f(U) = C + T, \\ (ii)\, N_0\, S^{(\nu+1)} + B f(S^{(\nu)}) = C, \nu = 0, 1, 2, \ldots \\ (iii)\, NE = T, N = N_0 + BG, G = diag(\frac{\partial f_i}{\partial u_i}), \end{array} \quad (40)$$

where $f(S^{(\nu)}) = f(x_i, S_i^{(\nu)}), i = 1, 2 \ldots, n$, N_0 and B are banded matrices and Also we have $f(U) - f(S) = GE$.

For the linear problem, where $f(x,u) = p(x) - g(x)u$, then Eq. (ii) in Eq.(40) is changed to:

$$NS = \bar{C}, \quad \text{where} \quad N = N_0 + B\bar{G} \quad \text{and} \quad \bar{G} = diag(g_i). \quad (41)$$

In the following, we will write the matrices associated with the variational problem given by Eqns ((25),(27)-(29)).
The matrix N_0 has the form:

$$N_0 = \begin{pmatrix} 9 & -\frac{9}{2} & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & & & \ddots & & & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 5 & -2 \\ & & & & 1 & -2 & 1 \end{pmatrix}, \quad (42)$$

and the matrix B has the form:

$$B = \begin{pmatrix} \omega^4 a_1 & \omega^4 a_2 & \omega^4 a_3 & \omega^4 a_5 & \omega^4 a_2 & & \\ \beta & \gamma & \beta & \alpha & & & \\ \alpha & \beta & \gamma & \beta & \alpha & & \\ & & & \ddots & & & \\ & & \alpha & \beta & \gamma & \beta & \alpha \\ & \omega^4 b_5 & \omega^4 b_4 & \omega^4 b_3 & \omega^4 b_2 & \omega^4 b_1 & \omega^4 b_0 \\ & \omega^4 c_5 & \omega^4 c_4 & \omega^4 c_3 & \omega^4 c_2 & \omega^4 c_1 & \omega^4 c_0 \end{pmatrix}. \quad (43)$$

For the vector C, we have:

$$c_i = \begin{cases} \frac{11}{2}A_1 + 3\omega A_2 + \omega^4 a_0 f_0, & i = 1 \\ -A_1 + \alpha f_0, & i = 2 \\ 0, & i = 3, 4, \ldots, n-3 \\ -\omega^2 B_1, & i = n-1 \\ \omega^2 B_1 - \omega^3 B_2, & i = n \end{cases} \quad (44)$$

Finally the vector $\bar{C}$ can be written from Eq. (41).

$$\bar{c_i} = \begin{cases} 5.5A_1 + 3hA_2 + h^4(\omega_0(g_0 - f_0 A_1) + \omega_1 g_1 + \omega_2 g_2) \\ \qquad + \omega_3 g_3 + \omega_4 g_4 + \omega_5 g_5, \quad i = 1 \\ -A_1 + h^4(\alpha(g_0 + g_4 - f_0 A_1) + \beta(g_1 + g_3) + \gamma g_2), \\ \quad i = 2 \\ h^4(\alpha(g_{i-2} + g_{i+2}) + \beta(g_{i-1} + g_{i+1}) + \gamma g_i), \\ \quad i = 3, 4, \ldots, n-3 \\ -h^2 B_1 + h^4(\alpha_5 g_{n-5} + \alpha_4 g_{n-4} + \alpha_3 g_{n-3} + \alpha_2 g_{n-2}) \\ (\qquad + \alpha_1 g_{n-1} + \alpha_0 g_n), \quad i = n-1 \\ h^2 B_1 - h^3 B_2 + h^4(\beta_5 g_{n-5} + \beta_4 g_{n-4} + \beta_3 g_{n-3} \\ \qquad + \beta_2 g_{n-2} + \beta_1 g_{n-1} + \beta_0 g_n), \quad i = n \end{cases} \quad (45)$$

Also, the matrices for the variational problem given by Eqns ((25),(27) and (30)) can be written by the same manner.

In order to derive a bound on $\|E\|$(where$\|.\|$refers to the infinite norm), we need the following lemma.
Lemma 3 see [18] If Q is square matrix of order n and $\|Q\| < 1$, then $(I+Q)^{-1}$exists and$\left\|(I+Q)^{-1}\right\| < \frac{1}{1-\|Q\|}$.

Recall to Eq.(40) and rewrite it as

$$E = N^{-1}T = (N_0 + BG)^{-1}T$$
$$= (I + N_0^{-1}BG)^{-1}N_0^{-1}T,$$

then

$$\|E\| \leq \frac{\|N_0^{-1}\| \, \|T\|}{1 - \|N_0^{-1}\| \, \|B\| \, \|G\|}, \tag{46}$$

Provided that $\|N_0^{-1}\| \, \|B\| \, G^* < 1$, $G^* = \max\left|\frac{\partial f}{\partial u}\right|$.

Lemma 4 [15], the matrix N_0 is nonsingular and its $\|N_0^{-1}\|$ satisfies that

$$\|N_0^{-1}\| = \frac{n}{24}[3n^3 + 4n^2 - n + 2].$$

Also, we have that
$\|N_0^{-1}\| = \frac{(b-a)^4}{8\omega^4}[1 + \frac{4\omega}{3(b-a)} - \frac{\omega^2}{3(b-a)^2} + \frac{2\omega^3}{3(b-a)^3}]$,
then $\|N_0^{-1}\| = \frac{(b-a)^4}{8\omega^4}[1 + O(\omega)]$, for small value of ω, we have

$$\|N_0^{-1}\| \leq \frac{(b-a)^4}{8\omega^4}. \tag{47}$$

Lemma 5 The discrete problem (ii) in Eq.(40) is uniquely solvable iff

$$G^* < \frac{8}{(b-a)^4 \bar{B}}, \tag{48}$$

where $\|B\| = \omega^4 \bar{B}$ and $\bar{B}$ is a finite number .

Proof

The proof follows directly from Lemma 3 and Lemma 4. then the discrete problem (ii) in Eq.(40) has a unique solution if $G^* < \frac{8}{(b-a)^4 \bar{B}}$.

The next result gives the order of convergence of our method.

Recall to Lemma 1 and Lemma 2, we get from Eq.(36) that:

Fourth order method 1

$$\|T\| = \frac{1}{6720}\omega^8 U_8 \quad \text{and} \quad \|B\| = \frac{136}{135}\omega^4$$

$$\text{where} \quad \mu = \frac{1}{\sqrt{3}} \quad \text{and} \quad U_8 = \max_{a \leq x \leq b}\left|u^{(8)}(x)\right|. \tag{49}$$

Then it follows from Eq.(46), Eq.(47) and Eq.(49) that

$$\|E\| \leq \frac{135(b-a)^4 U_8 \omega^4}{53760[135 - 7(b-a)^4 G^*]} = G_1\omega^4, \tag{50}$$

where

$$G_1 = \frac{135(b-a)^4 U_8}{53760[135 - 7(b-a)^4 G^*]}.$$

Fourth order method 2

Also, from Lemma 1 and Lemma 2, we get from Eq.(37) that:

$$\|T\| = \frac{1}{2160}\omega^8 U_8 \quad \text{and} \quad \|B\| = \frac{136}{135}\omega^4$$

$$\text{where} \quad \mu = \frac{1}{\sqrt{3}} \quad \text{and} \quad U_8 = \max_{a \leq x \leq b}\left|u^{(8)}(x)\right|. \tag{51}$$

Then it follows from Eq.(46), Eq.(47) and Eq.(51) that

$$\|E\| \leq \frac{135(b-a)^4 U_8 \omega^4}{17280[135 - 7(b-a)^4 G^*} = G_2\omega^4, \tag{52}$$

where

$$G_2 = \frac{135(b-a)^4 U_8}{17280[135 - 7(b-a)^4 G^*]}.$$

The above results can be summarized in the following theorem:

Theorem 6

Let $u(x)$ be the exact solution of the variational problem (7) along with first class of boundary conditions (9) or with second class of boundary conditions (10) and let u_i, $i = 0, 1, 2, \ldots, n$, satisfy the discrete problem (40). Further, if $e_i = u_i - s_i$, then $\|E\| \leq G_j\omega^4$, $\mu = \frac{1}{\sqrt{3}}$, where G_j, $j = 1, 2$ are constants and $\|E\| \leq G_3\omega^2$, $\mu \neq \frac{1}{\sqrt{3}}$, G_3 is a constant, neglecting all errors due to round off.

5. Numerical examples

We will consider some numerical illustrations for the solution of the variational problem using discrete quintic spline methods. All calculations are implemented with MATLAB R2014b.

Example 1: Consider the following variational problem

$$minJ[u(x)] = \int_0^1 (\frac{1}{2}((u^{(2)})^2 + xu^2) + (8 + 7x + x^3)e^x u)dx, \tag{53}$$

together with boundary conditions

$$u(0) = u^{(1)}(0) - 1 = u^{(2)}(1) + 4e = u^{(3)}(1) + 9e = 0. \tag{54}$$

The analytical solution is

$$u(x) = x(1-x)e^x. \tag{55}$$

The results for our fourth order methods are given in Table 1.

Example 2: Consider the following variational problem

$$minJ[u(x)] = \int_0^1 (\frac{1}{2}(u^{(2)})^2 + \frac{1}{3}u^3 + (8x\cos x$$

$$-(x^2 - 13)\sin x - (x^4 - 2x^2 + 1)\sin^2 x)u)dx, \tag{56}$$

along with the following boundary conditions

$$u(0) = u^{(1)}(0) + 1 = u^{(2)}(1) - 4\cos(1) - 2\sin(1)$$

$$= u^{(3)}(1) - 6\cos(1) + 6\sin(1) = 0. \tag{57}$$

Table 1: The observed maximum errors (in absolute value) for $\mu = \frac{1}{\sqrt{3}}$ and $\omega = 2^{-m}$ of Example 1

m	Fourth order method 1	Fourth order method 2
3	2.3896974E-05	4.7495949E-07
4	7.9317283E-07	7.1111918E-08
5	2.3175701E-08	5.5279230E-09
6	5.1559823E-10	4.0083136E-10
7	4.1745806E-10	5.9799276E-11

The analytical solution of (56) is

$$u(x) = (x^2 - 1)\sin(x), \tag{58}$$

The results for our fourth order methods are tabulated in Table 3.

Example 3: Consider the following variational problem

$$minJ[u(x)] = \int_{-1}^{1}(\tfrac{1}{2}((u^{(2)})^2 - xu^2) + (11 + 9x + x^2 - x^3)e^x u)dx, \tag{59}$$

together with boundary conditions

$$u(-1) = u^{(1)}(-1) - \frac{2}{e} = u(1) = u^{(1)}(1) + 2e = 0. \tag{60}$$

The analytical solution of (59) is

$$u(x) = (1 - x^2)e^x. \tag{61}$$

The numerical results for our fourth order methods are summarized in Table 5.

The observed maximum errors for the linear and nonlinear problems for our fourth order methods are listed in Tables 1 , 3 and 5,. Table 2 shows that our methods have high accuracy compared with the nonpolynomial spline method [26] , the fourth order finite difference method, fourth order shooting method and second order method introduced by Tien and Usmani [15] and Usmani [16]. Also, Table 4 declares that our methods are superior to the results in Zahra [18]. Finally in Table 5, the numerical results confirm that our methods are better than the methods in Khan et. al. [14]. These results are encouraging and suggest that our methods are practical when dealing with boundary value problems arising in beam and plate deflection theory.

Table 2: The observed maximum errors (in absolute value) for $\mu = \frac{1}{\sqrt{3}}$ and $\omega = 2^{-m}$ of Example 1

m	Fourth order method 1	Fourth order method 2	Zahra [18]	Tien and Usmani [15]	Usmani [16]
3	2.3E-5	4.7E-7	2.2E-5	2.7E-5	4.6E-5
4	7.9E-7	7.1E-8	9.8E-7	1.0E-6	2.9E-6
5	2.3E-8	5.5E-9	4.3E-8	4.5E-8	1.8E-7
6	5.1E-10	4.0E-10	2.0E-9	2.0E-9	1.1E-8
7	4.1E-10	5.9E-11	4.9E-10	2.3E-10	7.4E-10

Table 3: The observed maximum errors (in absolute value) for $\mu = \frac{1}{\sqrt{3}}$ and $\omega = 2^{-m}$ of Example 2

m	Fourth order method 1	Fourth order method 2
3	9.3337500E-06	4.4177794E-07
4	2.9435635E-07	3.0606159E-08
5	8.2413365E-09	2.2033219E-09
6	1.9522872E-10	1.3564260E-10
7	9.4502183E-11	1.0490452E-10

Table 4: The observed maximum errors (in absolute value) for $\mu = \frac{1}{\sqrt{3}}$ and $\omega = 2^{-m}$ of Example 2

m	Fourth order method 1	Fourth order method 2	Zahra [18]
3	9.33E-06	4.41E-07	1.06E-05
4	2.94E-07	3.06E-08	4.12E-07
5	8.24E-09	2.20E-09	1.70E-08
6	1.95E-10	1.35E-10	7.91E-10
7	9.45E-11	1.04E-10	2.57E-10

Table 5: The observed maximum errors (in absolute value) for $\mu = \frac{1}{\sqrt{3}}$ and $\omega = 2^{-m}$ of Example 3

m	Fourth order method 1	Fourth order method 2	Khan et. al.[14]
3	7.6E-05	7.4E-05	4.8E-03
4	8.7E-06	8.4E-06	2.7E-03
5	1.0E-06	9.9E-07	8.1E-04
6	1.2E-07	1.2E-07	2.0E-04
7	1.5E-08	1.4E-08	5.2E-05

4 Conclusion

Two new methods of order four based on discrete quintic spline are presented for solving variational problems. A simple algorithm for constructing discrete quintic spline with high order accuracy is obtained. The results obtained by the developed technique show that these methods maintain a high accuracy. Comparisons are given to demonstrate the validity and the effectivity of the proposed technique.

Acknowledgement

The work of W. K. Zahra is supported and funded by Cultural Affairs and Mission Sector, Ministry of Higher Education, Egypt.

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